Module 9.5: Static Risk Measures Geometric Brownian Motion-Based Compound Option Valuation Models

Learning objectives

- Explore static risk measures related to GBM compound options
- Illustrates the complex task of finding analytic Greeks
- Highlights ease of using numerical Greeks

See *Ch 9.5 SRM Compound Options*.

Module overview

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Compound option valuation model

Recall the compound option pricing model (CO) observed at time t under geometric Brownian motion based on an underlying instrument (S_t) with the compound option exercise price (X_C) expiring at time 2 (T_1) and the underlying option exercise price (X_U) expiring at time 1 ($T_2 > T_1$) can be expressed as

$$\begin{aligned} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right) \\ &-\iota_C\iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right) - \iota_C X_C B_{t,T_1,r} N\left(\iota_C\iota_U d_{21}\right) \end{aligned}, \quad (9.5.1)$$

where indicator functions denote

$$\iota_C = \begin{cases} +1 \text{ if compound call option} \\ -1 \text{ if compound put option} \end{cases} \text{ and} \quad (9.5.2)$$

$$\iota_U = \begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option} \end{cases}. \quad (9.5.3)$$

Recall a default-free, zero coupon, $1 par bond be expressed as

$$B_{t,T,x} = e^{-x(T-t)}, \quad (9.5.4)$$

and the bivariate cumulative standard normal distribution

$$N_2\left(a,b;\rho\right) \equiv \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{\exp\left\{-\dfrac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}} dz_1\, dz_2 . \quad (9.5.5)$$

Using a generic time to maturity, *T*, the periodic standard deviation are

$$\sigma_{t,T} = \sigma\sqrt{T-t} . \quad (9.5.6)$$

The correlation coefficient used in the bivariate distribution is

$$\rho = \frac{\sqrt{T_1 - t}}{\sqrt{T_2 - t}}, \quad (9.5.7)$$

and thus

$$\sqrt{1-\rho^2} = \frac{\sqrt{T_2 - T_1}}{\sqrt{T_2 - t}}. \tag{9.5.8}$$

Let $S_{T_1}^*$ be defined such that underlying option is at-the-money or

$$\iota_U S_{T_1}^* B_{T_1,T_2,\delta-\hat{q}} N_1\left(\iota_U d_{1,T_1,T_2}^*\right) - \iota_U X_U B_{T_1,T_2,r-\hat{q}} N_1\left(\iota_U d_{2,T_1,T_2}^*\right) - X_C = 0, \tag{9.5.9}$$

where

$$d_{2,T_1,T_2}^* = \frac{\ln\left(\frac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) - \frac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}}, \tag{9.5.10}$$

$$d_{1,T_1,T_2}^* = \frac{\ln\left(\frac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}} = d_{2,T_1,T_2}^* + \sigma_{T_1,T_2}, \text{ and} \tag{9.5.11}$$

$$N_1\left(d\right) = \int_{-\infty}^{d} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx. \tag{9.5.12}$$

Let d_{ij} denote the upper bound of the bivariate normal cumulative distribution function where $i = 1, 2$ denotes whether the volatility term is added ($i = 1$) or subtracted ($i = 2$) and $j = 1, 2$ denotes whether the evaluation is S^* at T_1 ($j = 1$) or X_U at T_2 ($j = 2$). We define

$$d_{21} \equiv \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right) - \frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}, \tag{9.5.13}$$

$$d_{11} \equiv \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right) + \frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}} = d_{21} + \sigma_{t,T_1}, \tag{9.5.14}$$

$$d_{22} \equiv \frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right) - \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}, \text{ and} \tag{9.5.15}$$

$$d_{12} \equiv \frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}} = d_{22} + \sigma_{t,T_2}. \tag{9.5.16}$$

We now turn to identify the analytic Greeks. The related proofs will be details later in this chapter should you wish to understand one method to solve for them. An effort was made to be as transparent as possible, although every step is not explained in detail.[1]

[1]See Brooks (2019) for more details.

The Greeks
Recall the value of the compound option can be expressed as an indirect function of the underlying instrument S as

$$CO_I = CO_{Indirect}\left[O(S,t),t\right], \tag{9.5.17}$$

where the underlying instrument is embedded in the underlying option $O(S,t)$. The value of the compound option can also be expressed as a direct function of the underlying instrument S as

$$CO_D = CO_{Direct}(S,t). \tag{9.5.18}$$

The underlying option is clearly a direct function of the underlying instrument and is expressed as

$$O = O(S,t). \tag{9.5.19}$$

Note the value of the compound option remains the same regardless of how it is represented or

$$CO_I\left[O(S,t),t\right] = CO_D(S,t). \tag{9.5.20}$$

We approach the Greek derivations assuming the compound option is a direct function.

Delta
The compound option delta is

$$\Delta_{CO_D} \equiv \frac{\partial CO_D(S,t,T_1,T_2)}{\partial S} = \iota_C \iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right), \tag{9.5.21}$$

and the underlying option delta is

$$\Delta_O \equiv \frac{\partial O(S,t,T_2)}{\partial S} = \iota_U B_{t,T_2,\delta-\hat{q}} N_1\left(\iota_U d_1\right). \tag{9.5.22}$$

Gamma
The compound option gamma is

$$\begin{aligned}\Gamma_{CO_D} &= \frac{\partial^2 CO_D(S,t,T_1,T_2)}{\partial S^2} \\ &= \frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right],\end{aligned} \tag{9.5.23}$$

and the underlying option gamma is

$$\Gamma_O \equiv \frac{\partial^2 O(S,t,T_2)}{\partial S^2} = \frac{B_{t,T_2,\delta-\hat{q}} n_1(d_1)}{S_t \sigma_{t,T_2}}. \tag{9.5.24}$$

Theta
The compound option theta is

$$
\begin{aligned}
\Theta_{CO_D} &\equiv \frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right) \\
&= -\frac{\sigma^2}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \left\{ N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_1}}\right]\right\} \\
&+\iota_C \iota_U \hat{q} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) \\
&-\iota_C \iota_U r B_{t,T_2,r} X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C r B_{t,T_1,r} X_C N_1\left(\iota_C d_{21}\right)
\end{aligned} \quad .(9.5.25)
$$

and the underlying option theta is

$$
\begin{aligned}
\Theta_O &\equiv \frac{\partial O\left(S,t,T_2\right)}{\partial t} \\
&= \iota_U\left(\delta-\hat{q}\right) S_t B_{\delta-\hat{q}} N\left(\iota_U d_1\right) - \iota_U\left(r-\hat{q}\right) X B_{r-\hat{q}} N\left(\iota_U d_2\right). \\
&-\frac{\sigma^2 S_t B_{\delta-\hat{q}}}{2\sigma} n\left(d_1\right)
\end{aligned} \qquad (9.5.26)
$$

It is important to note that we can ignore t embedded within rho. Rho is assumed constant as an input parameter once computed.

Vega

The compound option vega is

$$
\frac{\partial}{\partial \sigma} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \begin{bmatrix} \sqrt{T_1-t}\, N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right) n_1\left(d_{11}\right) \\ +\iota_C \sqrt{T_2-t}\, N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right) n_1\left(d_{12}\right) \end{bmatrix}, \qquad (9.5.27)
$$

and the underlying option vega is

$$
v_O \equiv \frac{\partial O}{\partial \sigma} = S B_{t,T_2,\delta} n\left(d_1\right)\sqrt{T_2-t} = X B_{t,T_2,r} n\left(d_2\right)\sqrt{T_2-t}\,. \qquad (9.5.28)
$$

Rho

The compound option rho is

$$
\begin{aligned}
\frac{\partial}{\partial r} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C \iota_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}}\left(T_2-t\right) X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) \\
&+\iota_C B_{t,T_1,r}\left(T_1-t\right) X_C N\left(\iota_C \iota_U d_{21}\right)
\end{aligned}, \qquad (9.5.29)
$$

and the underlying option rho is

$$
\rho_O \equiv \frac{\partial O}{\partial r} = \iota_U X\left(T_2-t\right) B_{t,T_2,2} N\left(\iota_U d_2\right). \qquad (9.5.30)
$$

Prior to working through the details of the Greek proofs, we will rely on several lemmas.

Important lemmas
We rely on the following lemmas for deriving the Greeks. Proofs for these lemmas follow the derivations of the Greeks.

Lemma 1: Leibniz integral rule for double integral applied to bivariate normal
Assuming $N_2\left[a(x),b(x);\rho\right]$ is a standard bivariate normal cumulative distribution function, then

$$\frac{dN_2\left[a(x),b(x);\rho\right]}{dx}=\frac{dN_2\left[a(x),b(x);\rho\right]}{db(x)}\frac{db(x)}{dx}+\frac{dN_2\left[a(x),b(x);\rho\right]}{da(x)}\frac{da(x)}{dx}. \tag{9.5.31}$$

Lemma 2: Compound option partial with respect to d21

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=0, \tag{9.5.32}$$

and

$$\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}}. \tag{9.5.33}$$

Lemma 3: Compound option partial with respect to d22

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)=0, \tag{9.5.34}$$

and

$$\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}}. \tag{9.5.35}$$

We now turn to sketch the proofs of the Greeks.

Compound option delta proof

$$\Delta_{CO_D}=\frac{\partial CO_D\left(S,t,T_1,T_2\right)}{\partial S}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right). \tag{9.5.36}$$

Proof: Note based on Equation (9.5.1), we have

$$\begin{aligned}\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)&=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial}{\partial S}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial}{\partial S}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C X_C B_{t,T_1,r}\frac{\partial}{\partial S}N_1\left(\iota_C d_3\right)\end{aligned}. \tag{9.5.37}$$

Based on Lemma 1 (Leibniz integral rule), we have

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\rho\right]}{\partial S}\\&=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{aligned}, \tag{9.5.38}$$

and

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\rho\right]}{\partial S}\\&=\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\frac{\partial d_{21}(S)}{\partial S}+\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\frac{\partial d_{22}(S)}{\partial S}.\end{aligned} \tag{9.5.39}$$

Also, from the standard normal cumulative distribution function, we note

$$\frac{\partial N_1(\iota_C d_{21})}{\partial S}=\frac{\partial N_1(\iota_C d_{21})}{\partial d_{21}}\frac{\partial d_{21}}{\partial S}. \tag{9.5.40}$$

Substituting

$$\begin{aligned}\frac{\partial}{\partial S}CO_D(S,t,T_1,T_2)&=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho)\\&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}\\&+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{aligned}\right\}\\&-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\frac{\partial d_{21}(S)}{\partial S}\\&+\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\frac{\partial d_{22}(S)}{\partial S}\end{aligned}\right\}\\&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1(\iota_C d_{21})}{\partial d_{21}}\frac{\partial d_{21}}{\partial S}\end{aligned}. \tag{9.5.41}$$

Recall

$$d_{11}=d_{21}+\sigma_{t,T_1}\text{ and} \tag{9.5.42}$$

$$d_{12}=d_{22}+\sigma_{t,T_2}. \tag{9.5.43}$$

Thus

$$\frac{\partial d_{11}}{\partial S}=\frac{\partial d_{21}}{\partial S}\text{ and} \tag{9.5.44}$$

$$\frac{\partial d_{12}}{\partial S}=\frac{\partial d_{22}}{\partial S}. \tag{9.5.45}$$

Substituting these derivatives

$$\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)$$
$$+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{array}{l}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{21}(S)}{\partial S}\\ +\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{22}(S)}{\partial S}\end{array}\right\}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\begin{array}{l}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\frac{\partial d_{21}(S)}{\partial S}\\ +\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\frac{\partial d_{22}(S)}{\partial S}\end{array}\right\}. \quad (9.5.46)$$
$$-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial S}$$

Rearranging

$$\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)$$
$$+\frac{\partial d_{21}(S)}{\partial S}\left\{\begin{array}{l}\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\\ -\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\end{array}\right\}. \quad (9.5.47)$$
$$+\frac{\partial d_{22}(S)}{\partial S}\left\{\begin{array}{l}\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\\ -\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\end{array}\right\}$$

Note based on Lemma 2,

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)} \text{ and} \quad (9.5.48)$$

$$\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}. \quad (9.5.49)$$
$$=\frac{\partial}{\partial d_{22}}CO\left(S,t,T_1,T_2\right)=0$$

And based on Lemma 3,

$$\begin{aligned}&\iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right)}{\partial d_{12}(S)}\\&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right)}{\partial d_{22}(S)}=\frac{\partial}{\partial d_{22}}CO\left(S,t,T_1,T_2\right)=0\end{aligned} \quad (9.5.50)$$

Therefore,

$$\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right). \quad (9.5.51)$$

Compound option gamma proof

$$\Gamma_{CO_D}=\frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2}=\frac{B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}}{S_t}\begin{bmatrix}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\\+\iota_C N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_2}}\end{bmatrix}. \quad (9.5.52)$$

Proof: Note based on Equation (9.5.51), we have

$$\Gamma_{CO_D}=\frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial}{\partial S}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right). \quad (9.5.53)$$

Based on Lemma 1, we have

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial S}\\&=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{aligned}. \quad (9.5.54)$$

Thus

$$\Gamma_{CO_D}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\begin{Bmatrix}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}\\+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{Bmatrix}, \quad (9.5.55)$$

and recall

$$d_{11}=\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}}\right)+\frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}}=d_{21}+\sigma_{t,T_1}, \quad (9.5.56)$$

$$\frac{\partial d_{11}}{\partial S}=\frac{1}{S_t\sigma_{t,T_1}}=\frac{\partial d_{21}}{\partial S}, \quad (9.5.57)$$

$$d_{12} = \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})} B_{T_1,T_2,-(r-\delta)}}{X_C}\right) + \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}} = d_{22} + \sigma_{t,T_2}, \text{ and} \tag{9.5.58}$$

$$\frac{\partial d_{12}}{\partial S} = \frac{1}{S_t \sigma_{t,T_2}} = \frac{\partial d_{22}}{\partial S}. \tag{9.5.59}$$

Note

$$\begin{aligned}
&\frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial d_{11}(S)} = \frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial\left[\iota_C \iota_U d_{11}(S)\right]} \frac{\partial\left[\iota_C \iota_U d_{11}(S)\right]}{\partial d_{11}(S)} \\
&= \iota_C \iota_U \frac{\partial}{\partial\left[\iota_C \iota_U d_{11}(S)\right]} \int_{-\infty}^{\iota_C \iota_U d_{11}(S)} N_1\left[\frac{\iota_U d_{12}(S) - \iota_C \rho z_2}{\sqrt{1-(\iota_C \rho)^2}}\right] n_1(z_2)\, dz_2 \\
&= \iota_C \iota_U N_1\left[\frac{\iota_U d_{12}(S) - \iota_C \rho \iota_C \iota_U d_{11}(S)}{\sqrt{1-(\iota_C \rho)^2}}\right] n_1\left[\iota_C \iota_U d_{11}(S)\right] \\
&= \iota_C \iota_U N_1\left[\frac{\iota_U d_{12}(S) - \rho \iota_U d_{11}(S)}{\sqrt{1-\rho^2}}\right] n_1\left[\iota_C \iota_U d_{11}(S)\right]
\end{aligned}, \tag{9.5.60}$$

and also

$$\begin{aligned}
&\frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial d_{12}(S)} = \frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial\left[\iota_U d_{12}(S)\right]} \frac{\partial\left[\iota_U d_{12}(S)\right]}{\partial d_{12}(S)} \\
&= \iota_U \frac{\partial}{\partial\left[\iota_U d_{12}\right]} \int_{-\infty}^{\iota_U d_{12}(S)} N_1\left[\frac{\iota_C \iota_U d_{11}(S) - \iota_C \rho z_1}{\sqrt{1-(\iota_C \rho)^2}}\right] n_1(z_1)\, dz_1 \\
&= \iota_U N_1\left[\frac{\iota_C \iota_U d_{11}(S) - \iota_C \rho \iota_U d_{12}(S)}{\sqrt{1-(\iota_C \rho)^2}}\right] n_1\left[\iota_U d_{12}(S)\right]
\end{aligned}. \tag{9.5.61}$$

Thus,

$$\Gamma_{CO_D} = \iota_C \iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \left\{ \begin{array}{l} \dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C\rho\right]}{\partial d_{11}(S)} \dfrac{\partial d_{11}(S)}{\partial S} \\ + \dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C\rho\right]}{\partial d_{12}(S)} \dfrac{\partial d_{12}(S)}{\partial S} \end{array} \right\}$$

$$= \iota_C \iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \left\{ \begin{array}{l} \iota_C\iota_U N_1\left[\dfrac{\iota_U d_{12} - \rho \iota_U d_{11}}{\sqrt{1-\rho^2}}\right] n_1\left[\iota_C\iota_U d_{11}\right] \dfrac{1}{S_t \sigma_{t,T_1}} \\ + \iota_U N_1\left[\dfrac{\iota_C\iota_U d_{11} - \iota_C \rho \iota_U d_{12}}{\sqrt{1-(\iota_C\rho)^2}}\right] n_1\left[\iota_U d_{12}\right] \dfrac{1}{S_t \sigma_{t,T_2}} \end{array} \right\}. \quad (9.5.62)$$

Therefore,

$$\Gamma_{CO_D} = \frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t} \left[N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right) \frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C\iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right) \frac{n_1(d_{12})}{\sigma_{t,T_2}} \right]. \quad (9.5.63)$$

Compound option theta proof

$$\begin{array}{l} \Theta_{CO_D} \equiv \dfrac{\partial}{\partial t} CO_D\left(S, t, T_1, T_2, \iota_C, \iota_U\right) \\ = -\dfrac{\sigma^2}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \left\{ N_1\left(\iota_U \dfrac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right) \left[\dfrac{n_1(d_{11})}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C\iota_U \dfrac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right) \left[\dfrac{n_1(d_{12})}{\sigma_{t,T_1}}\right] \right\} \\ + \iota_C\iota_U \hat{q} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right) \\ - \iota_C\iota_U r B_{t,T_2,r} X_U N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C\rho\right) - \iota_C r B_{t,T_1,r} X_C N_1\left(\iota_C d_{21}\right) \end{array}. \quad (9.5.64)$$

