Module 8.1: SRM GBM-Based Binomial Models

Learning objectives

- Computing call and put option Greeks using the GBM binomial option valuation approach with the standard, enhanced, and numerical methods
- Contrast European-style and American-style call and put option Greeks using the GBM binomial option valuation approach

Executive summary

Based on the material presented in Module 5.2, we illustrate computing option Greeks within the GBM binomial option valuation model for both European-style and American-style options.

Central finance concepts

The main idea is once we have a robust valuation model, we are now able to explore various static risk measures. After reviewing the valuation models introduced in Module 5.2, we explore option Greeks that are simply SRMs.

GBM-based European-style binomial option valuation models

Recall the GBM-based binomial option framework is designed to converge to a lognormal distribution in the limit to be consistent with the GBMOVM. This binomial framework has several objectives:

1. Multiplicative
2. Recombining
3. Incorporate dividends
4. Address early exercise with American-style options

Multiplicative and recombining are incorporated using *u* and *d* parameters at each node.

There are several GBM-based multiperiod valuation models including when there are no dividends, when a dividend yield is assumed, and when discrete dividends are assumed. Further, there are several alternative ways to frame these models such as based on digital valuation models.

GBM-based American-style binomial option valuation models

For American-style options, he early exercise potential must be incorporated. As discussed below, the approach typically taken is known as backward induction. At each node, we must compare the following values, the model option value, the early exercise value, and the lower boundary condition. The existence of various forms of dividends simply changes the required formulas.

Binomial option valuation model Greeks

In the quantitative materials below, we explore delta, gamma, theta, vega, and rho, also known as the Greeks. Delta measures an option value's sensitivity to changes in the underlying instrument's price. Gamma measures the delta's sensitivity to changes in the underlying instrument's price. Vega (also known as kappa, lambda, and sigma) measures an option price's sensitivity to changes in the underlying asset's volatility. Theta measures an option price's sensitivity to changes in the time to maturity. Rho measures an option price's sensitivity to changes in the interest rate.

Quantitative finance materials

After a detailed review of various valuation models, we take a deep dive into SRMs related to GBM-based binomial option valuation models.

GBM-based European-style binomial option valuation models

Recall the GBM-based binomial framework has several objectives including multiplicative-based, recombining, able to incorporate dividends, and able to address early exercise with American-style options.

We now first review several versions of the GBM-based BOVM depending on the nature of dividend payments.

No dividend yield multiperiod valuation equation

The current value of an option is equal to the present value of the expected terminal payout as we assume European-style options. The multiperiod binomial valuation equation can be expressed as

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U S_0 Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U}, \tag{9.1.1}$$

where the binomial summations are

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{9.1.2}$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{9.1.3}$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{9.1.4}$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{9.1.5}$$

where the indicator function denotes

$$\iota_U = \begin{cases} +1 \text{ if call option} \\ -1 \text{ if put option} \end{cases}, \tag{9.1.6}$$

$$\Delta t = \frac{T}{n}, \tag{9.1.7}$$

$$\pi = \frac{e^{r\Delta t} - d}{u - d}, \tag{9.1.8}$$

$$A \equiv \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}, \tag{9.1.9}$$

$$Den \equiv \pi e^A + \left(1-\pi\right), \tag{9.1.10}$$

$$\inf\left\{\text{int } j : u^j d^{n-j} S_0 > X\right\} > a = \frac{-\ln\left(\frac{S}{X}\right) - rT + n\ln\left(Den\right)}{A} \tag{9.1.11}$$

$$\pi_1 = \frac{\pi e^A}{Den}, \text{ and} \tag{9.1.12}$$

$$\pi_2 = \pi = \frac{e^{r\Delta t} - d}{u - d}. \tag{9.1.13}$$

Alternatively, the binomial option valuation model can be expressed as

$$O_0 = PV_r\left[\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0, \iota_U u^j d^{n-j} S_0 - \iota_U X\right)\right].$$

where u and d are defined as

$$u = \frac{e^{r\Delta t + A}}{Den} \text{ and} \tag{9.1.14}$$

$$d = \frac{e^{r\Delta t}}{Den}. \tag{9.1.15}$$

Dividend yield multiperiod period valuation equation

As before, the current value of an option is equal to the present value of the expected terminal payout as we assume European-style options where the underlying instrument is adjusted for a continuously compounded cash flow yield.

