Module 8.5

Static Risk Measures
GBM-Based Compound
Option Valuation Models

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Overview

- Review COVM model
- Derive delta, gamma, and theta
- Explore sensitivities with respect to
  - Underlying
  - Volatility
  - Option Yield
  - Compound option maturity

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Central Finance Concepts

- Detailed graphical analysis
  - Underlying
  - Greeks
  - CaCall, CaPut, PuCall, and PuPut
- Sensitivity to underlying stock price
- Sensitivity to volatility
- Sensitivity to option yield
- Sensitivity to compound option maturity

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Underlying Instrument

- Sensitivity to underlying instrument price
- Sensitivity to Greeks
  - Delta
  - Gamma
  - Theta
  - Vega
  - Rho

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Figure 9.5.1. Compound Option Value Sensitivity to the Underlying Instrument

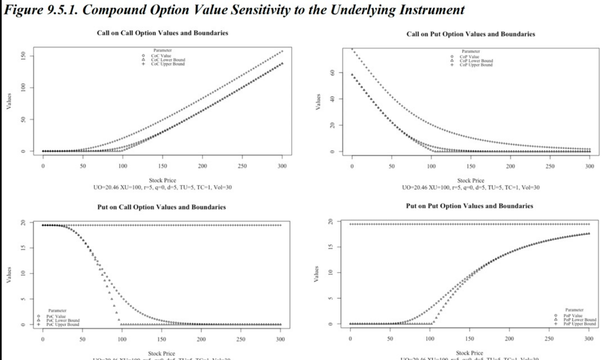

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Figure 9.5.2. Compound Option Delta Sensitivity to the Underlying Instrument

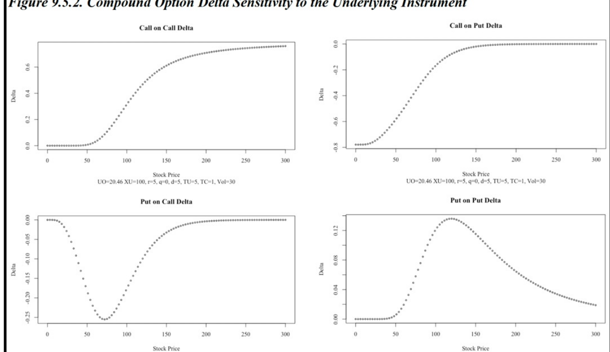

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Figure 9.5.3. Compound Option Gamma Sensitivity to the Underlying Instrument

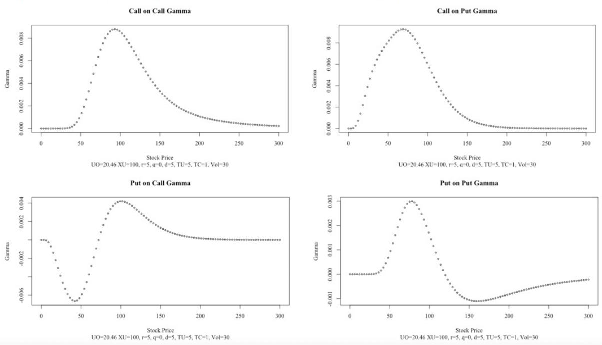

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Figure 9.5.4. Compound Option Theta Sensitivity to the Underlying Instrument

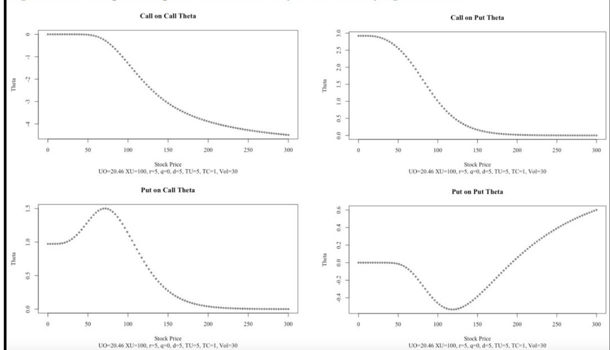

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Figure 9.5.5. Compound Option Vega Sensitivity to the Underlying Instrument