Proof: There are several variables that are a function of calendar time t. Note if

$$B_{t,T,x} = e^{-x(T-t)}, \quad (9.5.65)$$

then

$$\frac{\partial}{\partial t} B_{t,T,x} = x e^{-x(T-t)} = x B_{t,T,x}. \quad (9.5.66)$$

Recall from Equation (9.5.1),

$$\begin{array}{l} CO_D\left(S, t, T_1, T_2, \iota_C, \iota_U\right) = \iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right) \\ - \iota_C\iota_U X_U B_{t,T_2,r} N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C\rho\right) - \iota_C X_C B_{t,T_1,r} N_1\left(\iota_C d_{21}\right) \end{array}. \quad (9.5.67)$$

Thus,

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\
&+\iota_C\iota_U S_t B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\frac{\partial}{\partial t}B_{t,T_1,\hat{q}}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C\iota_U X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial t}B_{t,T_2,r}\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial}{\partial t}N_1\left(\iota_C d_{21}\right)-\iota_C X_C N_1\left(\iota_C d_{21}\right)\frac{\partial}{\partial t}B_{t,T_1,r}
\end{aligned}
. \qquad (9.5.68)
$$

We now work with each partial derivative with respect to calendar time (numbered in order of appearance in the equation above). Based on Lemma 1, we have

(1) $$\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial\left(\iota_C\iota_U d_{11}\right)}\frac{\partial\left(\iota_C\iota_U d_{11}\right)}{\partial t}+\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial\left(\iota_U d_{12}\right)}\frac{\partial\left(\iota_U d_{12}\right)}{\partial t}. \qquad (9.5.69)$$

Note

$$
\begin{aligned}
\frac{\partial\left(\iota_C\iota_U d_{11}\right)}{\partial t}&=\iota_C\iota_U\frac{\partial d_{11}}{\partial t}=\iota_C\iota_U\frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S_{T_1}^*}\right)+\frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}\right]\\
&=\iota_C\iota_U\frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t}{S_{T_1}^*}\right)+\left(r-\hat{q}+\frac{\sigma^2}{2}\right)\left(T_1-t\right)}{\sigma\sqrt{T_1-t}}\right]
\end{aligned}
. \qquad (9.5.70)
$$

Let

$$A\left(t\right)\equiv\ln\left(\frac{S_t}{S_{T_1}^*}\right)+\left(r-\hat{q}+\frac{\sigma^2}{2}\right)\left(T_1-t\right), \qquad (9.5.71)$$

then

$$\frac{\partial}{\partial t}A\left(t\right)=-\left(r-\hat{q}+\frac{\sigma^2}{2}\right), \qquad (9.5.72)$$

and

$$B\left(t\right)=\sigma\sqrt{T_1-t}, \qquad (9.5.73)$$

then

$$\frac{\partial}{\partial t}B\left(t\right)=\frac{\partial}{\partial t}\sigma\sqrt{T_1-t}=\frac{\sigma}{2}\left(T_1-t\right)^{-1/2}=\frac{\sigma}{2\sqrt{T_1-t}}. \qquad (9.5.74)$$

Therefore,

$$\frac{\partial\left(\iota_C\iota_U d_{11}\right)}{\partial t}=\iota_C\iota_U\frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t}{S_{T_1}^*}\right)+\left(r-\hat{q}+\frac{\sigma^2}{2}\right)\left(T_1-t\right)}{\sigma\sqrt{T_1-t}}\right]=\iota_C\iota_U\frac{\partial}{\partial t}\left[\frac{A(t)}{B(t)}\right]$$

$$=\iota_C\iota_U\left[\frac{B(t)\frac{\partial}{\partial t}A(t)-A(t)\frac{\partial}{\partial t}B(t)}{B^2(t)}\right]. \quad (9.5.75)$$

$$=\iota_C\iota_U\left[\frac{-\sigma\sqrt{T_1-t}\left(r-\hat{q}+\frac{\sigma^2}{2}\right)+\left[\ln\left(\frac{S_t}{S_{T_1}^*}\right)+\left(r-\hat{q}+\frac{\sigma^2}{2}\right)\left(T_1-t\right)\right]\left(\frac{\sigma}{2\sqrt{T_1-t}}\right)}{\sigma^2\left(T_1-t\right)}\right]$$

Reducing,

$$\frac{\partial\left(\iota_C\iota_U d_{11}\right)}{\partial t}=\iota_C\iota_U\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{11}}{2\left(T_1-t\right)}\right]. \quad (9.5.76)$$

Following a similar approach,

$$\frac{\partial\left(\iota_U d_{12}\right)}{\partial t}=\iota_U\frac{\partial d_{12}}{\partial t}=\iota_U\frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}{X_U}\right)+\frac{\sigma^2\left(T_2-t\right)}{2}}{\sigma\sqrt{T_2-t}}\right]. \quad (9.5.77)$$

$$=\iota_C\iota_U\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{12}}{2\left(T_2-t\right)}\right]$$

Therefore, (note we can cancel the indicator functions)

$$\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{11}}{2\left(T_1-t\right)}\right]. \quad (9.5.78)$$

$$+\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{12}}{2\left(T_2-t\right)}\right]$$

(2) $$\frac{\partial B_{t,T_1,\hat{q}}}{\partial t} = \hat{q} B_{t,T_1,\hat{q}}. \quad (9.5.79)$$

(3) $$\frac{\partial}{\partial t} N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) = \frac{\partial N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial\left(\iota_C \iota_U d_{21}\right)} \frac{\partial\left(\iota_C \iota_U d_{21}\right)}{\partial t} + \frac{\partial N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial\left(\iota_U d_{22}\right)} \frac{\partial\left(\iota_U d_{22}\right)}{\partial t}. \quad (9.5.80)$$

Note following a similar approach as above,

$$\frac{\partial\left(\iota_C \iota_U d_{21}\right)}{\partial t} = \iota_C \iota_U \frac{\partial d_{21}}{\partial t} = \iota_C \iota_U \frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}}\right) - \frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}}\right], \text{ and} \quad (9.5.81)$$

$$= \iota_C \iota_U \left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\left(T_1 - t\right)} + \frac{d_{21}}{2\left(T_1 - t\right)}\right]$$

$$\frac{\partial\left(\iota_U d_{22}\right)}{\partial t} = \iota_U \frac{\partial d_{22}}{\partial t} = \iota_U \frac{\partial}{\partial t}\left[\frac{\ln\left(S_t B_{t,T_1,-(r-\hat{q})} B_{T_1,T_2,-(r-\delta)} \big/ X_U\right) - \frac{\sigma^2_{t,T_2}}{2}}{\sigma_{t,T_2}}\right]. \quad (9.5.82)$$

$$= \iota_U \left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2 - t}} + \frac{d_{22}}{2\left(T_2 - t\right)}\right]$$

Therefore, (note we can cancel the indicator functions)

$$\frac{\partial}{\partial t} N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) = \frac{\partial N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial d_{21}}\left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\left(T_1 - t\right)} + \frac{d_{21}}{2\left(T_1 - t\right)}\right]. \quad (9.5.83)$$

$$+ \frac{\partial N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial d_{22}}\left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2 - t}} + \frac{d_{22}}{2\left(T_2 - t\right)}\right]$$

(4) $$\frac{\partial B_{t,T_2,r}}{\partial t} = r B_{t,T_2,r}. \quad (9.5.84)$$

(5) $$\frac{\partial}{\partial t} N_1\left(\iota_C d_{21}\right) = \frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial\left(\iota_C d_{21}\right)} \frac{\partial\left(\iota_C d_{21}\right)}{\partial t} = \frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1 - t}} + \frac{d_{21}}{2\left(T_1 - t\right)}\right]. \quad (9.5.85)$$

(6) $$\frac{\partial B_{t,T_1,r}}{\partial t} = rB_{t,T_1,r}. \tag{9.5.86}$$

Substituting these results into the original equation,

$$\begin{aligned}
&\frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right) \\
&= \iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \left\{ \begin{array}{l} \dfrac{\partial N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)}{\partial d_{11}} \left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}} + \dfrac{d_{11}}{2\left(T_1-t\right)} \right] \\ + \dfrac{\partial N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)}{\partial d_{12}} \left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}} + \dfrac{d_{12}}{2\left(T_2-t\right)} \right] \end{array} \right\} \\
&+ \iota_C \iota_U S_t B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) \hat{q} B_{t,T_1,\hat{q}} \\
&- \iota_C \iota_U X_U B_{t,T_2,r} \left\{ \begin{array}{l} \dfrac{\partial N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial d_{21}} \left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)} + \dfrac{d_{21}}{2\left(T_1-t\right)} \right] \\ + \dfrac{\partial N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial d_{22}} \left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}} + \dfrac{d_{22}}{2\left(T_2-t\right)} \right] \end{array} \right\} \\
&- \iota_C \iota_U X_U N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) r B_{t,T_2,r} \\
&- \iota_C X_C B_{t,T_1,r} \frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}} \left[-\frac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}} + \frac{d_{21}}{2\left(T_1-t\right)} \right] \\
&- \iota_C X_C N_1\left(\iota_C d_{21}\right) r B_{t,T_1,r}
\end{aligned} \quad . \tag{9.5.87}$$

Rearranging to exploit Lemmas 2 and 3,

$$\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)$$

$$\begin{aligned}
&=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{11}}{2\left(T_1-t\right)}\right] \\
&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{12}}{2\left(T_2-t\right)}\right] \\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)}+\frac{d_{21}}{2\left(T_1-t\right)}\right] \\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}}{2\left(T_2-t\right)}\right] \\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}}{2\left(T_1-t\right)}\right] \\
&+\iota_C\iota_U S_t B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\hat{q}B_{t,T_1,\hat{q}} \\
&-\iota_C\iota_U X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)rB_{t,T_2,r} \\
&-\iota_C X_C N_1\left(\iota_C d_{21}\right)rB_{t,T_1,r}
\end{aligned} \quad . \qquad (9.5.88)$$

Substituting for d_{11} and d_{12} and related partials, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right) \\
&= \left\{
\begin{aligned}
&\iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \frac{\partial N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)}{\partial d_{21}} \left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}} + \frac{d_{21}+\sigma\sqrt{T_1-t}}{2\left(T_1-t\right)} \right] \\
&-\iota_C \iota_U X_U B_{t,T_2,r} \frac{\partial N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial d_{21}} \left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)} + \frac{d_{21}}{2\left(T_1-t\right)} \right] \\
&-\iota_C X_C B_{t,T_1,r} \frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}} \left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}} + \frac{d_{21}}{2\left(T_1-t\right)} \right]
\end{aligned}
\right\} \\
&+ \left\{
\begin{aligned}
&\iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \frac{\partial N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)}{\partial d_{22}} \left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}} + \frac{d_{22}+\sigma\sqrt{T_2-t}}{2\left(T_2-t\right)} \right] \\
&-\iota_C \iota_U X_U B_{t,T_2,r} \frac{\partial N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right)}{\partial d_{22}} \left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}} + \frac{d_{22}}{2\left(T_2-t\right)} \right]
\end{aligned}
\right\} \\
&+\iota_C \iota_U S_t B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) \hat{q} B_{t,T_1,\hat{q}} \\
&-\iota_C \iota_U X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) r B_{t,T_2,r} \\
&-\iota_C X_C N_1\left(\iota_C d_{21}\right) r B_{t,T_1,r}
\end{aligned}
\quad (9.5.89)
$$

Note that

$$
-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}} + \frac{d_{21}+\sigma\sqrt{T_1-t}}{2\left(T_1-t\right)} = -\frac{\left(r-\hat{q}-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}} + \frac{d_{21}}{2\left(T_1-t\right)} - \frac{\sigma}{2\sqrt{T_1-t}} \quad (9.5.90)
$$

and

$$
-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}} + \frac{d_{22}+\sigma\sqrt{T_2-t}}{2\left(T_2-t\right)} = -\frac{\left(r-\hat{q}-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}} + \frac{d_{22}}{2\left(T_2-t\right)} - \frac{\sigma}{2\sqrt{T_2-t}}. \quad (9.5.91)
$$

Substituting these results, based on Lemmas 2 and 3, the remaining terms are (substituting back for d_{21} and d_{22})

$$\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=-\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left(\frac{\sigma}{2\sqrt{T_1-t}}\right)$$
$$-\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left(\frac{\sigma}{2\sqrt{T_2-t}}\right) \quad . \quad (9.5.92)$$
$$\iota_C\iota_U\hat{q}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)$$
$$-\iota_C\iota_U rB_{t,T_2,r}X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C rB_{t,T_1,r}X_C N_1\left(\iota_C d_{21}\right)$$

Recall

$$\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}=\iota_C\iota_U N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right), \quad (9.5.93)$$

and also

$$\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}=\iota_U N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right). \quad (9.5.94)$$

Thus,

$$\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=-S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\left(\frac{\sigma}{2\sqrt{T_1-t}}\right)$$
$$-\iota_C S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\left(\frac{\sigma}{2\sqrt{T_2-t}}\right) \quad . \quad (9.5.95)$$
$$\iota_C\iota_U\hat{q}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)$$
$$-\iota_C\iota_U rB_{t,T_2,r}X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C rB_{t,T_1,r}X_C N_1\left(\iota_C d_{21}\right)$$

And finally,

$$\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=-\frac{\sigma^2}{2}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{array}{l}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\right]\\-\iota_C N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_1}}\right]\end{array}\right\}$$
$$+\iota_C\iota_U\hat{q}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right) \quad . \quad (9.5.96)$$
$$-\iota_C\iota_U rB_{t,T_2,r}X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C rB_{t,T_1,r}X_C N_1\left(\iota_C d_{21}\right)$$

Compound option vega proof

$$\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=S_tB_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\begin{array}{l}\sqrt{T_1-t}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\\+\iota_C\sqrt{T_2-t}N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\end{array}\right]. \quad (9.5.97)$$

Proof: There are several variables that are a function of volatility, σ. Highlighting this dependency, we have

$$\begin{aligned}&\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_US_tB_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial\sigma}N_2\left[\iota_C\iota_Ud_{11}\left(\sigma\right),\iota_Ud_{12}\left(\sigma\right);\iota_C\rho\right]\\&-\iota_C\iota_UX_UB_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial\sigma}N_2\left[\iota_C\iota_Ud_{21}\left(\sigma\right),\iota_Ud_{22}\left(\sigma\right);\iota_C\rho\right]-\iota_CX_CB_{t,T_1,r}\frac{\partial}{\partial\sigma}N\left[\iota_C\iota_Ud_{21}\left(\sigma\right)\right]\end{aligned}, \quad (9.5.98)$$

Based on lemma 1, we have

$$\begin{aligned}&\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_US_tB_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left\{\begin{array}{l}\frac{\partial N_2\left[\iota_C\iota_Ud_{11}\left(\sigma\right),\iota_Ud_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{11}}\frac{\partial d_{11}}{\partial\sigma}\\+\frac{\partial N_2\left[\iota_C\iota_Ud_{11}\left(\sigma\right),\iota_Ud_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{12}}\frac{\partial d_{12}}{\partial\sigma}\end{array}\right\}\\&-\iota_C\iota_UX_UB_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\left\{\begin{array}{l}\frac{\partial N_2\left[\iota_C\iota_Ud_{21}\left(\sigma\right),\iota_Ud_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}\\+\frac{\partial N_2\left[\iota_C\iota_Ud_{21}\left(\sigma\right),\iota_Ud_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{22}}\frac{\partial d_{22}}{\partial\sigma}\end{array}\right\}\\&-\iota_CX_CB_{t,T_1,r}\frac{\partial N\left[\iota_C\iota_Ud_{21}\left(\sigma\right)\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}\end{aligned}, \quad (9.5.99)$$

Rearranging to exploit Lemmas 2 and 3, we have

$$\begin{aligned}&\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_US_tB_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_Ud_{11}\left(\sigma\right),\iota_Ud_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{11}}\frac{\partial d_{11}}{\partial\sigma}\\&-\iota_C\iota_UX_UB_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_Ud_{21}\left(\sigma\right),\iota_Ud_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}\\&-\iota_CX_CB_{t,T_1,r}\frac{\partial N\left[\iota_C\iota_Ud_{21}\left(\sigma\right)\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}\\&+\iota_C\iota_US_tB_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_Ud_{11}\left(\sigma\right),\iota_Ud_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{12}}\frac{\partial d_{12}}{\partial\sigma}\\&-\iota_C\iota_UX_UB_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_Ud_{21}\left(\sigma\right),\iota_Ud_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{22}}\frac{\partial d_{22}}{\partial\sigma}\end{aligned}, \quad (9.5.100)$$

Based on Lemma 2

$$\frac{\partial}{\partial \sigma} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = \iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{21}} \frac{\partial d_{11}}{\partial \sigma}$$

$$-\iota_C \iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right]}{\partial d_{21}} \frac{\partial d_{21}}{\partial \sigma}$$

$$-\iota_C X_C B_{t,T_1,r} \frac{\partial N\left[\iota_C \iota_U d_{21}\right]}{\partial d_{21}} \frac{\partial d_{21}}{\partial \sigma} \quad , \qquad (9.5.101)$$

$$+\iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{22}} \frac{\partial d_{12}}{\partial \sigma}$$

$$-\iota_C \iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right]}{\partial d_{22}} \frac{\partial d_{22}}{\partial \sigma}$$

Note

$$\frac{\partial d_{11}}{\partial \sigma} = \frac{\partial\left(d_{21} + \sigma\sqrt{T_1 - t}\right)}{\partial \sigma} = \frac{\partial d_{21}}{\partial \sigma} + \sqrt{T_1 - t} \text{ and} \qquad (9.5.102)$$

$$\frac{\partial d_{12}}{\partial \sigma} = \frac{\partial\left(d_{22} + \sigma\sqrt{T_2 - t}\right)}{\partial \sigma} = \frac{\partial d_{22}}{\partial \sigma} + \sqrt{T_2 - t}\,. \qquad (9.5.103)$$