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U S e^{-\delta T} Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U}, \tag{9.1.16}$$

where the binomial summations are

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{9.1.17}$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{9.1.18}$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{9.1.19}$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{9.1.20}$$

where the terms are as defined before except

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u-d}, \tag{9.1.21}$$

$$\pi_2 = \pi = \frac{e^{(r-\delta)\Delta t} - d}{u-d}. \tag{9.1.22}$$

Generically, the binomial option valuation model can be expressed as

$$O_0 = PV_r\left[\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0,\iota_U u^j d^{n-j} S_0 - \iota_U X\right)\right], \tag{9.1.23}$$

where u and d are defined as

$$u = \frac{e^{(r-\delta)\Delta t + A}}{Den} \text{ and} \tag{9.1.24}$$

$$d = \frac{e^{(r-\delta)\Delta t}}{Den}. \tag{9.1.25}$$

Alternative expression of multi-period binomial option valuation model

The multi-period binomial option valuation model is simply the present value of the expected terminal payout. For plain vanilla call and put options, we have

$$c = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0,u^j d^{n-j} S_0 - X\right) \text{ and} \tag{9.1.26}$$

$$p = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0, X - u^j d^{n-j} S_0\right). \tag{9.1.27}$$

For cash-or-nothing digital call and put options, we have

$$c_{CoN} = e^{-rT} DP \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} I_{u^j d^{n-j} S_0 > X} \text{ and} \tag{9.1.28}$$

$$p_{CoN} = e^{-rT} DP \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} I_{u^j d^{n-j} S_0 < X}. \tag{9.1.29}$$

where *DP* denotes the digital cash payout if the option expires in-the-money.

For asset-or-nothing digital call and put options, we have

$$c_{AoN} = e^{-rT} \sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!} \right) \pi^j \left(1-\pi\right)^{n-j} u^j d^{n-j} S_0 I_{u^j d^{n-j} S_0 > X} = c + c_{CoN}\left(DP = X\right) \text{ and} \quad (9.1.30)$$

$$p_{AoN} = e^{-rT} \sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!} \right) \pi^j \left(1-\pi\right)^{n-j} u^j d^{n-j} S_0 I_{u^j d^{n-j} S_0 < X} = p + p_{CoN}\left(DP = X\right). \quad (9.1.31)$$

GBM-based American-style binomial option valuation models

We briefly review the setup with and without dividends.

No dividends model

The current value of an option is no longer equal to the present value of the expected terminal payout as we assume American-style options. The early exercise potential must be incorporated. The approach typically taken is known as backward induction. "Backward induction is the process of reasoning backwards in time, from the end of a problem or situation, to determine a sequence of optimal actions. It proceeds by first considering the last time a decision might be made and choosing what to do in any situation at that time. Using this information, one can then determine what to do at the second-to-last time of decision. This process continues backwards until one has determined the best action for every possible situation (i.e., for every possible information set) at every point in time."[1] Thus, at the maturity of the option, we know

$$O_{n,j} = \max\left[0, \iota_U\left(S_{n,j} - X\right)\right] = \max\left[0, \iota_U\left(u^j d^{n-j} S_0 - X\right)\right] : j = 0, \ldots, n, \quad (9.1.32)$$

where j denotes the number of up moves for the underlying over the option life. The indicator function denotes

$$\iota_U = \begin{cases} +1 \text{ if call option} \\ -1 \text{ if put option} \end{cases} \text{ and} \quad (9.1.33)$$

$$n = \frac{T}{\Delta t} \text{ (total number of time steps over option life).} \quad (9.1.34)$$

Based on our single period results, we know that at time i for j up moves, the binomial model value (denoted with B superscript) can be expressed as

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + \left(1-\pi\right) O_{i+1,j}\right], \quad (9.1.35)$$

where $PV_{r,i,\Delta t}\left(\ \right)$ denotes the present value at time i for the next Δt period based on the continuously compounded rate r and as defined before $\pi = \frac{e^{r\Delta t} - d}{u - d}$, $A \equiv \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}$, $Den \equiv \pi e^A + \left(1-\pi\right)$, $u = \frac{e^{r\Delta t + A}}{Den}$ and $d = \frac{e^{r\Delta t}}{Den}$. With constant interest rates, we have $PV_{r,i,\Delta t}\left(1\right) = e^{-r\Delta t}$. The binomial model value, however, may be lower than the early exercise value (denoted with superscript X) that can be expressed as

$$O_{i,j}^X = \max\left[0, \iota_U\left(S_{i,j} - X\right)\right]. \quad (9.1.36)$$