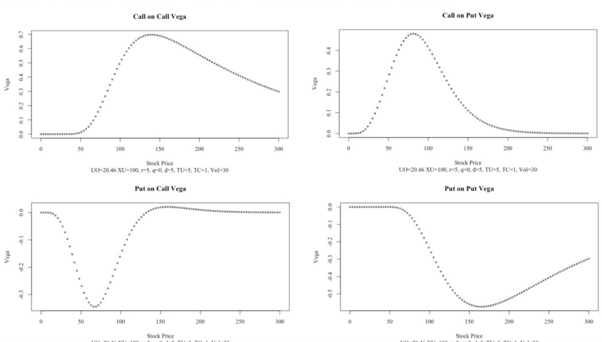

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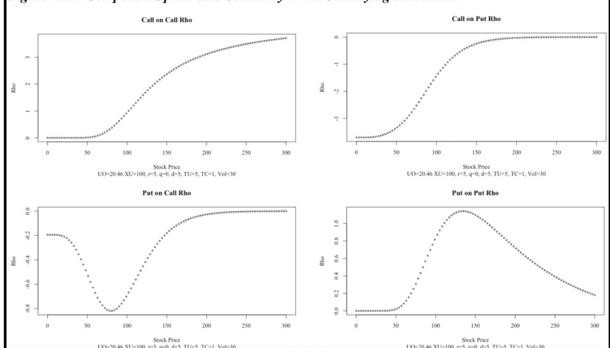
Figure 9.5.6. Compound Option Rho Sensitivity to the Underlying Instrument



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Sensitivity to Volatility

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Figure 9.5.7. Compound Option Value Sensitivity to Volatility

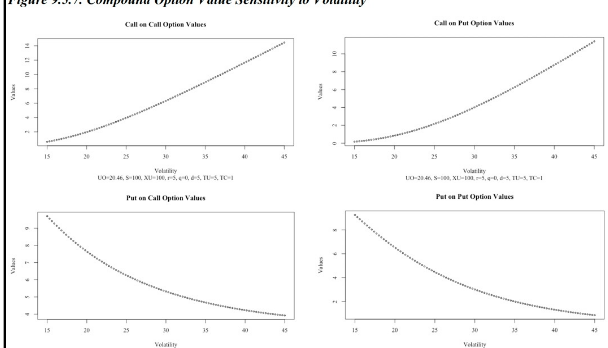

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Figure 9.5.8. Compound Option Delta Sensitivity to Volatility

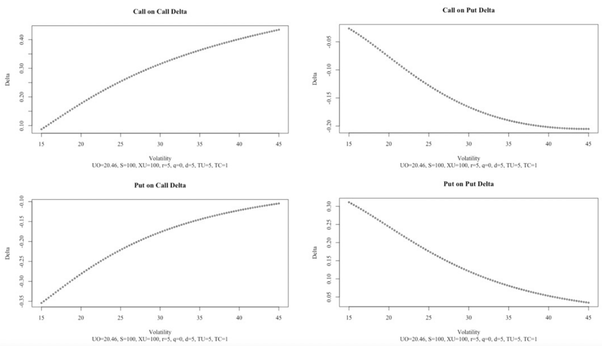

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Figure 9.5.9. Compound Option Gamma Sensitivity to Volatility