Substituting for these partials and rearranging, we have

$$\frac{\partial}{\partial \sigma} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = \frac{\partial d_{21}}{\partial \sigma} \left\{ \begin{aligned} &\iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{21}} \\ &-\iota_C \iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right]}{\partial d_{21}} \\ &-\iota_C X_C B_{t,T_1,r} \frac{\partial N\left[\iota_C \iota_U d_{21}\right]}{\partial d_{21}} \end{aligned} \right\}$$

$$+\frac{\partial d_{22}}{\partial \sigma} \left\{ \begin{aligned} &\iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{22}} \\ &-\iota_C \iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right]}{\partial d_{22}} \end{aligned} \right\} \quad . \qquad (9.5.104)$$

$$+\iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{21}} \sqrt{T_1 - t}$$

$$+\iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{22}} \sqrt{T_2 - t}$$

Again, based on Lemma 2 and 3, we have

$$\frac{\partial}{\partial \sigma} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = \iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{11}} \sqrt{T_1 - t}$$

$$+\iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} \frac{\partial N_2\left[\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right]}{\partial d_{12}} \sqrt{T_2 - t} \quad . \qquad (9.5.105)$$

Note

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{11}}=\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial\left[\iota_C\iota_U d_{11}\right]}\frac{\partial\left[\iota_C\iota_U d_{11}\right]}{\partial d_{11}}$$
$$=\iota_C\iota_U\frac{\partial}{\partial\left[\iota_C\iota_U d_{11}\right]}\int_{-\infty}^{\iota_C\iota_U d_{11}}N_1\left[\frac{\iota_U d_{12}-\iota_C\rho z_2}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_2\right)dz_2$$
$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{12}-\iota_C\rho\iota_C\iota_U d_{11}}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(\iota_C\iota_U d_{11}\right) \quad , \quad (9.5.106)$$
$$=\iota_C\iota_U N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)$$

and also

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{12}}=\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial\left[\iota_U d_{12}\right]}\frac{\partial\left[\iota_U d_{12}\right]}{\partial d_{12}}$$
$$=\iota_U\frac{\partial}{\partial\left[\iota_U d_{12}\right]}\int_{-\infty}^{\iota_U d_{12}}N_1\left[\frac{\iota_C\iota_U d_{11}-\iota_C\rho z_1}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_1\right)dz_1 \quad . \quad (9.5.107)$$
$$=\iota_U N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)$$

Substituting,

$$\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\iota_C\iota_U N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\right]\sqrt{T_1-t}$$
$$+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\iota_U N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\right]\sqrt{T_2-t} \quad . \quad (9.5.108)$$

Therefore, the compound option vega can be expressed as

$$\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\begin{array}{l}\sqrt{T_1-t}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\\+\iota_C\sqrt{T_2-t}N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\end{array}\right]. \quad (9.5.109)$$

Compound option rho proof

$$\begin{aligned}\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\left(T_2-t\right)X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\\+\iota_C B_{t,T_1,r}\left(T_1-t\right)X_C N\left(\iota_C\iota_U d_{21}\right)\end{aligned}. \qquad (9.5.110)$$

Proof: There are several variables that are a function of the interest rate, *r*.

$$\begin{aligned}&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial r}N_2\left[\iota_C\iota_U d_{11}\left(r\right),\iota_U d_{12}\left(r\right);\iota_C\rho\right]\\&-\iota_C\iota_U X_U B_{t,T_2,r}\left(r\right)B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial r}N_2\left[\iota_C\iota_U d_{21}\left(r\right),\iota_U d_{22}\left(r\right);\iota_C\rho\right]\\&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left[\iota_C\iota_U d_{21}\left(r\right),\iota_U d_{22}\left(r\right);\iota_C\rho\right]\frac{\partial}{\partial r}B_{t,T_2,r}\left(r\right)\\&-\iota_C X_C B_{t,T_1,r}\left(r\right)\frac{\partial}{\partial r}N\left[\iota_C\iota_U d_{21}\left(r\right)\right]\\&-\iota_C X_C N\left[\iota_C\iota_U d_{21}\left(r\right)\right]\frac{\partial}{\partial r}B_{t,T_1,r}\left(r\right)\end{aligned}, \qquad (9.5.111)$$

From lemma 1, we note

$$\begin{aligned}&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\begin{bmatrix}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\frac{\partial d_{11}}{\partial r}\\+\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\frac{\partial d_{12}}{\partial r}\end{bmatrix}\\&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\begin{bmatrix}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\+\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r}\end{bmatrix}\\&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}\\&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\&-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}\end{aligned}, \qquad (9.5.112)$$

Rearranging to exploit Lemmas 2 and 3, we have

$$\begin{aligned}
&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\frac{\partial d_{11}}{\partial r}\\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\frac{\partial d_{12}}{\partial r}\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r}\\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}
\end{aligned}, \qquad (9.5.113)$$

Based on Lemma 2

$$\begin{aligned}
&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r}\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}
\end{aligned}, \qquad (9.5.114)$$

Note

$$\frac{\partial d_{11}}{\partial r}=\frac{\partial\left(d_{21}+\sigma\sqrt{T_1-t}\right)}{\partial r}=\frac{\partial d_{21}}{\partial r}\text{ and} \qquad (9.5.115)$$

$$\frac{\partial d_{12}}{\partial r}=\frac{\partial\left(d_{22}+\sigma\sqrt{T_2-t}\right)}{\partial r}=\frac{\partial d_{22}}{\partial r}. \qquad (9.5.116)$$

Substituting for these partials and rearranging, we have

$$\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\frac{\partial d_{21}}{\partial r}\begin{bmatrix}\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\dfrac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}}\\ -\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\dfrac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\\ -\iota_C X_C B_{t,T_1,r}\dfrac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\end{bmatrix}$$

$$+\frac{\partial d_{22}}{\partial r}\begin{bmatrix}\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\dfrac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}}\\ -\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\dfrac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\end{bmatrix}, \quad (9.5.117)$$

$$-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}$$

Again, based on Lemma 2 and 3 and partials with respect to *B*, we have the compound option rho as

$$\begin{aligned}\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)&=\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\left(T_2-t\right)B_{t,T_2,r},\\&+\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\left(T_1-t\right)B_{t,T_1,r}\end{aligned} \quad (9.5.118)$$

Validation of partial differential equation

Recall the compound option partial differential equation can be expressed as

$$r(t)CO_D=\frac{\partial CO_D}{\partial t}+\left[r(t)-\hat{q}(t)\right]\frac{\partial CO_D}{\partial S}S+\frac{1}{2}\sigma^2(t)S\frac{\partial^2 CO_D}{\partial S^2}. \quad (9.5.119)$$

We validate this equation by solving for theta or

$$\frac{\partial CO_D}{\partial t}=r(t)CO_D-\left[r(t)-\hat{q}(t)\right]\frac{\partial CO_D}{\partial S}S-\frac{1}{2}\sigma^2(t)S\frac{\partial^2 CO_D}{\partial S^2}. \quad (9.5.120)$$

Recall

$$\begin{aligned}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)&=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\&-\iota_C\iota_U X_U B_{t,T_2,r}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C X_C B_{t,T_1,r}N\left(\iota_C d_{21}\right)\end{aligned}, \quad (9.5.121)$$

$$\Delta_{CO_D}\equiv\frac{\partial CO_D\left(S,t,T_1,T_2\right)}{\partial S}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right), \text{ and} \quad (9.5.122)$$

$$\Gamma_{CO_D}=\frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2}=\frac{B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}+\iota_C N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_2}}\right]. \quad (9.5.123)$$

Thus, substituting these results into the rearranged partial differential equation, we have

$$\frac{\partial CO_D}{\partial t} = r(t)\begin{bmatrix} \iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho) \\ -\iota_C \iota_U X_U B_{t,T_2,r} N_2(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho) - \iota_C X_C B_{t,T_1,r} N(\iota_C d_{21}) \end{bmatrix}$$
$$-\left[r(t) - \hat{q}(t)\right] S\left[\iota_C \iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho)\right] \quad . \quad (9.5.124)$$
$$-\frac{1}{2}\sigma^2(t) S\left[\frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C \iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right]\right]$$

Cancelling terms,

$$\frac{\partial CO_D}{\partial t} = -\iota_C \iota_U r(t) X_U B_{t,T_2,r} N_2(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho) - \iota_C r(t) X_C B_{t,T_1,r} N(\iota_C d_{21})$$
$$+\hat{q}(t) S\left[\iota_C \iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho)\right] \quad . \quad (9.5.125)$$
$$-\frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{2}\sigma^2(t)\left[N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C \iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right]$$

Recall

$$\Theta_{CO_D} \equiv \frac{\partial}{\partial t} CO_D(S, t, T_1, T_2, \iota_C, \iota_U)$$
$$= -\frac{\sigma^2(t)}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\left\{N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1(d_{11})}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C \iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1(d_{12})}{\sigma_{t,T_1}}\right]\right\} \quad .$$
$$+\iota_C \iota_U \hat{q}(t) S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho) - \iota_C \iota_U r(t) B_{t,T_2,r} X_U N_2(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho) - \iota_C r(t) B_{t,T_1,r} X_C N_1(\iota_C d_{21})$$

(9.5.126)

Thus, the PDE obtained from hedging with the underlying instrument is satisfied.

Lemma 1: Leibniz integral rule for double integral applied to bivariate normal

Assuming $N_2(a(x), b(x); \rho)$ is a standard bivariate normal cumulative distribution function, then

$$\frac{dN_2(a(x), b(x); \rho)}{dx} = \frac{dN_2(a(x), b(x); \rho)}{db(x)}\frac{db(x)}{dx} + \frac{dN_2(a(x), b(x); \rho)}{da(x)}\frac{da(x)}{dx}. \quad (9.5.127)$$

Lemma 1 Proof: Assuming $f(x, z_1, z_2)$ is a continuous function where the needed derivatives exists, then

$$y(x, z_1, z_2) \equiv \int_{l_2(x)}^{u_2(x)} \int_{l_1(x)}^{u_1(x)} f(x, z_1, z_2)\, dz_1\, dz_2 = \int_{l_2(x)}^{u_2(x)} g(x, z_2)\, dz_2, \quad (9.5.128)$$

where

$$g(x, z_2) \equiv \int_{l_1(x)}^{u_1(x)} f(x, z_1, z_2)\, dz_1, \quad (9.5.129)$$

and note

$$\frac{\partial y(x,z_1,z_2)}{\partial u_1(x)}=\int_{l_2(x)}^{u_2(x)} f\left(x,u_1(x),z_2\right)dz_2, \tag{9.5.130}$$

$$\frac{\partial y(x,z_1,z_2)}{\partial u_2(x)}=\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,u_2(x)\right)dz_1, \tag{9.5.131}$$

$$\frac{\partial y(x,z_1,z_2)}{\partial l_1(x)}=-\int_{l_2(x)}^{u_2(x)} f\left(x,l_1(x),z_2\right)dz_2, \text{ and} \tag{9.5.132}$$

$$\frac{\partial y(x,z_1,z_2)}{\partial l_2(x)}=-\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,l_2(x)\right)dz_1. \tag{9.5.133}$$

Applying the Leibniz integral rule to the outer integral

$$\begin{aligned}\frac{dy(x,z_1,z_2)}{dx}&=\int_{l_2(x)}^{u_2(x)}\frac{d}{dx}\left[g(x,z_2)\right]dz_2+\frac{du_2(x)}{dx}g\left(x,u_2(x)\right)-\frac{dl_2(x)}{dx}g\left(x,l_2(x)\right)\\&=\int_{l_2(x)}^{u_2(x)}\frac{d}{dx}\left[\int_{l_1(x)}^{u_1(x)} f(x,z_1,z_2)dz_1\right]dz_2+\frac{du_2(x)}{dx}\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,u_2(x)\right)dz_1-\frac{dl_2(x)}{dx}\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,l_2(x)\right)dz_1\end{aligned}, \tag{9.5.134}$$

and also applying the Leibniz integral rule to the inner integral (first term in the equation above)

$$\frac{d}{dx}\left[\int_{l_1(x)}^{u_1(x)} f(x,z_1,z_2)dz_1\right]=\int_{l_1(x)}^{u_1(x)}\frac{df(x,z_1,z_2)}{dx}dz_1+\frac{du_1(x)}{dx}f\left(x,u_1(x),z_2\right)-\frac{dl_1(x)}{dx}f\left(x,l_1(x),z_2\right). \tag{9.5.135}$$

Substituting this result into the previous equation, we have

$$\begin{aligned}\frac{dy(x,z_1,z_2)}{dx}&=\int_{l_2(x)}^{u_2(x)}\left[\int_{l_1(x)}^{u_1(x)}\frac{df(x,z_1,z_2)}{dx}dz_1+\frac{du_1(x)}{dx}f\left(x,u_1(x),z_2\right)-\frac{dl_1(x)}{dx}f\left(x,l_1(x),z_2\right)\right]dz_2\\&+\frac{du_2(x)}{dx}\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,u_2(x)\right)dz_1-\frac{dl_2(x)}{dx}\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,l_2(x)\right)dz_1\end{aligned}. \tag{9.5.136}$$

Thus

$$\begin{aligned}\frac{dy(x,z_1,z_2)}{dx}&=\int_{l_2(x)}^{u_2(x)}\int_{l_1(x)}^{u_1(x)}\frac{df(x,z_1,z_2)}{dx}dz_1\,dz_2\\&+\frac{du_1(x)}{dx}\int_{l_2(x)}^{u_2(x)} f\left(x,u_1(x),z_2\right)dz_2-\frac{dl_1(x)}{dx}\int_{l_2(x)}^{u_2(x)} f\left(x,l_1(x),z_2\right)dz_2,\\&+\frac{du_2(x)}{dx}\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,u_2(x)\right)dz_1-\frac{dl_2(x)}{dx}\int_{l_1(x)}^{u_1(x)} f\left(x,z_1,l_2(x)\right)dz_1\end{aligned} \tag{9.5.137}$$

or

$$\frac{dy(x,z_1,z_2)}{dx}=\int_{l_2(x)}^{u_2(x)}\int_{l_1(x)}^{u_1(x)}\frac{df(x,z_1,z_2)}{dx}dz_1\,dz_2+\frac{\partial y(x,z_1,z_2)}{\partial u_1(x)}\frac{du_1(x)}{dx}+\frac{\partial y(x,z_1,z_2)}{\partial l_1(x)}\frac{dl_1(x)}{dx}$$
$$+\frac{\partial y(x,z_1,z_2)}{\partial u_2(x)}\frac{du_2(x)}{dx}+\frac{\partial y(x,z_1,z_2)}{\partial l_2(x)}\frac{dl_2(x)}{dx}. \quad (9.5.138)$$

QED

Example: Bivariate standard normal cumulative distribution function
Consider

$$N_2\left(a(x),b(x);\rho\right)\equiv\int_{-\infty}^{a(x)}\int_{-\infty}^{b(x)}\frac{\exp\left\{-\frac{z_1^2-2\rho z_1z_2+z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_1\,dz_2=\int_{-\infty}^{a(x)}\int_{-\infty}^{b(x)}f(z_1,z_2)dz_1\,dz_2. \quad (9.5.139)$$

Based on Lemma 1, we have

$$\frac{dN_2\left(a(x),b(x);\rho\right)}{dx}=\int_{-\infty}^{a(x)}\int_{-\infty}^{b(x)}\frac{df(z_1,z_2)}{dx}dz_1\,dz_2+\frac{db(x)}{dx}\int_{-\infty}^{a(x)}f\left(b(x),z_2\right)dz_2-\frac{d(-\infty)}{dx}\int_{-\infty}^{a(x)}f(-\infty,z_2)dz_2$$
$$+\frac{da(x)}{dx}\int_{-\infty}^{b(x)}f\left(z_1,a(x)\right)dz_1-\frac{d(-\infty)}{dx}\int_{-\infty}^{b(x)}f(z_1,-\infty)dz_1 .\quad (9.5.140)$$

Thus,

$$\frac{dN_2\left(a(x),b(x);\rho\right)}{dx}=\frac{db(x)}{dx}\int_{-\infty}^{a(x)}f\left(b(x),z_2\right)dz_2+\frac{da(x)}{dx}\int_{-\infty}^{b(x)}f\left(z_1,a(x)\right)dz_1$$
$$=\frac{db(x)}{dx}\int_{-\infty}^{a(x)}\frac{\exp\left\{-\frac{b(x)^2-2\rho b(x)z_2+z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_2+\frac{da(x)}{dx}\int_{-\infty}^{b(x)}\frac{\exp\left\{-\frac{z_1^2-2\rho z_1a(x)+a(x)^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_1, \quad (9.5.141)$$

or

$$\frac{dN_2\left(a(x),b(x);\rho\right)}{dx}=\frac{dN_2\left(a(x),b(x);\rho\right)}{db(x)}\frac{db(x)}{dx}+\frac{dN_2\left(a(x),b(x);\rho\right)}{da(x)}\frac{da(x)}{dx}. \quad (9.5.142)$$

Lemma 2: Compound option partial with respect to d_{21}
We assert

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=0, \quad (9.5.143)$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}. \quad (9.5.144)$$

Lemma 2 Proof: Recall

$$d_{11} = d_{21} + \sigma_{t,T_1}, \tag{9.5.145}$$

and therefore

$$\frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial d_{11}(S)} = \frac{\partial N_2\left\{\iota_C \iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right], \iota_U d_{12}(S); \iota_C \rho\right\}}{\partial\left[d_{21}(S)+\sigma_{t,T_1}\right]} \frac{\partial\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\partial d_{21}(S)}$$
$$= \frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial d_{21}(S)}. \tag{9.5.146}$$