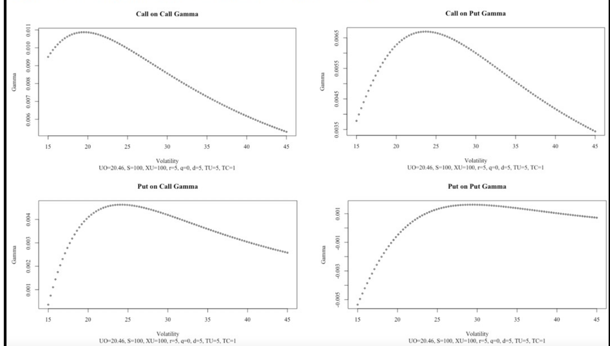

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Figure 9.5.10. Compound Option Theta Sensitivity to Volatility

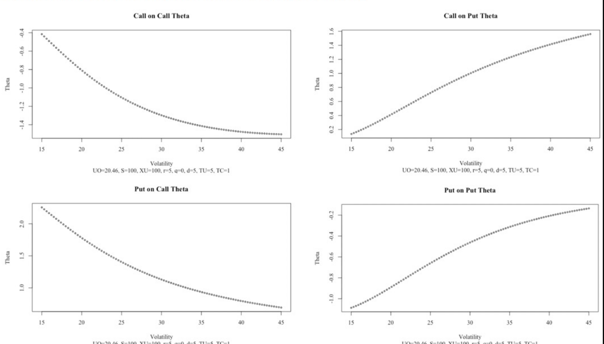

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Figure 9.5.11. Compound Option Vega Sensitivity to Volatility

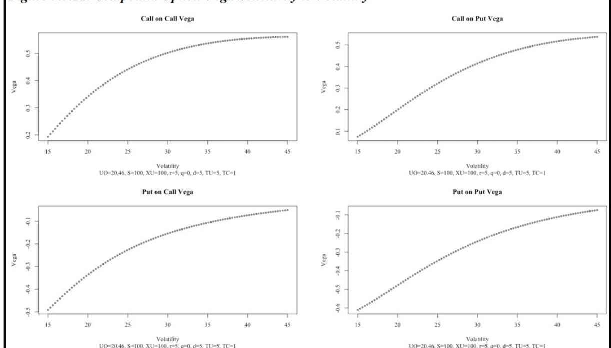

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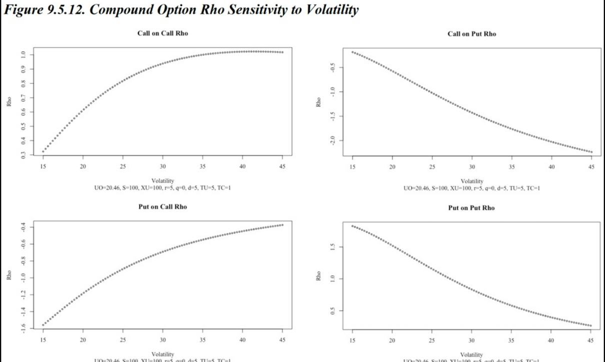
Figure 9.5.12. Compound Option Rho Sensitivity to Volatility



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Sensitivity to Option Yield

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Figure 9.5.13. Compound Option Value Sensitivity to Option Yield

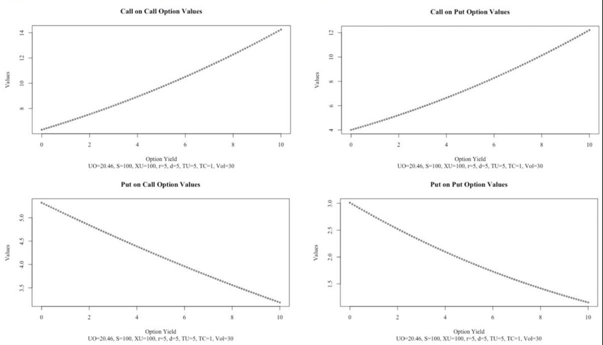

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Figure 9.5.14. Compound Option Delta Sensitivity to Option Yield

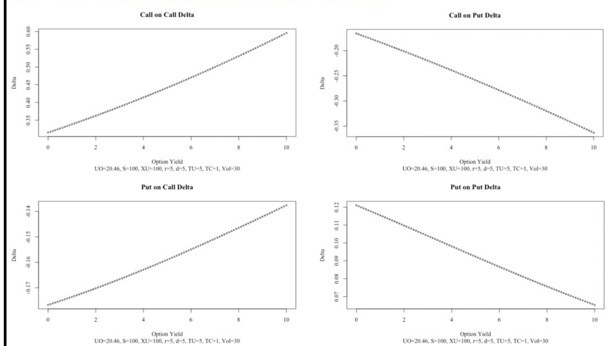

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Figure 9.5.15. Compound Option Gamma Sensitivity to Option Yield

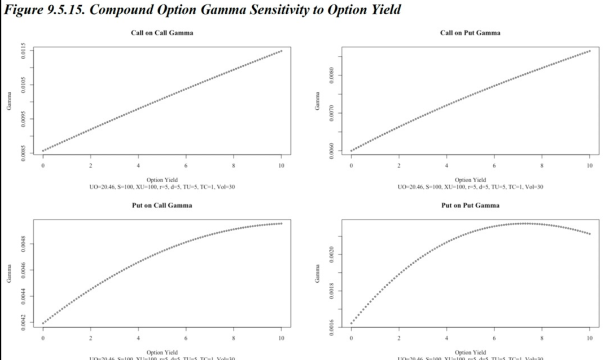

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Figure 9.5.16. Compound Option Theta Sensitivity to Option Yield

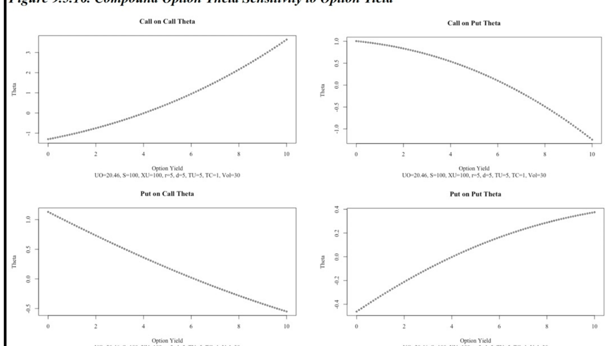

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Figure 9.5.17. Compound Option Vega Sensitivity to Option Yield

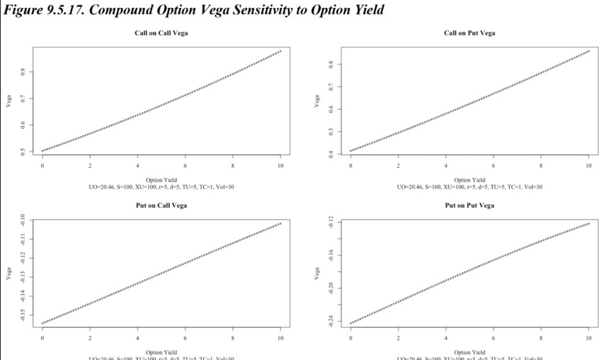

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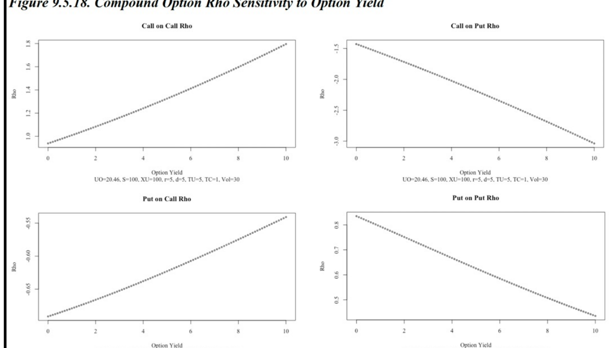
Figure 9.5.18. Compound Option Rho Sensitivity to Option Yield



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Sensitivity to Compound Option Maturity

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Figure 9.5.19. Compound Option Value Sensitivity to Compound Option Maturity