Taking the derivative, we have

$$\frac{\partial}{\partial d_{21}} CO_D\left(S,t,T_1,T_2\right) = \iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \frac{\partial N_2\left[\iota_C \iota_U d_{11}(S), \iota_U d_{12}(S); \iota_C \rho\right]}{\partial d_{21}(S)}$$
$$-\iota_C \iota_U X_U B_{t,T_2,r} \frac{\partial N_2\left[\iota_C \iota_U d_{21}(S), \iota_U d_{22}(S); \iota_C \rho\right]}{\partial d_{21}(S)} - \iota_C X_C B_{t,T_1,r} \frac{\partial N_1\left[\iota_C d_{21}(S)\right]}{\partial d_{21}}. \tag{9.5.147}$$

We now focus on $\dfrac{\partial N_2\left[\iota_C \iota_U d_{21}(S), \iota_U d_{22}(S); \iota_C \rho\right]}{\partial d_{21}(S)}$. Note

$$N_2\left[\iota_C \iota_U d_{21}(S), \iota_U d_{22}(S); \iota_C \rho\right] = \int_{-\infty}^{\iota_C \iota_U d_{21}(S)} \int_{-\infty}^{\iota_U d_{22}(S)} n_2\left(z_1, z_2\right) dz_2\, dz_1$$
$$= \int_{-\infty}^{\iota_C \iota_U d_{21}(S)} \left[\int_{-\infty}^{\iota_U d_{22}(S)} n_1\left(z_2 \middle| z_1\right) dz_2\right] n_1\left(z_1\right) dz_1 = \int_{-\infty}^{\iota_U d_{22}(S)} \left[\int_{-\infty}^{\iota_C \iota_U d_{21}(S)} n_1\left(z_1 \middle| z_2\right) dz_1\right] n_1\left(z_2\right) dz_2. \tag{9.5.148}$$

Thus

$$N_2\left[\iota_C \iota_U d_{21}(S), \iota_U d_{22}(S); \iota_C \rho\right] = \int_{-\infty}^{\iota_C \iota_U d_{21}(S)} \left[\int_{-\infty}^{\iota_U d_{22}(S)} n_1\left(z_2 \middle| z_1\right) dz_2\right] n_1\left(z_1\right) dz_1$$
$$= \int_{-\infty}^{\iota_C \iota_U d_{21}(S)} N_1\left[\frac{\iota_U d_{22}(S) - \iota_C \rho z_1}{\sqrt{1-\left(\iota_C \rho\right)^2}}\right] n_1\left(z_1\right) dz_1, \tag{9.5.149}$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial\left[\iota_C\iota_U d_{21}(S)\right]}\frac{\partial\left[\iota_C\iota_U d_{21}(S)\right]}{\partial d_{21}(S)}$$

$$=\iota_C\iota_U\frac{\partial}{\partial\left[\iota_C\iota_U d_{21}(S)\right]}\int_{-\infty}^{\iota_C\iota_U d_{21}(S)} N_1\left[\frac{\iota_U d_{22}(S)-\iota_C\rho z_1}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1(z_1)dz_1$$

$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\iota_C\rho\iota_C\iota_U d_{21}(S)}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1\left[\iota_C\iota_U d_{21}(S)\right] \qquad (9.5.150)$$

$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\rho\iota_U d_{21}(S)}{\sqrt{1-\rho^2}}\right]n_1\left[\iota_C\iota_U d_{21}(S)\right]$$

We now focus on $\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}$. Note

$$N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\int_{-\infty}^{\iota_U d_{12}(S)} n_2(z_1,z_2)dz_2\,dz_1$$

$$=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\left[\int_{-\infty}^{\iota_U d_{12}(S)} n_1(z_2|z_1)n_1(z_1)dz_2\right]dz_1=\int_{-\infty}^{\iota_U d_{12}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{11}(S)} n_1(z_1|z_2)n_1(z_2)dz_1\right]dz_2 \qquad (9.5.151)$$

And

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}=\frac{\partial N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}}{\partial d_{21}(S)}, \qquad (9.5.152)$$

$$N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_2(z_1,z_2)dz_2\,dz_1$$

$$=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\left[\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_1(z_2|z_1)dz_2\right]n_1(z_1)dz_1 \text{, and} \qquad (9.5.153)$$

$$=\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]}\left[\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]} n_1(z_1|z_2)dz_1\right]n_1(z_2)dz_2$$

$$\frac{\partial}{\partial d_{21}} N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}$$

$$=\frac{\partial}{\partial\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}} N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}\frac{\partial\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}}{\partial d_{21}}$$

$$=\iota_C\iota_U\frac{\partial}{\partial d_{21}}\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\left[\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_1\left(z_2\middle|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1 \quad (9.5.154)$$

$$=\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\iota_C\rho\left(\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}$$

$$=\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\rho\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\sqrt{1-\rho^2}}\right\}n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}$$

Finally

$$\frac{\partial}{\partial d_{21}} N_1\left[\iota_C d_{21}\left(S\right)\right]=\iota_C n_1\left[\iota_C d_{21}\left(S\right)\right]. \quad (9.5.155)$$

Therefore,

$$\frac{\partial}{\partial d_{21}} CO_D\left(S,t,T_1,T_2\right)$$

$$=\iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\left(\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\rho\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\sqrt{1-\rho^2}}\right\}n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}\right). \quad (9.5.156)$$

$$-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\rho\iota_U d_{21}(S)}{\sqrt{1-\rho^2}}\right]n_1\left[\iota_C\iota_U d_{21}(S)\right]\right\}$$

$$-\iota_C X_C B_{t,T_1,r}\left\{\iota_C n_1\left[\iota_C d_{21}\left(S\right)\right]\right\}$$

Note

$$n_1\left[\iota_C d_{21}\left(S\right)\right]=\frac{\exp\left\{-\frac{\left[\iota_C d_{21}\left(S\right)\right]^2}{2}\right\}}{\sqrt{2\pi}}=\frac{\exp\left\{-\frac{\left[\iota_C\iota_U d_{21}\left(S\right)\right]^2}{2}\right\}}{\sqrt{2\pi}}\text{, and} \quad (9.5.157)$$

$$=\frac{\exp\left[-\frac{d_{21}^2\left(S\right)}{2}\right]}{\sqrt{2\pi}}=n_1\left[\iota_C\iota_U d_{21}\left(S\right)\right]=n_1\left[d_{21}\left(S\right)\right]$$

$$n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}=\frac{\exp\left(-\frac{\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}^2}{2}\right)}{\sqrt{2\pi}}=\frac{\exp\left\{-\frac{\left[d_{21}(S)+\sigma_{t,T_1}\right]^2}{2}\right\}}{\sqrt{2\pi}}$$

$$=n_1\left[d_{21}(S)\right]\exp\left[-\frac{2\sigma_{t,T_1}d_{21}(S)+\sigma_{t,T_1}^2}{2}\right]=n_1\left[d_{21}(S)\right]\exp\left[-\sigma_{t,T_1}d_{21}(S)-\frac{\sigma_{t,T_1}^2}{2}\right]$$

$$=n_1\left[d_{21}(S)\right]\exp\left(-\frac{\sigma_{t,T_1}^2}{2}\right)\exp\left[-\sigma_{t,T_1}d_{21}(S)\right]$$

$$=n_1\left[d_{21}(S)\right]\exp\left(-\frac{\sigma_{t,T_1}^2}{2}\right)\exp\left[-\sigma_{t,T_1}\frac{\ln\left(\frac{S_tB_{t,T_1,-(r-\hat{q})}}{S_{T_1}^*}\right)-\frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}\right]$$

$$=\frac{S_{T_1}^*B_{t,T_1,r-\hat{q}}}{S_t}n_1\left[d_{21}(S)\right] \quad . \quad (9.5.158)$$

Therefore,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)$$

$$=S_tB_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\iota_C\iota_U\left(\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\rho\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\sqrt{1-\rho^2}}\right\}\frac{S_{T_1}^*B_{t,T_1,r-\hat{q}}}{S_t}n_1\left[d_{21}(S)\right]\right). \quad (9.5.159)$$

$$-X_UB_{t,T_2,r}\iota_C\iota_U\left\{\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\rho\iota_U d_{21}(S)}{\sqrt{1-\rho^2}}\right]n_1\left[d_{21}(S)\right]\right\}$$

$$-X_CB_{t,T_1,r}\iota_C\left\{\iota_C n_1\left[d_{21}(S)\right]\right\}$$

Eliminating squared indicator functions and rearranging,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=n_1\left[d_{21}(S)\right]B_{t,T_1,r}\left(\begin{array}{l}S_{T_1}^*B_{T_1,T_2,\delta}N_1\left\{\iota_U\frac{d_{22}(S)+\sigma_{t,T_2}-\rho\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\sqrt{1-\rho^2}}\right\}\\-X_UB_{T_1,T_2,r}N_1\left[\iota_U\frac{d_{22}(S)-\rho d_{21}(S)}{\sqrt{1-\rho^2}}\right]-X_C\end{array}\right). \quad (9.5.160)$$

Note

$$\frac{d_{22}-\rho d_{21}}{\sqrt{1-\rho^2}}=\frac{d_{22}-\frac{\sigma_{t,T_1}}{\sigma_{t,T_2}}d_{21}}{\frac{\sigma_{T_1,T_2}}{\sigma_{t,T_2}}}=\frac{\sigma_{t,T_2}d_{22}-\sigma_{t,T_1}d_{21}}{\sigma_{T_1,T_2}}$$

$$=\frac{\sigma_{t,T_2}\left[\frac{\ln\left(\frac{S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}{X_U}\right)-\frac{\sigma^2_{t,T_2}}{2}}{\sigma_{t,T_2}}\right]-\sigma_{t,T_1}\left[\frac{\ln\left(\frac{S_tB_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}}\right)-\frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}}\right]}{\sigma_{T_1,T_2}}$$

$$=\frac{\ln\left(\frac{S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}{X_U}\right)-\frac{\sigma^2_{t,T_2}}{2}-\ln\left(\frac{S_tB_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}}\right)+\frac{\sigma^2_{t,T_1}}{2}}{\sigma_{T_1,T_2}}=\frac{\ln\left(\frac{S_tB_{T_1,T_2,-(r-\delta)}S^*_{T_1}}{X_US_t}\right)-\left(\frac{\sigma^2_{t,T_2}}{2}-\frac{\sigma^2_{t,T_1}}{2}\right)}{\sigma_{T_1,T_2}}$$

$$=\frac{\ln\left(\frac{S^*_{T_1}B_{T_1,T_2,-(r-\delta)}}{X_U}\right)-\frac{\sigma^2_{T_1,T_2}}{2}}{\sigma_{T_1,T_2}}=d^*_{2,T_1,T_2} \qquad .(9.5.161)$$

And

$$\frac{d_{22}-\rho\left(d_{21}+\sigma_{t,T_1}\right)+\sigma_{t,T_2}}{\sqrt{1-\rho^2}}=\frac{d_{22}-\rho d_{21}}{\sqrt{1-\rho^2}}+\frac{\sigma_{t,T_2}-\rho\sigma_{t,T_1}}{\sqrt{1-\rho^2}}$$

$$=d^*_{2,T_1,T_2}+\frac{\sigma_{t,T_2}-\frac{\sigma_{t,T_1}}{\sigma_{t,T_2}}\sigma_{t,T_1}}{\frac{\sigma_{T_1,T_2}}{\sigma_{t,T_2}}}=d^*_{2,T_1,T_2}+\frac{\sigma^2_{t,T_2}-\sigma^2_{t,T_1}}{\sigma_{T_1,T_2}}=d^*_{2,T_1,T_2}+\frac{\sigma^2_{T_1,T_2}}{\sigma_{T_1,T_2}}=d^*_{1,T_1,T_2}. \qquad (9.5.162)$$

Therefore,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=n_1\left[d_{21}(S)\right]B_{t,T_1,r}\left\{S^*_{T_1}B_{T_1,T_2,\delta}N_1\left(\iota_Ud^*_{1,T_1,T_2}\right)-X_UB_{T_1,T_2,r}N_1\left(\iota_Ud^*_{2,T_1,T_2}\right)-X_C\right\}, \qquad (9.5.163)$$

and therefore

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=\iota_Un_1\left(d_{21}(S)\right)B_{t,T_1,r}\left\{\iota_US^*_{T_1}B_{T_1,T_2,\delta}N_1\left(\iota_Ud^*_{1,T_1,T_2}\right)-\iota_UX_UB_{T_1,T_2,r}N_1\left(\iota_Ud^*_{2,T_1,T_2}\right)-\iota_UX_C\right\}.(9.5.164)$$

Recall

$$\iota_US^*_{T_1}B_{T_1,T_2,\delta}N_1\left(\iota_Ud^*_{1,T_1,T_2}\right)-\iota_UX_UB_{T_1,T_2,r}N_1\left(\iota_Ud^*_{2,T_1,T_2}\right)-\iota_UX_C=0. \qquad (9.5.165)$$

Thus,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=0. \qquad (9.5.166)$$

Lemma 3: Compound option partial with respect to d_{22}
We assert

$$\frac{\partial}{\partial d_{22}} CO_D\left(S,t,T_1,T_2\right)=0, \tag{9.5.167}$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}. \tag{9.5.168}$$

Lemma 3 Proof: Recall

$$d_{12}=d_{22}+\sigma_{t,T_2}, \tag{9.5.169}$$

and follow same logic as the first part of Lemma 2. Taking the derivative,

$$\begin{aligned}&\frac{\partial}{\partial d_{22}} CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\\&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\end{aligned}. \tag{9.5.170}$$

We now focus on $\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$. Note

$$\begin{aligned}&N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\int_{-\infty}^{\iota_U d_{22}(S)} n_2\left(z_1,z_2\right)dz_2\,dz_1\\&=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\left[\int_{-\infty}^{\iota_U d_{22}(S)} n_1\left(z_2|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1=\int_{-\infty}^{\iota_U d_{22}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{21}(S)} n_1\left(z_1|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2\end{aligned}. \tag{9.5.171}$$

Thus

$$\begin{aligned}&N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_U d_{22}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{21}(S)} n_1\left(z_1|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2\\&=\int_{-\infty}^{\iota_U d_{22}(S)} N\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho z_2}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_2\right)dz_2\end{aligned}, \tag{9.5.172}$$

and

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial\left[\iota_U d_{22}(S)\right]}\frac{\partial\left[\iota_U d_{22}(S)\right]}{\partial d_{22}(S)}\\&=\iota_U\frac{\partial}{\partial\left[\iota_U d_{22}\right]}\int_{-\infty}^{\iota_U d_{22}(S)} N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho z_2}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_2\right)dz_2\end{aligned}. \tag{9.5.173}$$

Therefore,

$$\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}=\iota_U N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho\iota_U d_{22}(S)}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1\left[\iota_U d_{22}(S)\right]. \quad (9.5.174)$$

We now focus on $\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$. Note

$$\begin{aligned}N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]&=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\int_{-\infty}^{\iota_U d_{12}(S)} n_2(z_1,z_2)\,dz_2\,dz_1\\&=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\left[\int_{-\infty}^{\iota_U d_{12}(S)} n_1(z_2|z_1)\,dz_2\right]n_1(z_1)\,dz_1=\int_{-\infty}^{\iota_U d_{12}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{11}(S)} n_1(z_1|z_2)\,dz_1\right]n_1(z_2)\,dz_2\end{aligned}. \quad (9.5.175)$$

And

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}=\frac{\partial N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}}{\partial d_{22}(S)}, \quad (9.5.176)$$

$$\begin{aligned}&N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_2(z_1,z_2)\,dz_2\,dz_1\\&=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\left[\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_1(z_2|z_1)\,dz_2\right]n_1(z_1)\,dz_1\\&=\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]}\left[\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]} n_1(z_1|z_2)\,dz_1\right]n_1(z_2)\,dz_2\end{aligned}, \quad (9.5.177)$$

$$\begin{aligned}&\frac{\partial}{\partial d_{22}}N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}\\&=\frac{\partial}{\partial\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}}N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}\frac{\partial\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}}{\partial d_{22}}\\&=\iota_U\frac{\partial}{\partial d_{22}}\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]}\left[\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]} n_1(z_1|z_2)\,dz_1\right]n_1(z_2)\,dz_2\\&=\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-(\iota_C\rho)^2}}\right\}n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}\end{aligned}, \quad (9.5.178)$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$

$$=\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-(\iota_C\rho)^2}}\right\}n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}. \tag{9.5.179}$$

Substituting these two partial derivatives,

$$\frac{\partial}{\partial d_{22}}CO_D(S,t,T_1,T_2)=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$

$$-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$

$$=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-(\iota_C\rho)^2}}\right\}n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}. \tag{9.5.180}$$

$$-\iota_C\iota_U X_U B_{t,T_2,r}\iota_U N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho\iota_U d_{22}(S)}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1\left[\iota_U d_{22}(S)\right]$$

Note

$$n_1(\iota_U d_{22})=\frac{\exp\left[-\frac{(\iota_U d_{22})^2}{2}\right]}{\sqrt{2\pi}}=\frac{\exp\left(-\frac{d_{22}^2}{2}\right)}{\sqrt{2\pi}}=n_1(d_{22}), \tag{9.5.181}$$

and

$$n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}=\frac{\exp\left(-\frac{\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}^2}{2}\right)}{\sqrt{2\pi}}=n_1\left[d_{22}(S)+\sigma_{t,T_2}\right]$$

$$=\frac{\exp\left[-\frac{\left(d_{22}+\sigma_{t,T_2}\right)^2}{2}\right]}{\sqrt{2\pi}}=n_1\left(d_{21}\right)\exp\left(-\frac{2\sigma_{t,T_2}d_{22}+\sigma_{t,T_2}^2}{2}\right)=n_1\left(d_{21}\right)\exp\left(-\sigma_{t,T_2}d_{22}-\frac{\sigma_{t,T_2}^2}{2}\right)$$

$$=n_1\left(d_{22}\right)\exp\left(-\frac{\sigma_{t,T_2}^2}{2}\right)\exp\left(-\sigma_{t,T_2}d_{22}\right) \quad . \tag{9.5.182}$$

$$=n_1\left(d_{22}\right)\exp\left(-\frac{\sigma_{t,T_2}^2}{2}\right)\exp\left\{-\sigma_{t,T_2}\frac{\ln\left[S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}/X_U\right]-\frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}\right\}$$

$$=\frac{X_U}{S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}n_1\left(d_{22}\right)$$

Substituting

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)$$

$$=\iota_C\iota_US_tB_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\iota_UN_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}\frac{X_U}{S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}n_1\left(d_{22}\right). \tag{9.5.183}$$

$$-\iota_C\iota_UX_UB_{t,T_2,r}\iota_UN_1\left[\frac{\iota_C\iota_Ud_{21}(S)-\iota_C\rho\iota_Ud_{22}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(d_{22}\right)$$

Rearranging and cancelling terms,

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)$$

$$=\iota_CX_UB_{t,T_2,r}n_1\left(d_{22}\right)\left(\begin{array}{l}N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}\\-N_1\left[\frac{\iota_C\iota_Ud_{21}(S)-\iota_C\rho\iota_Ud_{22}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]\end{array}\right). \tag{9.5.184}$$

Thus,

$$\frac{\partial}{\partial d_{22}} CO_D\left(S,t,T_1,T_2\right)$$

$$= \iota_C X_U B_{t,T_2,r} n_1\left(d_{22}\right)\left(\begin{array}{l} N_1\left\{\iota_C \iota_U \frac{\left[d_{21}\left(S\right)+\sigma_{t,T_1}\right]-\rho\left[d_{22}\left(S\right)+\sigma_{t,T_2}\right]}{\sqrt{1-\rho^2}}\right\} \\ -N_1\left[\iota_C \iota_U \frac{d_{21}\left(S\right)-\rho d_{22}\left(S\right)}{\sqrt{1-\rho^2}}\right] \end{array}\right). \tag{9.5.185}$$

Note

$$\frac{d_{21}+\sigma_{t,T_1}-\rho\left(d_{22}+\sigma_{t,T_2}\right)}{\sqrt{1-\rho^2}} = \frac{d_{21}-\rho d_{22}}{\sqrt{1-\rho^2}} + \frac{\sigma_{t,T_1}-\rho\sigma_{t,T_2}}{\sqrt{1-\rho^2}}, \tag{9.5.186}$$

and recall $\rho = \frac{\sigma_{t,T_1}}{\sigma_{t,T_2}}$.