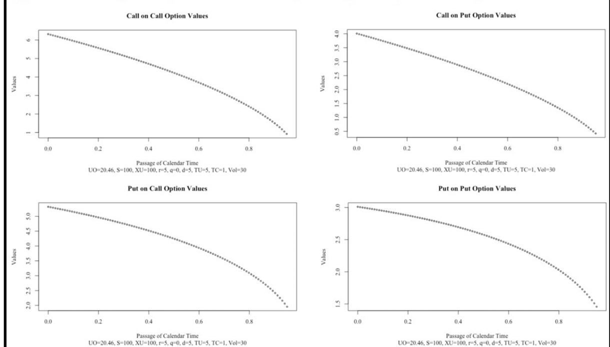

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Figure 9.5.20. Compound Option Delta Sensitivity to Compound Option Maturity

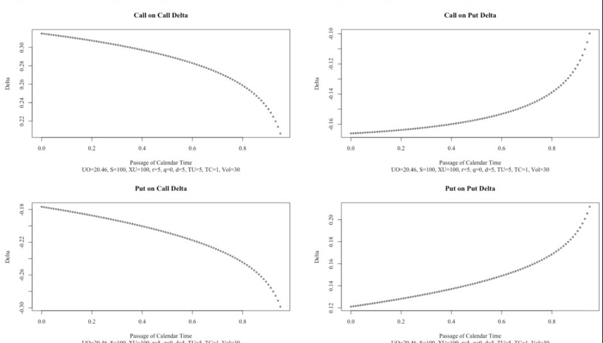

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Figure 9.5.21. Compound Option Gamma Sensitivity to Compound Option Maturity

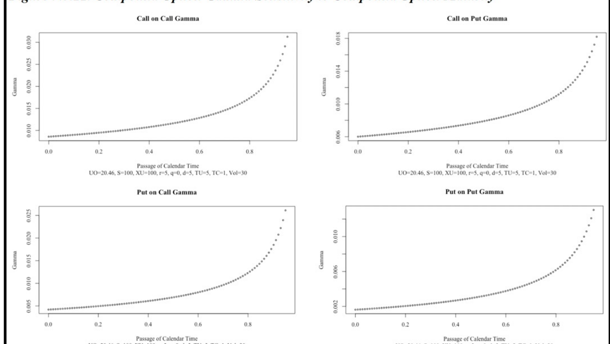

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Figure 9.5.22. Compound Option Theta Sensitivity to Compound Option Maturity

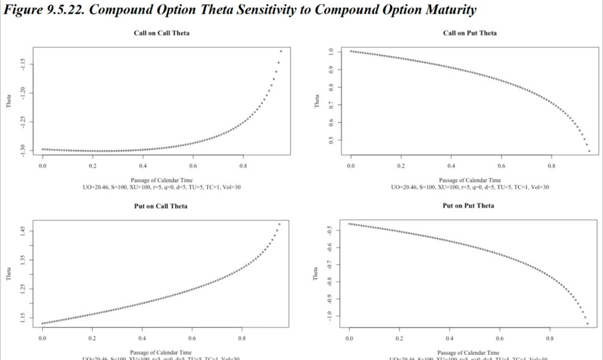

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Figure 9.5.23. Compound Option Vega Sensitivity to Calendar Time