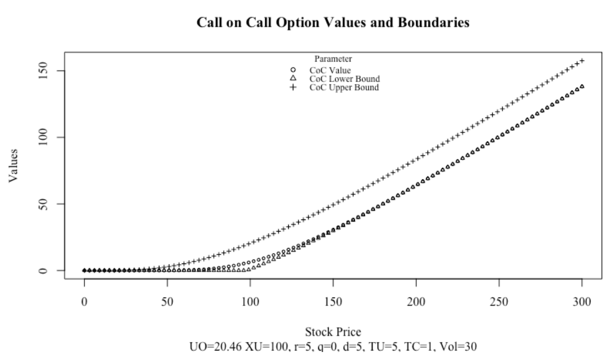
QED

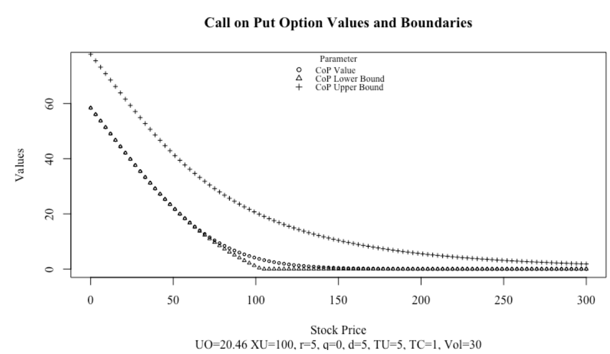
Selected Results

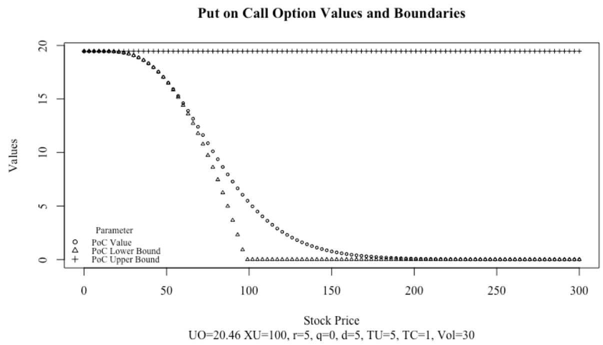
<< code needs to be audited >>

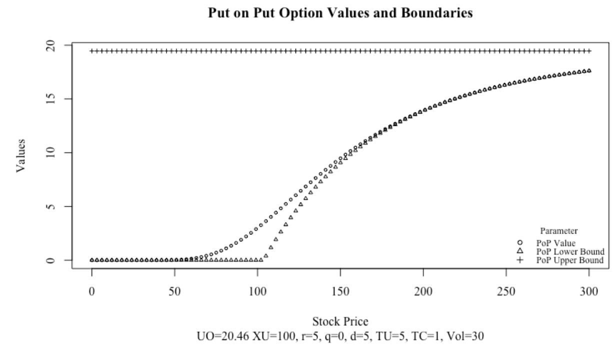
Sensitivity to underlying instrument

Compound option value sensitivity to the underlying instrument

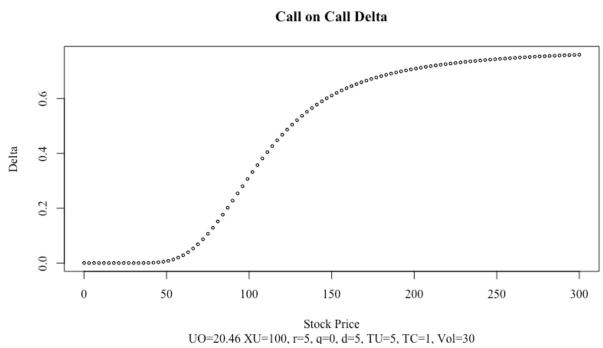


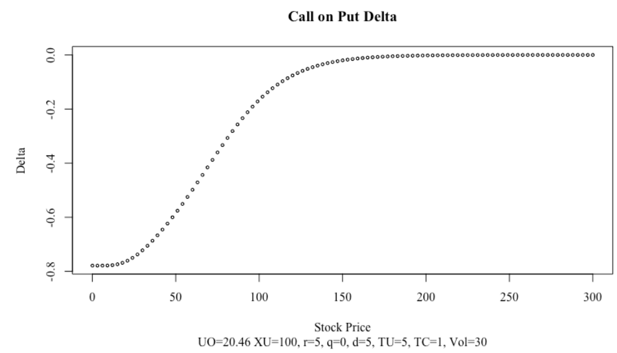


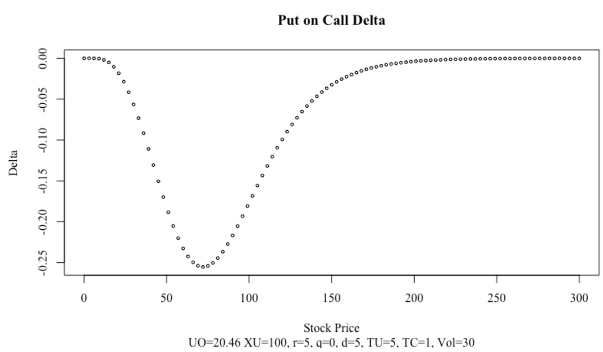


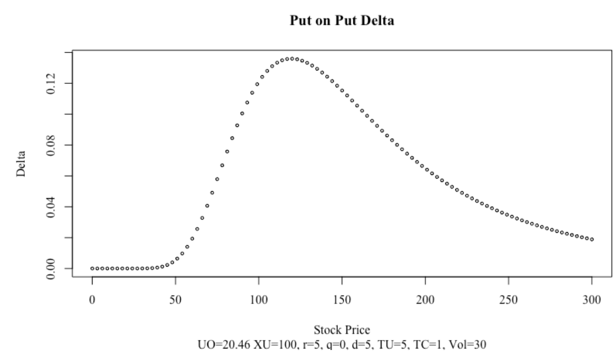


Compound option delta sensitivity to the underlying instrument

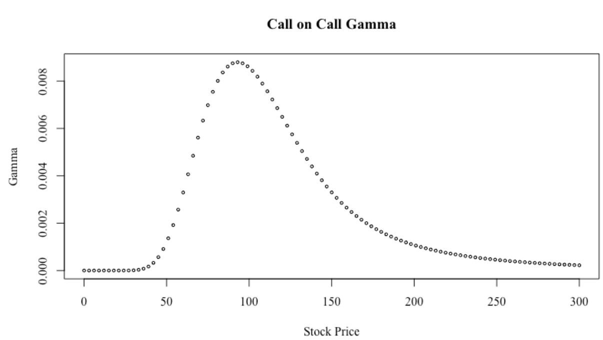


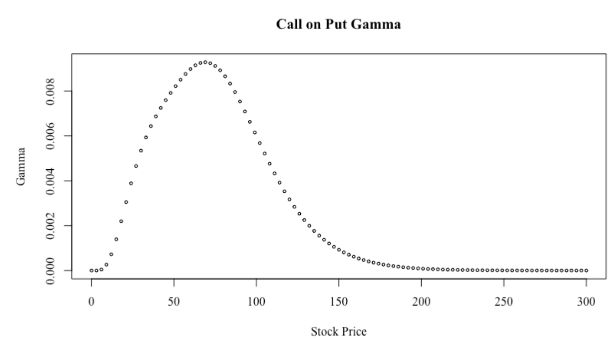


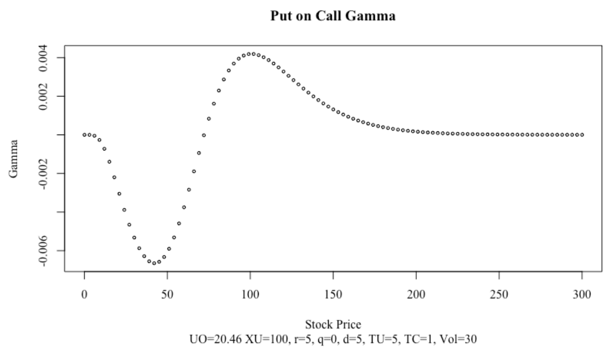


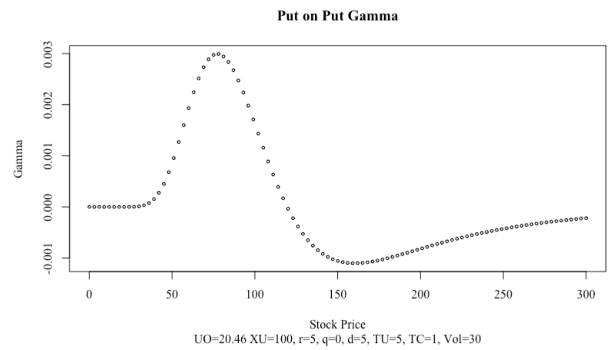


Compound option gamma sensitivity to the underlying instrument

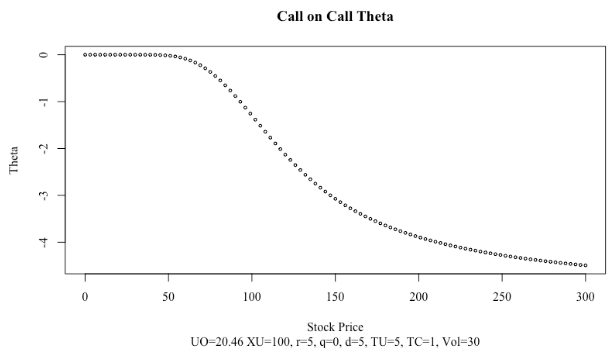


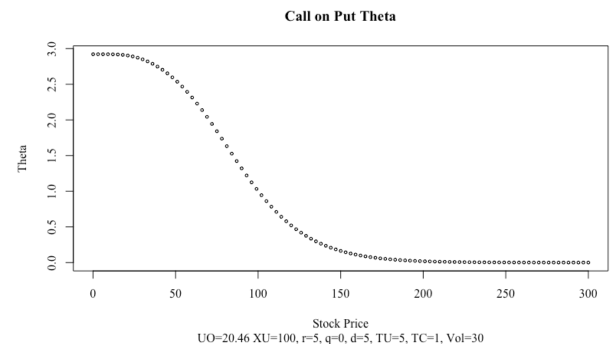


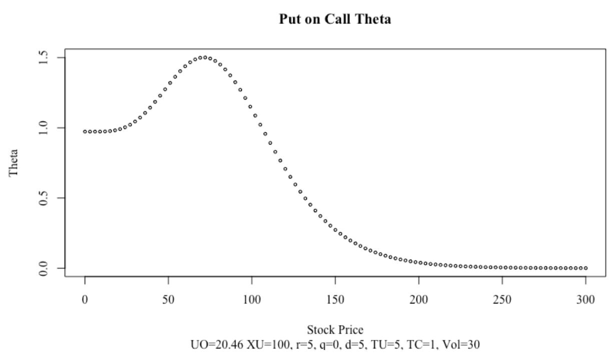


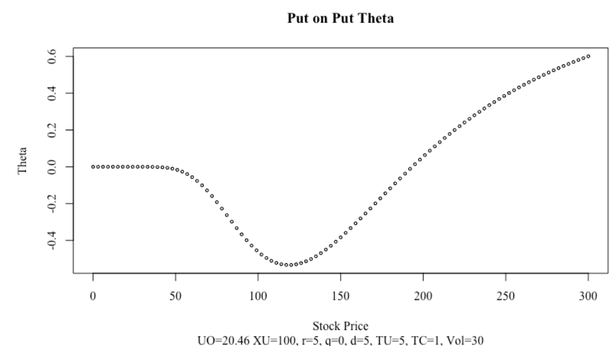


Compound option theta sensitivity to the underlying instrument

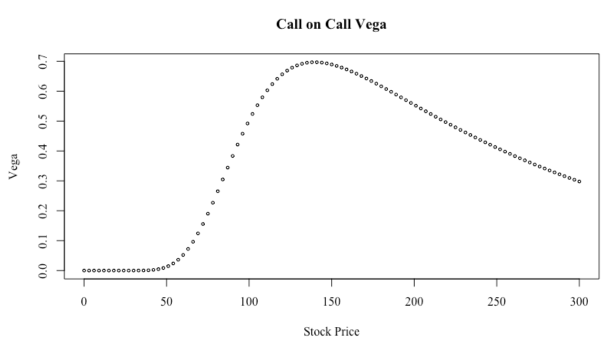


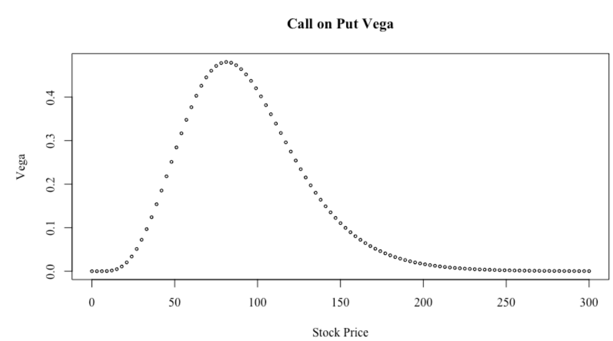


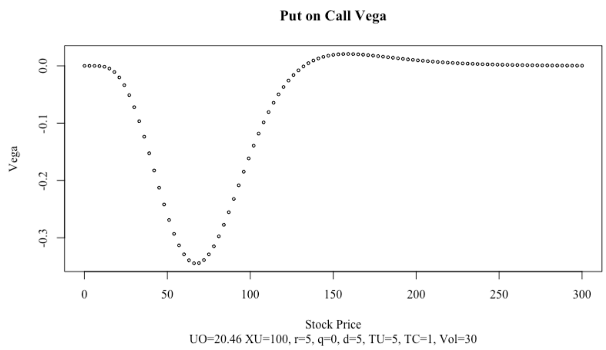


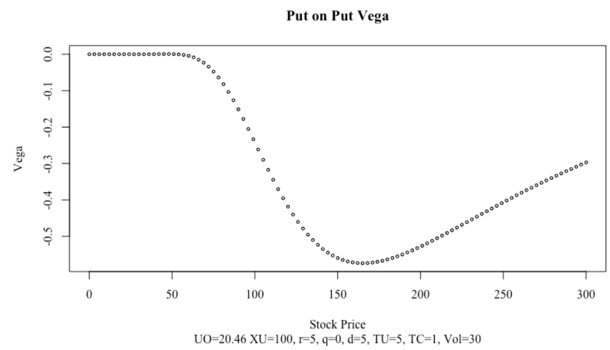


Compound option vega sensitivity to the underlying instrument

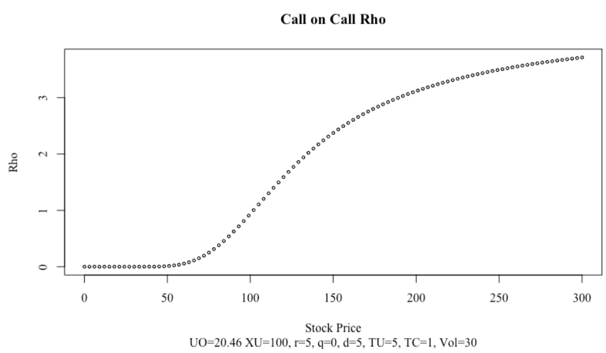


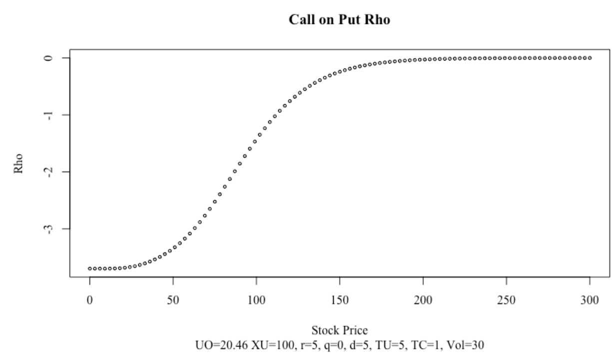


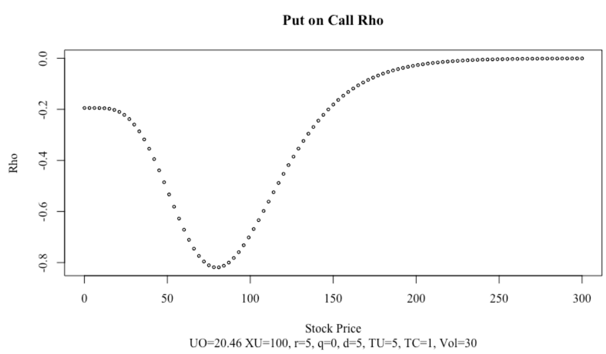


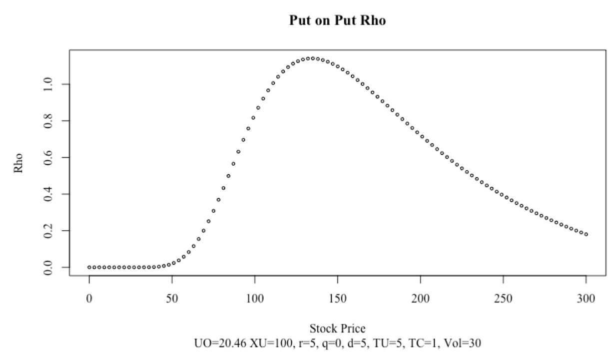


Compound option rho sensitivity to the underlying instrument



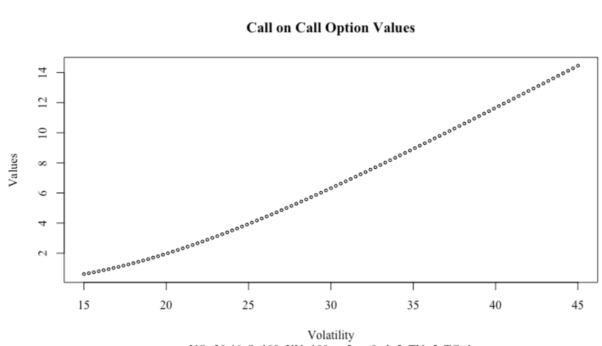


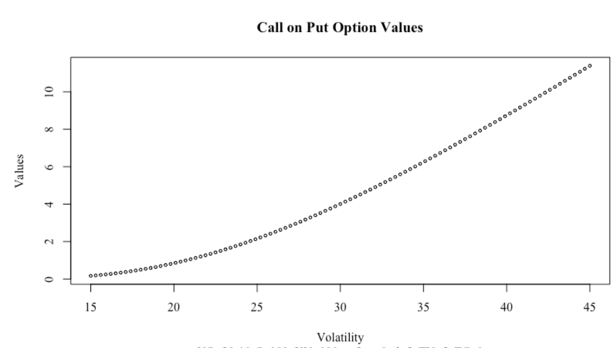


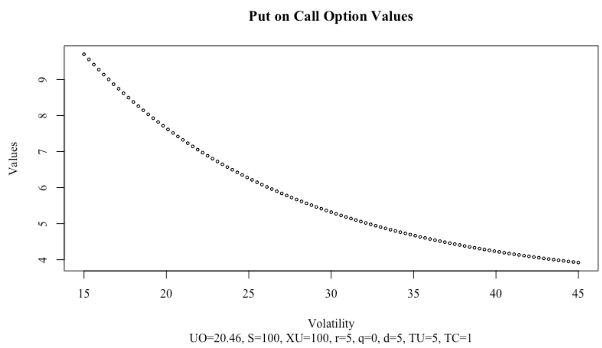


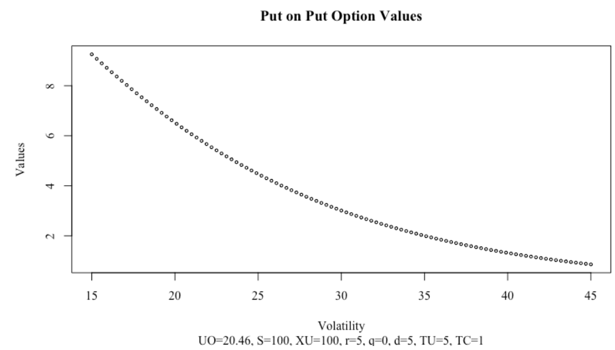
Sensitivity to volatility

Compound option value sensitivity to volatility

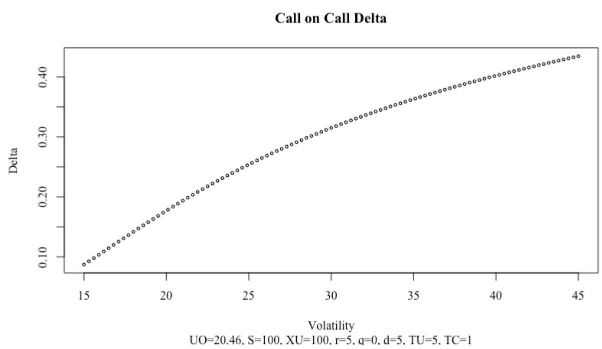


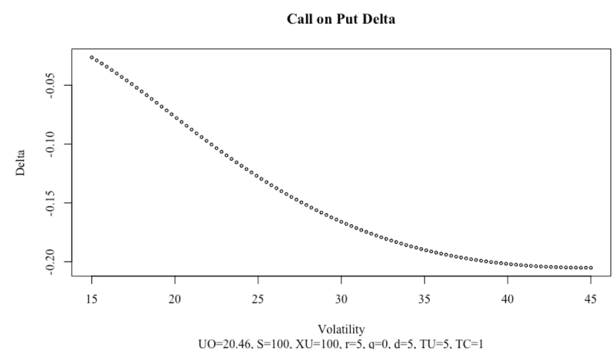


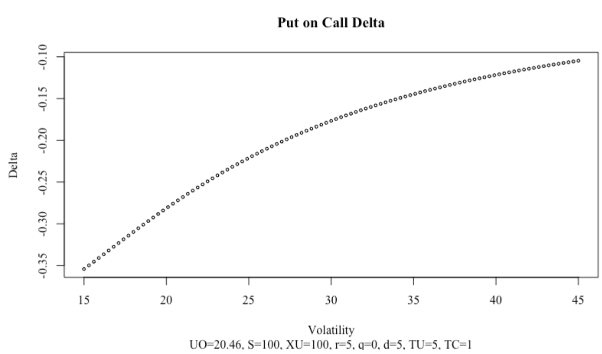


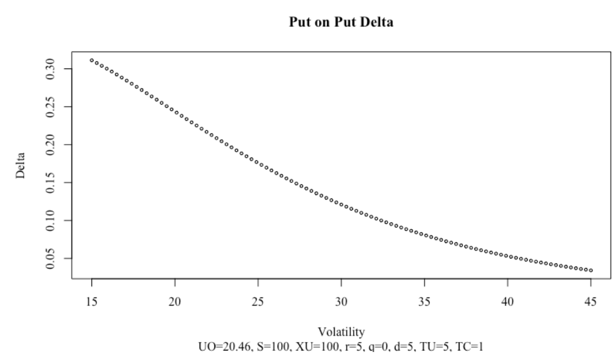


Compound option delta sensitivity to volatility

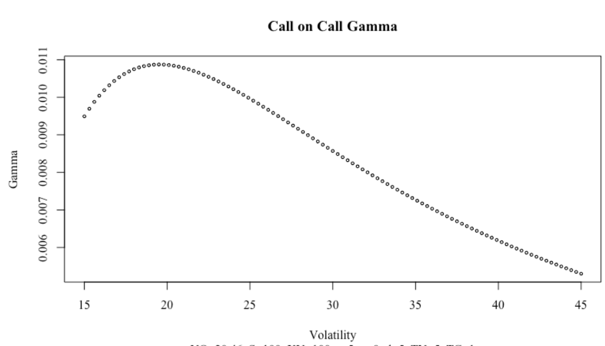


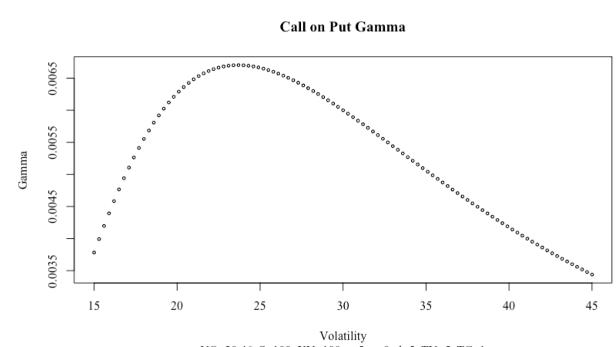


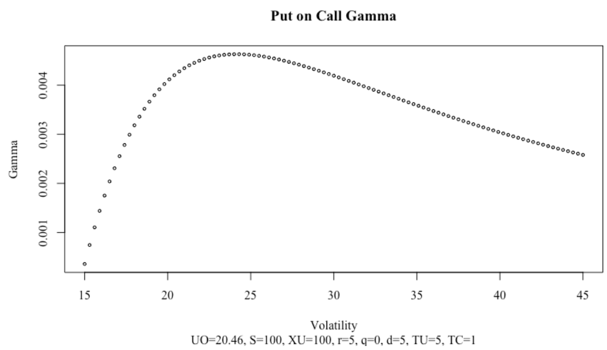


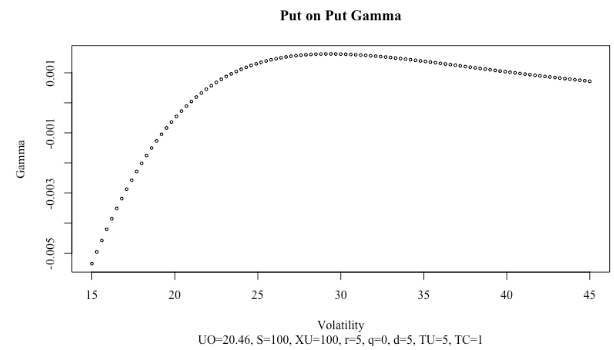


Compound option gamma sensitivity to volatility

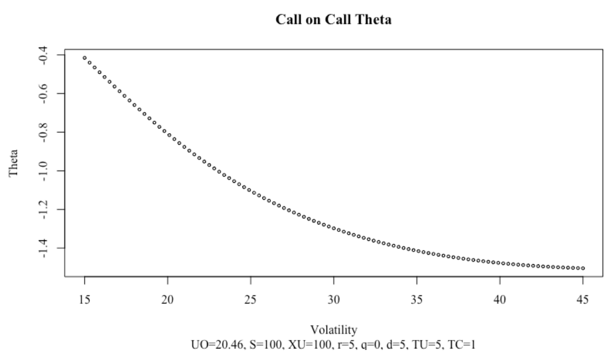


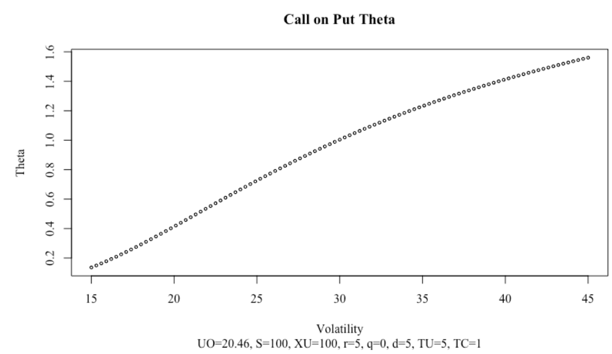


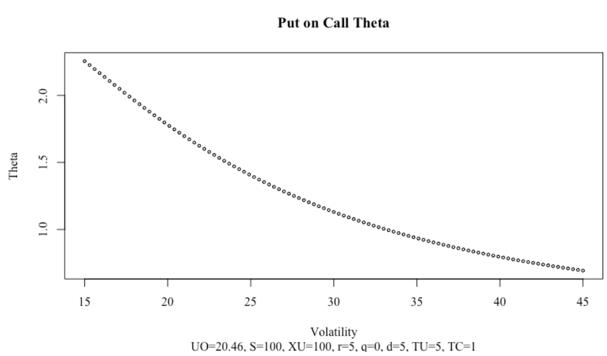


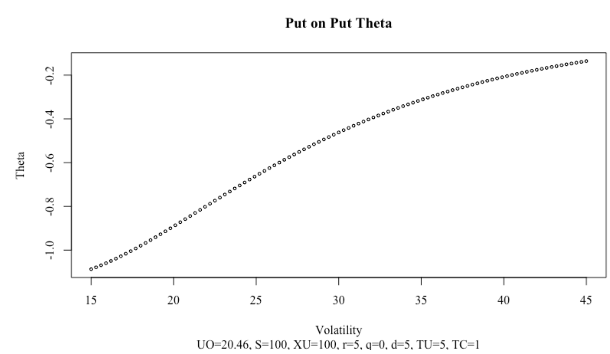


Compound option theta sensitivity to volatility

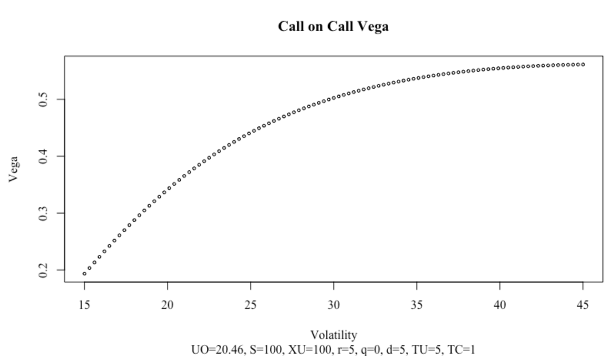


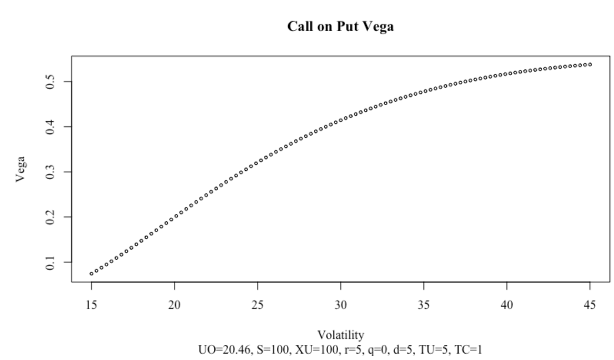


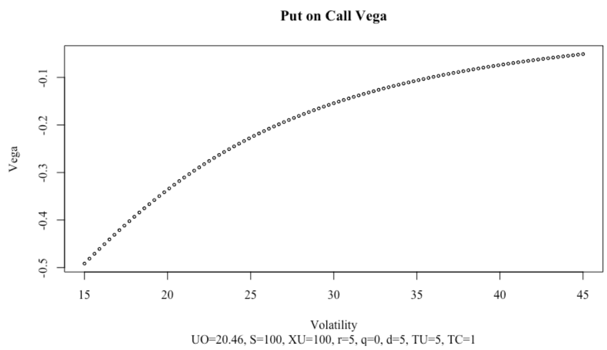


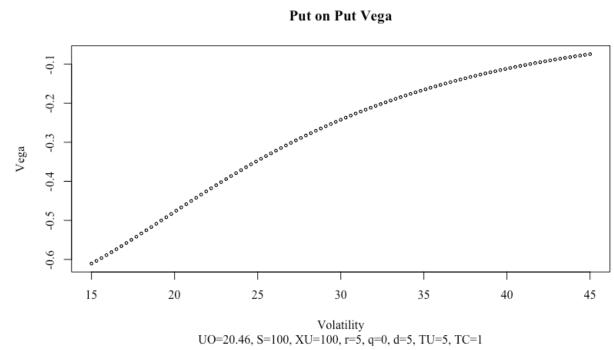


Compound option vega sensitivity to volatility

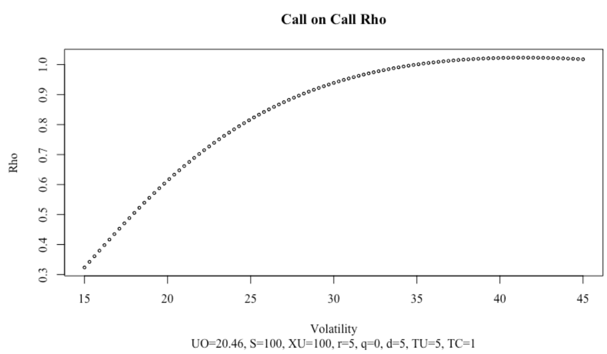


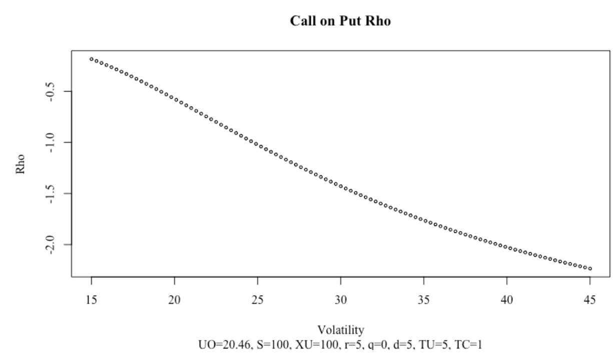


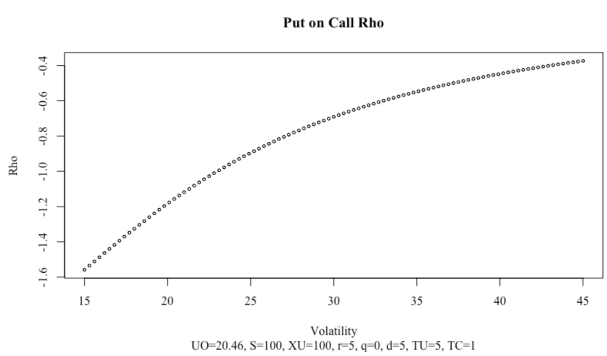


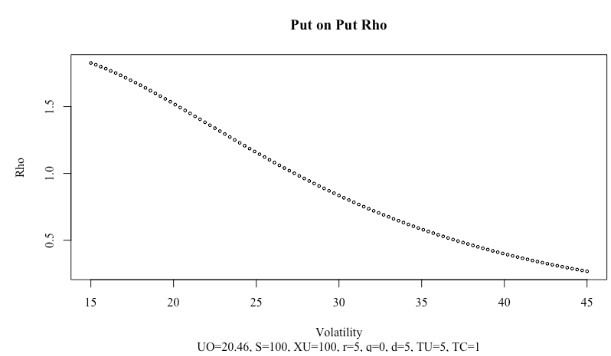


Compound option rho sensitivity to volatility



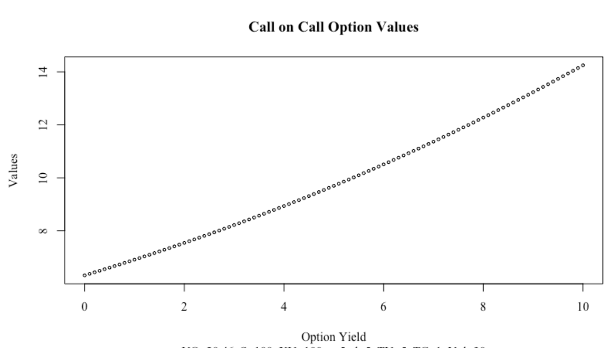


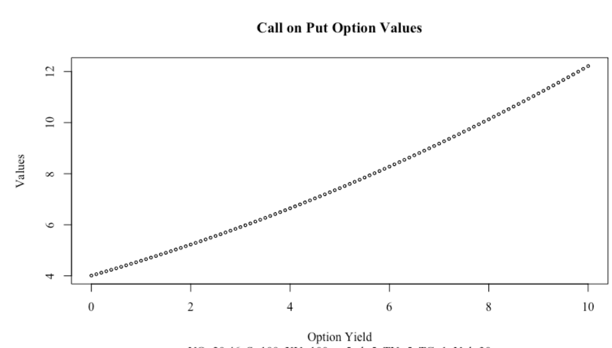


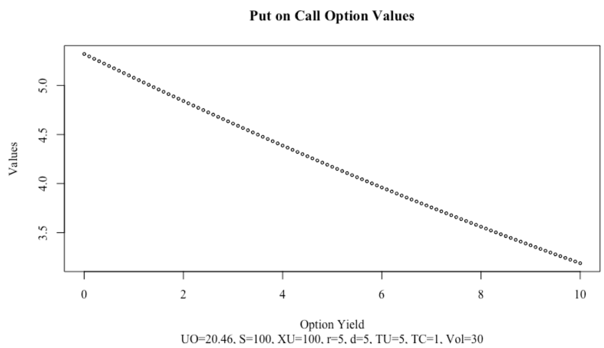


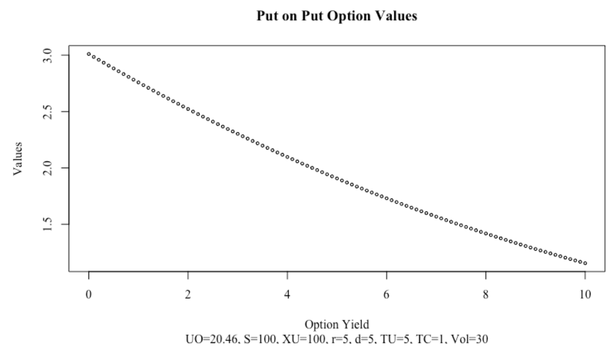
Sensitivity to option yield

Compound option value sensitivity to option yield

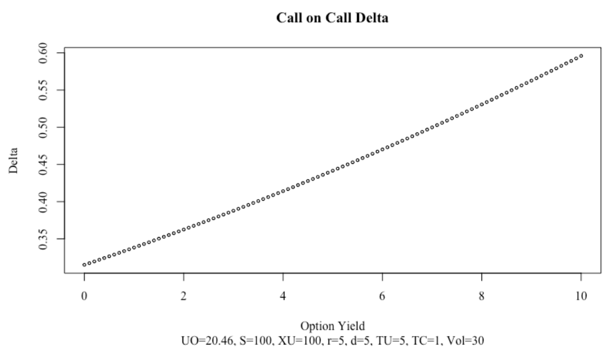


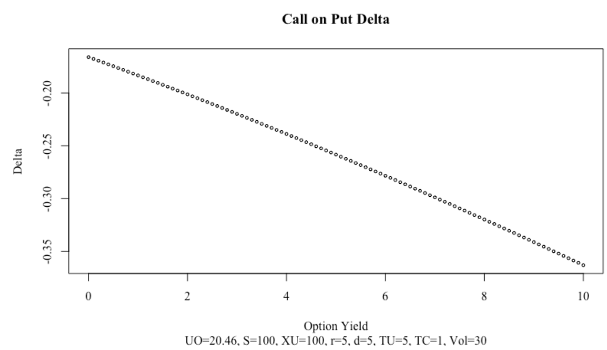