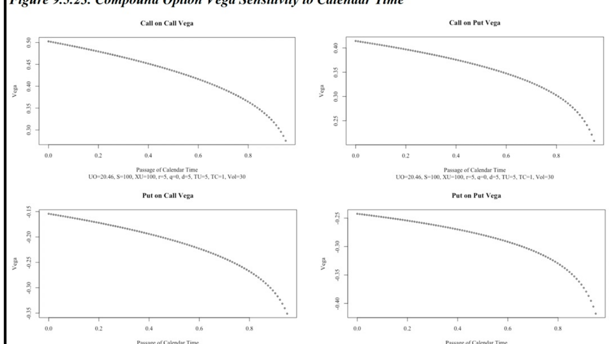

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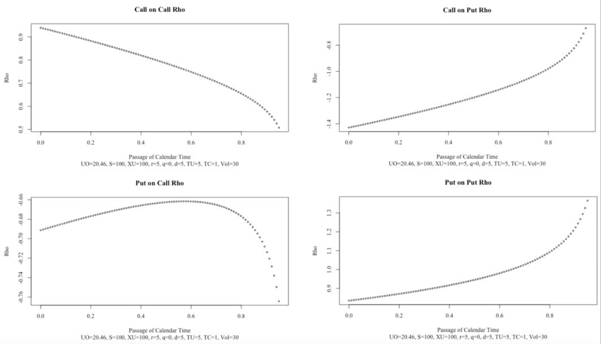
Figure 9.5.24. Compound Option Rho Sensitivity to Calendar Time



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Quantitative Finance Materials

- Review COVM
- Derivations
  - Delta
  - Gamma
  - Theta

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COVM

$$CO(S,t,T_1,T_2,\iota_C,\iota_U) = \iota_C\iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} N_2(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho)$$
$$-\iota_C\iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} N_2(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho) - \iota_C X_C B_{t,T_1,r} N(\iota_C\iota_U d_{21})$$

$$N_2(a,b;\rho) \equiv \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{\exp\left\{-\frac{z_1^2-2\rho z_1 z_2+z_2^2}{2(1-\rho^2)}\right\}}{2\pi\sqrt{1-\rho^2}} dz_1\, dz_2$$

$$d_{21} \equiv \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right) - \frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}} \qquad d_{11} \equiv \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right) + \frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}} = d_{21} + \sigma_{t,T_1}$$

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COVM

$$d_{22} \equiv \frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right) - \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}} \qquad d_{12} \equiv \frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}} = d_{22} + \sigma_{t,T_2}$$

$$\iota_U S_{T_1}^* B_{T_1,T_2,\delta-\hat{q}} N_1(\iota_U d_{1,T_1,T_2}^*) - \iota_U X_U B_{T_1,T_2,r-\hat{q}} N_1(\iota_U d_{2,T_1,T_2}^*) - X_C = 0$$

$$d_{2,T_1,T_2}^* = \frac{\ln\left(\frac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) - \frac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}} \qquad d_{1,T_1,T_2}^* = \frac{\ln\left(\frac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}} = d_{2,T_1,T_2}^* + \sigma_{T_1,T_2}$$

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Compound OVM (COVM)

- Option on option (Cacall, Caput, Pucall, and Puput)
- Two perspectives related to underlying
  - Indirect: $CO_I = CO_{Indirect}\left[O(S,t),t\right]$
  - Direct: $CO_D = CO_{Direct}(S,t)$
  - Same results: $CO_I\left[O(S,t),t\right] = CO_D(S,t)$

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Computing Greeks

- Analytic Greeks
  - Difficult to derive (time consuming)
  - Easy to code
  - Minimal machine error
  - Fast computational time
- Numerical Greeks
  - Fast to code
  - Potentially significant machine error
  - Slow computational time

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Delta

- COVM

$$\Delta_{CO_D} \equiv \frac{\partial CO_D\left(S,t,T_1,T_2\right)}{\partial S} = \iota_C \iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)$$

- GBMOVM

$$\Delta_O \equiv \frac{\partial O\left(S,t,T_2\right)}{\partial S} = \iota_U B_{t,T_2,\delta-\hat{q}} N_1\left(\iota_U d_1\right)$$