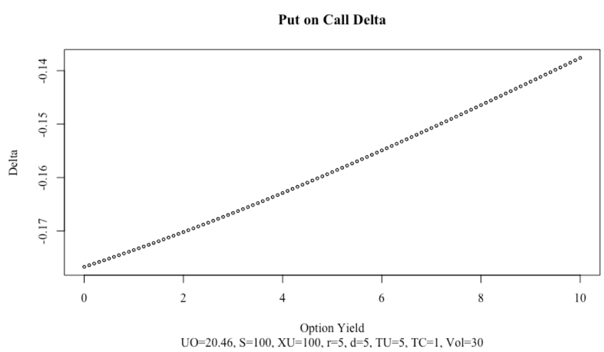


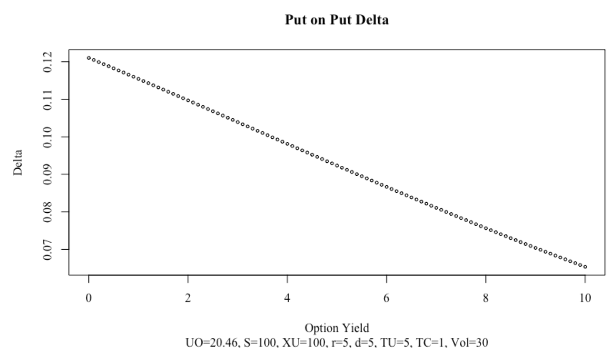


Compound option delta sensitivity to option yield

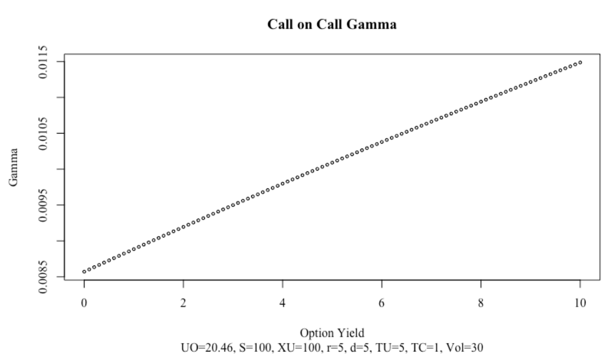


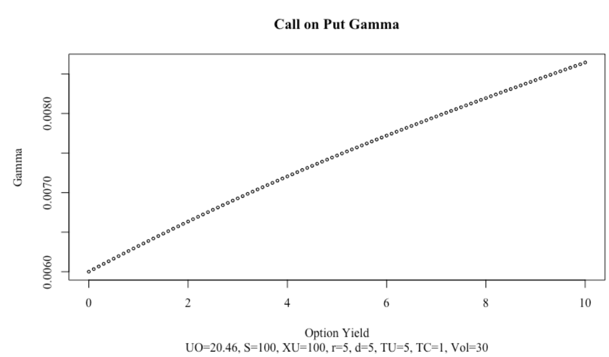




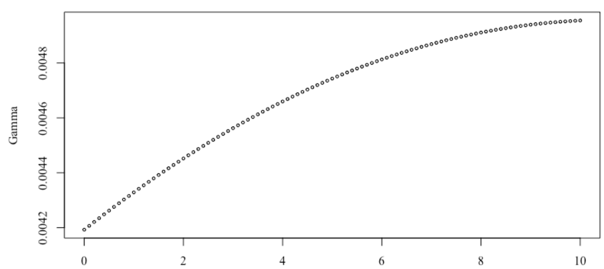


Compound option gamma sensitivity to option yield





Put on Call Gamma



Option Yield
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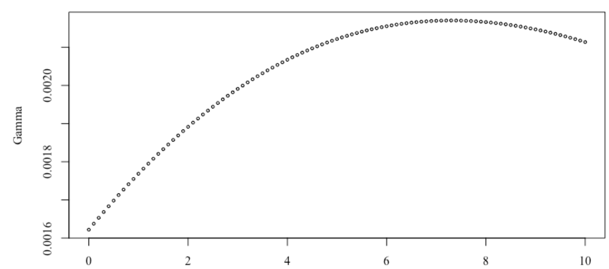
Put on Put Gamma



Option Yield
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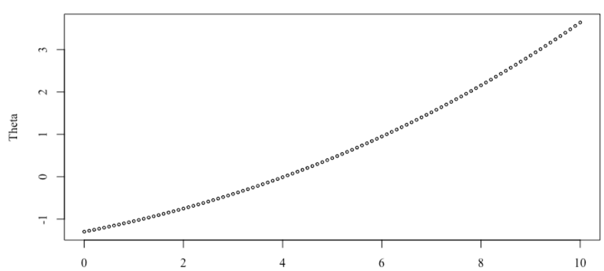
Compound option theta sensitivity to option yield

Call on Call Theta



Option Yield
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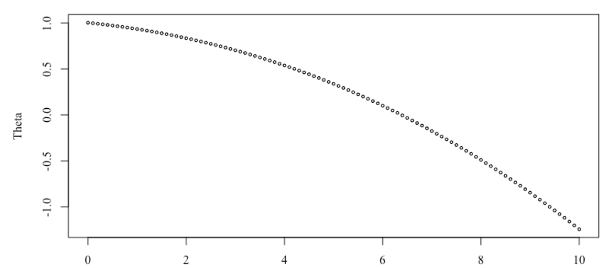
Call on Put Theta



Option Yield
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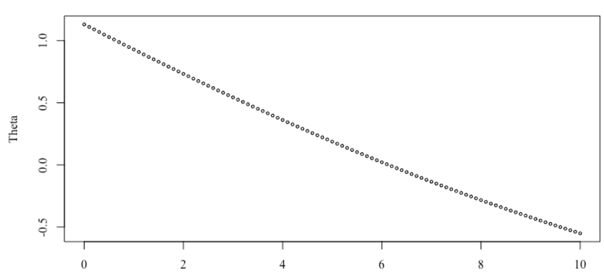
Put on Call Theta



Option Yield
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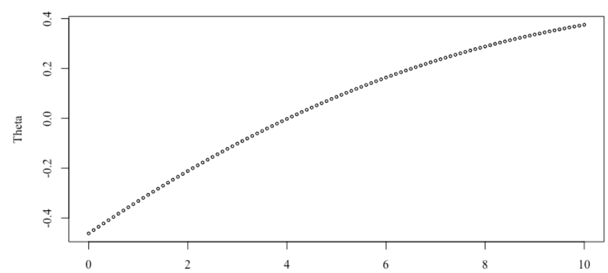
Put on Put Theta



Option Yield
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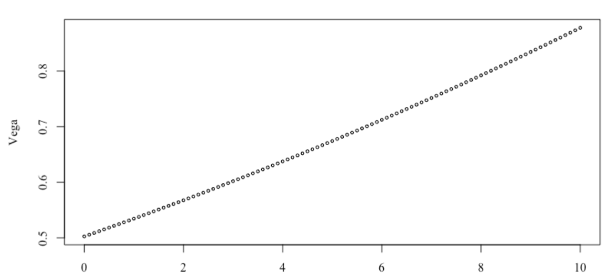
Compound option vega sensitivity to option yield

Call on Call Vega



Option Yield
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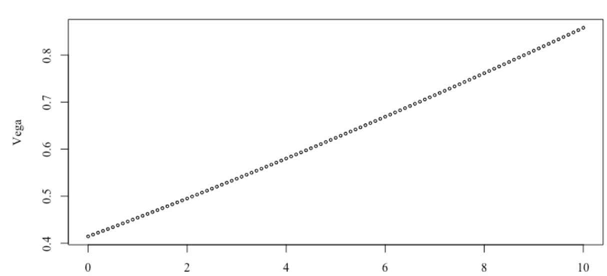
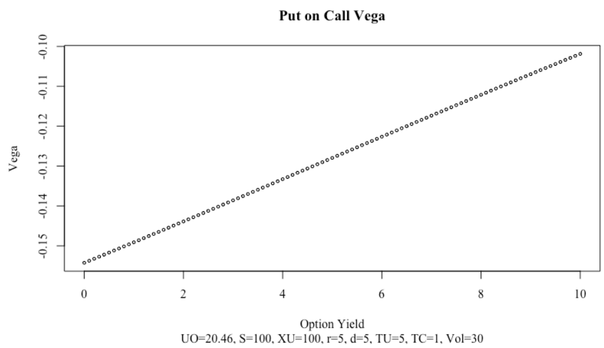
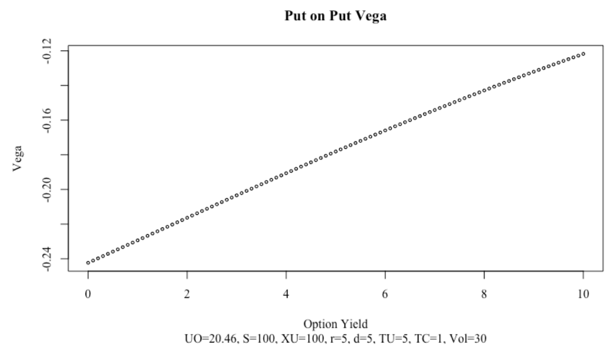
Call on Put Vega

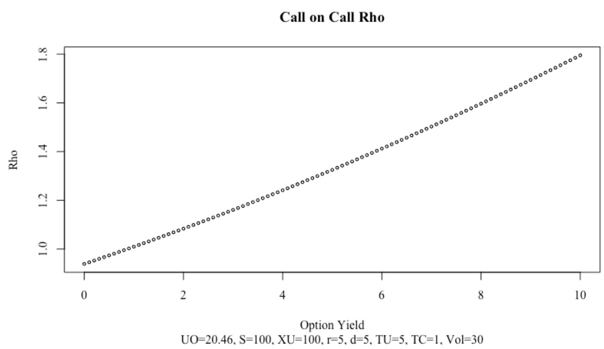


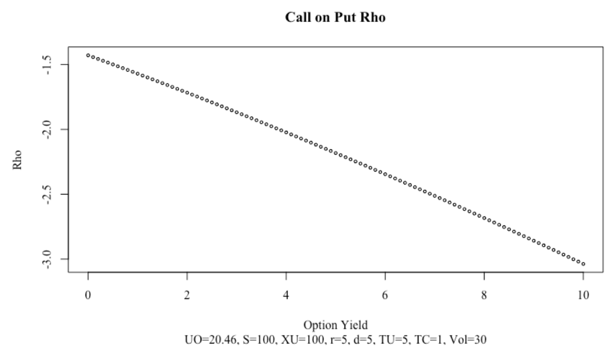
Option Yield
UO=20.46, S=100, XU=100, r=5, d=5, TU=5, TC=1, Vol=30