R code for analytic delta. Very difficult to derive, hard to code, but runs fast.

```
# CO Delta
CODelta <- function(C, L, U){
  with(C, {
    r <- r/100
    d <- d/100
    q <- q/100
    v <- v/100
    B2d <- exp(-d*TU)
    B12Nq <- exp(q*(TU - TC))
    B12q <- exp(-q*(TU - TC))
    B2r <- exp(-r*TU)
    B1r <- exp(-r*TC)
    B1q <- exp(-q*TC)
    B12d <- exp(-d*(TU - TC))
    d11 <- COd11(C, L, U)
    d21 <- COd21(C, L, U)
    d12 <- COd12(C)
    d22 <- COd22(C)
    mean1 <- rep(0,2)
    lower1 <- rep(-Inf,2)
    corr1 <- diag(2)
    corr1[lower.tri(corr1)] <- iC*sqrt(TC/TU)
    corr1[upper.tri(corr1)] <- iC*sqrt(TC/TU)
    upper1 <- c(iC*iU*d11,iU*d12)
    N2d11d12 <- pmvnorm(lower=lower1, upper=upper1, mean=mean1, corr=corr1)[1]
    Delta <- iC*iU*B12Nq*B2d*N2d11d12
    # Delta <- iC*iU*B1q*B12d*N2d11d12
    return(Delta)
  })
}
```


R code for numerical delta. Easy to derive, easy to code, but runs slow.

```
#
# Numeric Greeks
#
# Compound option delta
CONGDelta <- function(C, L, U){
  with(C, {
    # Increment <- 0.01
    Original <- S
    Change <- Increment*Original
    High <- Original + Change
    C$S <- High
    OHigh <- COValue(C, L, U)
    Low <- Original - Change
    C$S <- Low
    OLow <- COValue(C, L, U)
    C$S <- Original
    Answer <- (OHigh - OLow)/(High - Low)
    return(Answer)
  })
}
```


Gamma

- COVM

$$\Gamma_{CO_D} = \frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2}$$

$$= \frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_2}}\right]$$

- GBMOVM

$$\Gamma_O \equiv \frac{\partial^2 O\left(S,t,T_2\right)}{\partial S^2} = \frac{B_{t,T_2,\delta-\hat{q}} n_1\left(d_1\right)}{S_t \sigma_{t,T_2}}$$

Theta

- COVM

$$\Theta_{CO_D} \equiv \frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)$$

$$= -\frac{\sigma^2(t)}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\left\{N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_1}}\right]\right\}$$

$$+\iota_C \iota_U \hat{q}(t) S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)$$

$$-\iota_C \iota_U r(t) B_{t,T_2,r} X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C r(t) B_{t,T_1,r} X_C N_1\left(\iota_C d_{21}\right)$$

- GBMOVM

$$\Theta_O \equiv \frac{\partial O\left(S,t,T_2\right)}{\partial t}$$

$$= \iota_U\left[\delta(t)-\hat{q}(t)\right] S_t B_{\delta-\hat{q}} N\left(\iota_U d_1\right) - \iota_U\left[r(t)-\hat{q}(t)\right] X B_{r-\hat{q}} N\left(\iota_U d_2\right)$$

$$-\frac{\sigma^2(t) S_t B_{\delta-\hat{q}}}{2\sigma} n\left(d_1\right)$$

Important Lemmas for Proofs

$$\frac{dN_2\left[a(x),b(x);\rho\right]}{dx} = \frac{dN_2\left[a(x),b(x);\rho\right]}{db(x)}\frac{db(x)}{dx} + \frac{dN_2\left[a(x),b(x);\rho\right]}{da(x)}\frac{da(x)}{dx}$$

$$\frac{\partial}{\partial d_{21}} CO_D\left(S,t,T_1,T_2\right) = 0$$

$$\frac{\partial}{\partial d_{22}} CO_D\left(S,t,T_1,T_2\right) = 0$$

Summary

- Reviewed COVM model
- Derived delta, gamma, and theta
- Explored sensitivities with respect to
  - Underlying
  - Volatility
  - Option Yield
  - Calendar time

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