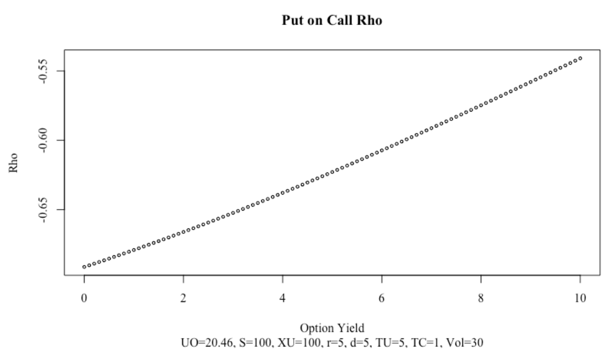


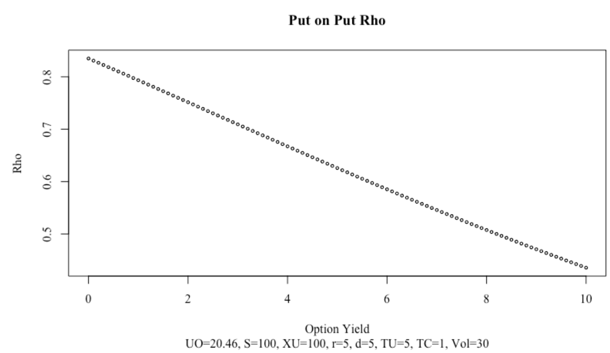


Compound option rho sensitivity to option yield



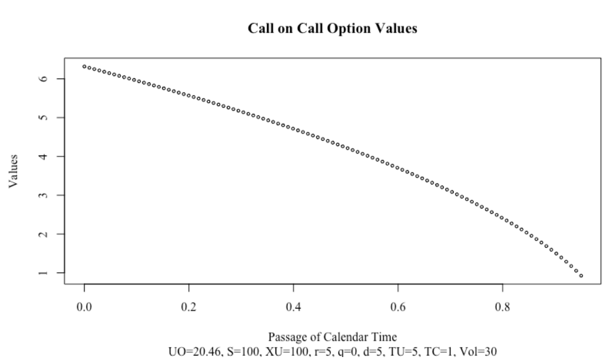


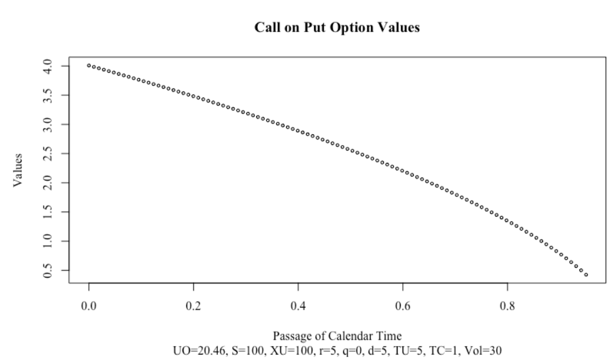


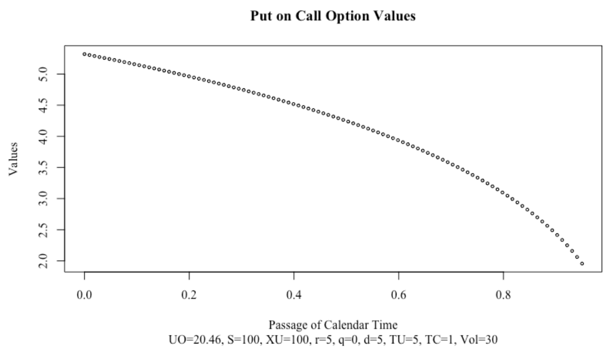


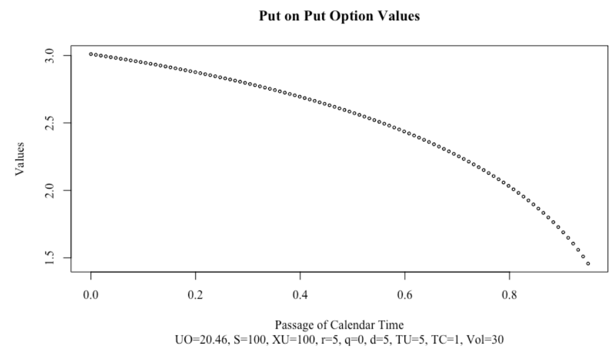
Sensitivity to compound option maturity

Compound option value sensitivity to compound option maturity

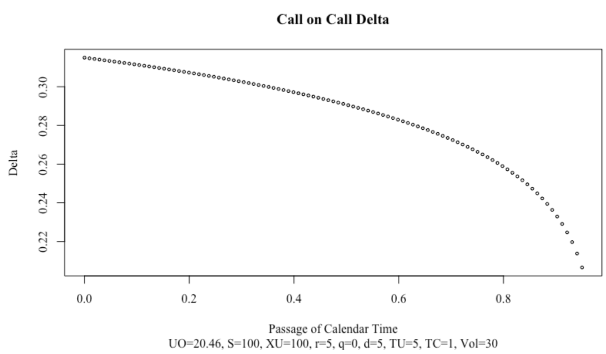


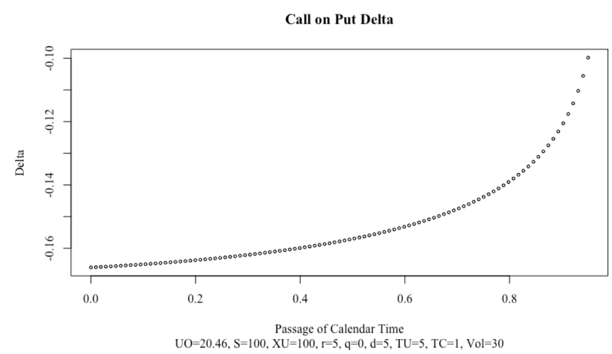


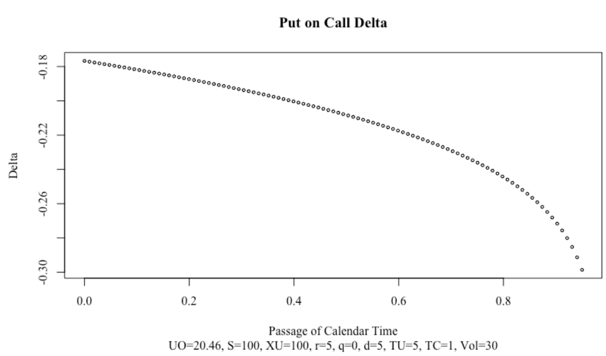


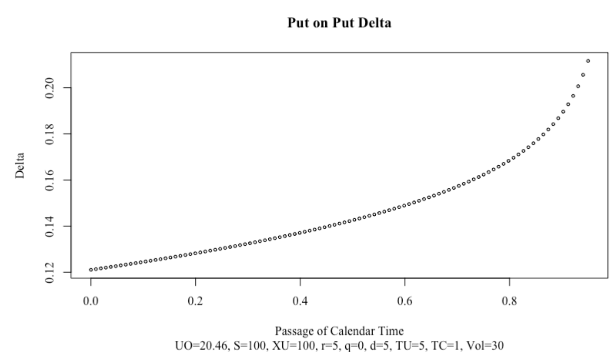


Compound option delta sensitivity to compound option maturity

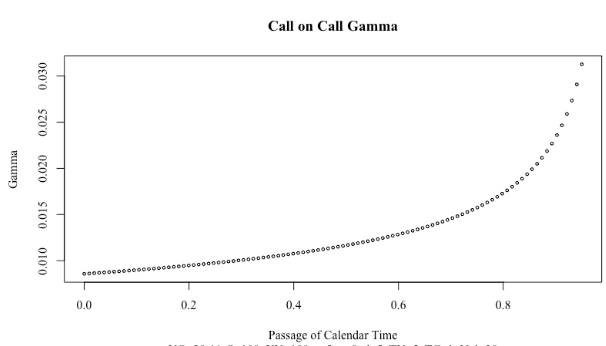


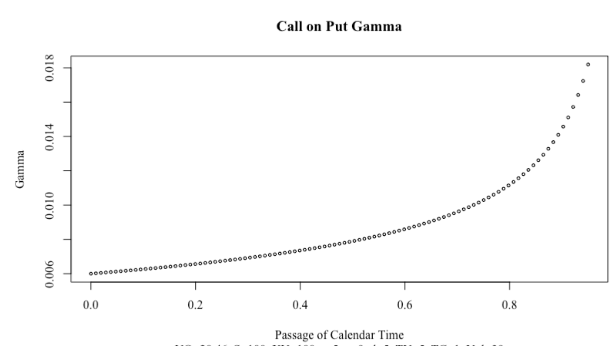


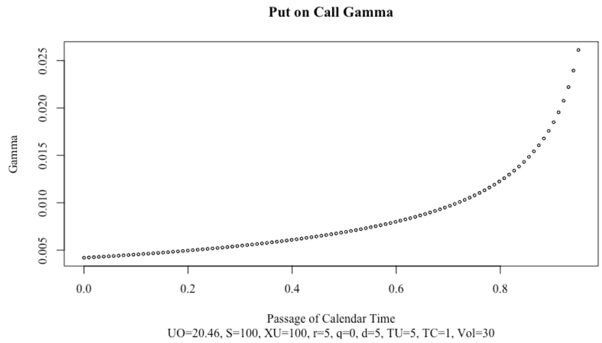


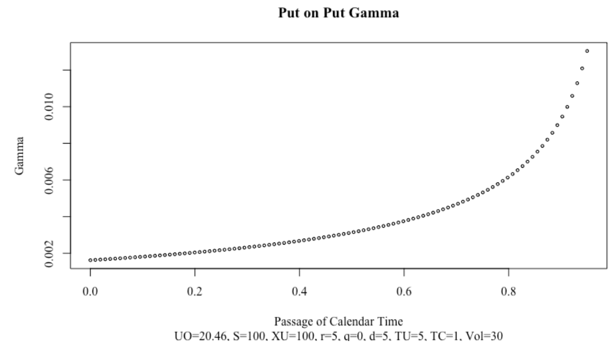


Compound option gamma sensitivity to compound option maturity

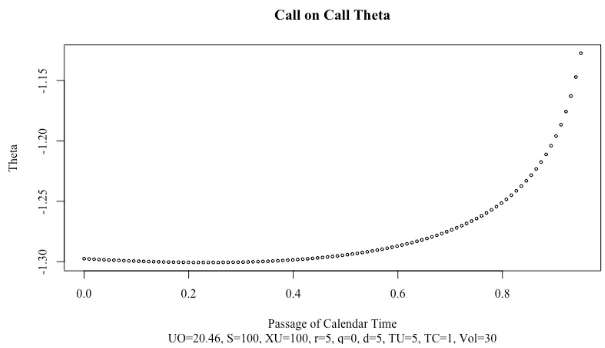


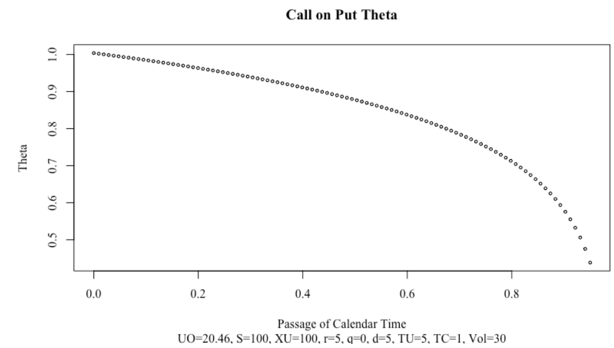


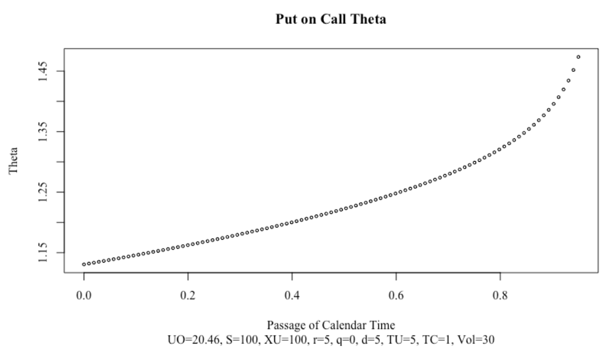


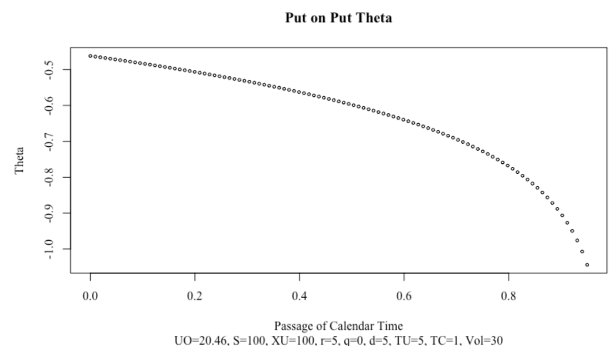


Compound option theta sensitivity to compound option maturity

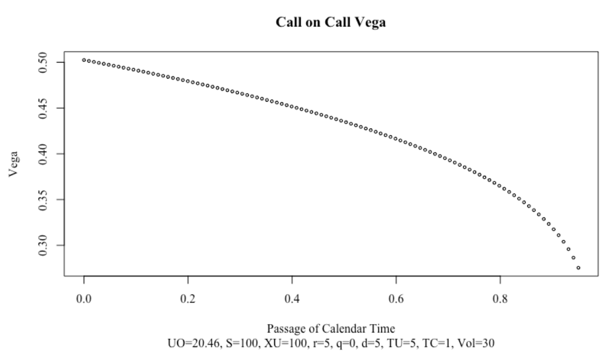


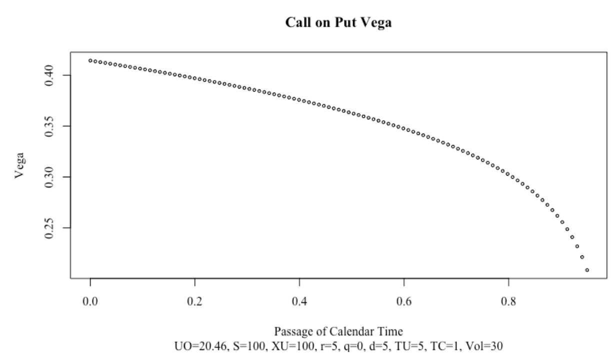


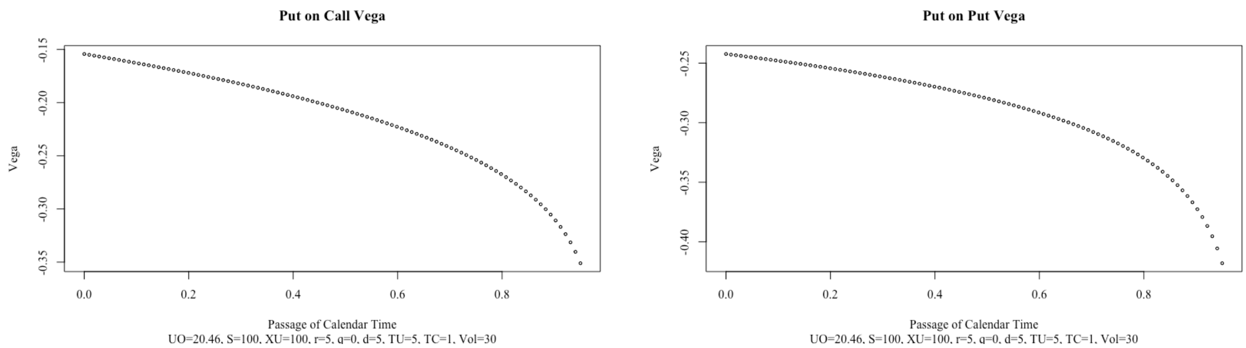




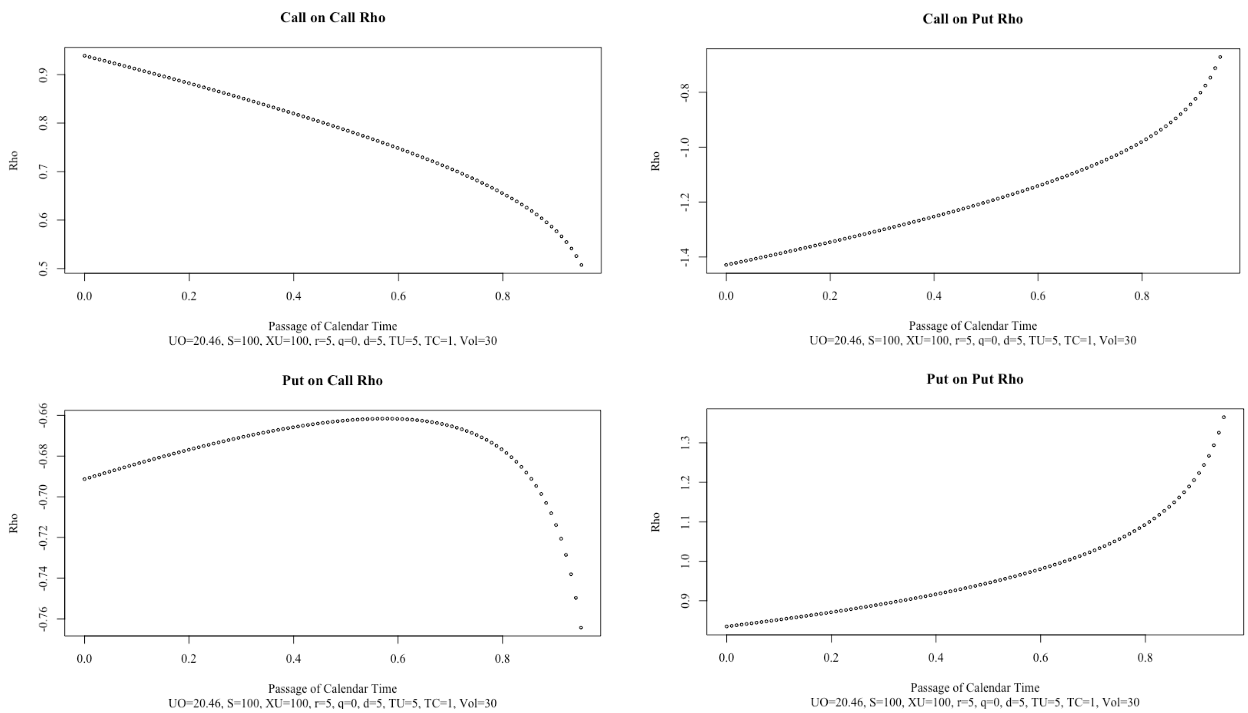
Compound option vega sensitivity to calendar time







Compound option rho sensitivity to calendar time



R code for analytic delta. Very difficult to derive, hard to code, but runs fast.

```
# CO Delta
CODelta <- function(C, L, U){
  with(C, {
    r <- r/100
    d <- d/100
    q <- q/100
    v <- v/100
    B2d <- exp(-d*TU)
    B12Nq <- exp(q*(TU - TC))
    B12q <- exp(-q*(TU - TC))
    B2r <- exp(-r*TU)
    B1r <- exp(-r*TC)
    B1q <- exp(-q*TC)
    B12d <- exp(-d*(TU - TC))
    d11 <- COd11(C, L, U)
```

```
    d21 <- COd21(C, L, U)
    d12 <- COd12(C)
    d22 <- COd22(C)
    mean1 <- rep(0,2)
    lower1 <- rep(-Inf,2)
    corr1 <- diag(2)
    corr1[lower.tri(corr1)] <- iC*sqrt(TC/TU)
    corr1[upper.tri(corr1)] <- iC*sqrt(TC/TU)
    upper1 <- c(iC*iU*d11,iU*d12)
    N2d11d12 <- pmvnorm(lower=lower1, upper=upper1, mean=mean1, corr=corr1)[1]
    Delta <- iC*iU*B12Nq*B2d*N2d11d12
    # Delta <- iC*iU*B1q*B12d*N2d11d12
    return(Delta)
  })
}
```

R code for numerical delta: Much easier to code, but takes longer to run and may be unstable.

```
#
# Numeric Greeks
#
# Compound option delta
CONGDelta <- function(C, L, U){
  with(C, {
    # Increment <- 0.01
    Original <- S
    Change <- Increment*Original
    High <- Original + Change
    C$S <- High
    OHigh <- COValue(C, L, U)
    Low <- Original - Change
    C$S <- Low
    OLow <- COValue(C, L, U)
    C$S <- Original
    Answer <- (OHigh - OLow)/(High - Low)
    return(Answer)
  })
}
```

Compound option rho sensitivity to calendar time