Recall the lower boundary condition (denoted with superscript L) is

$$O_{i,j}^L = \max\left\{0, \iota_U\left[S_{i,j} - PV_{r,i,n-i}\left(X\right)\right]\right\}. \quad (9.1.37)$$

Thus, the fair value of the American-style option at time *i* with *j* up moves is

$$O_{i,j} = \max\left(O_{i,j}^B, O_{i,j}^X, O_{i,j}^L\right). \quad (9.1.38)$$

Note assuming positive interest rates and no dividends $O_{i,j}^L \ge O_{i,j}^X$ for call options and $O_{i,j}^L \le O_{i,j}^X$ for put options. The initial option value is obtained through backward induction along the binomial lattice for the underlying instrument. With European-style options, the fair value at time *i* with *j* up moves is

[1]Wikipedia, "Backward Induction," observed on February 20, 2017.

$$O_{i,j} = \max\left(O_{i,j}^{B}, O_{i,j}^{L}\right). \tag{9.1.39}$$

Dividend yield model

The process to compute the option value is the same at the no dividend case except

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u - d}. \tag{9.1.40}$$

$$u = \frac{e^{(r-\delta)\Delta t + A}}{Den} \text{ and} \tag{9.1.41}$$

$$d = \frac{e^{(r-\delta)\Delta t}}{Den}. \tag{9.1.42}$$

Based on our single period results, we know that at time i for j up moves, the binomial model value (denoted with B superscript) can be expressed as

$$O_{i,j}^{B} = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + (1-\pi) O_{i+1,j}\right], \tag{9.1.43}$$

Recall the lower boundary condition (denoted with superscript L) is

$$O_{i,j}^{L} = \max\left\{0, \iota_U\left[PV_{\delta,i,n-i}\left(S_{i,j}\right) - PV_{r,i,n-i}\left(X\right)\right]\right\}. \tag{9.1.44}$$

Thus, the fair value of the option at time i with j up moves is

$$O_{i,j} = \max\left(O_{i,j}^{B}, O_{i,j}^{X}, O_{i,j}^{L}\right). \tag{9.1.45}$$

The initial option value is obtained through backward induction along the binomial lattice for the underlying instrument. We now explore the binomial option valuation model Greeks

Binomial option valuation model Greeks

We now cover what are called the Greeks. Specifically, we focus here on understanding delta, gamma, theta, vega, and rho. Recall delta measures an option value's sensitivity to changes in the underlying instrument's price. Gamma measures the delta's sensitivity to changes in the underlying instrument's price. Vega (also known as kappa, lambda, and sigma) measures an option price's sensitivity to changes in the underlying asset's volatility. Theta measures an option price's sensitivity to changes in the time to maturity. Rho measures an option price's sensitivity to changes in the interest rate.

Delta

Mathematically, delta is defined as

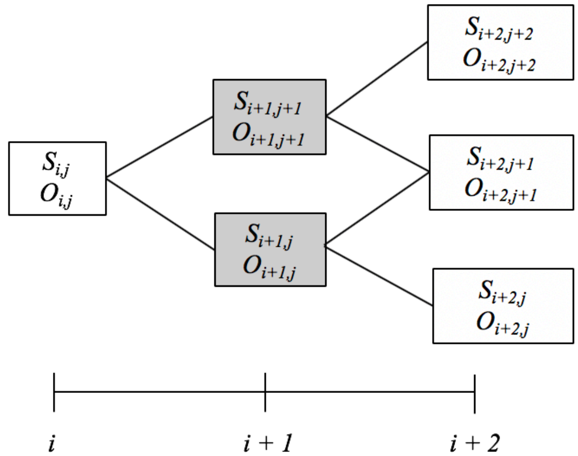
$$\Delta_O \equiv \frac{\partial O}{\partial S}. \tag{9.1.46}$$

Within the binomial lattice, delta can be estimated as

$$\Delta_{O,i,j} = \frac{O_{i+1,j+1} - O_{i+1,j}}{S_{i+1,j+1} - S_{i+1,j}}. \text{ (\textbf{Standard Binomial Method})} \tag{9.1.47}$$

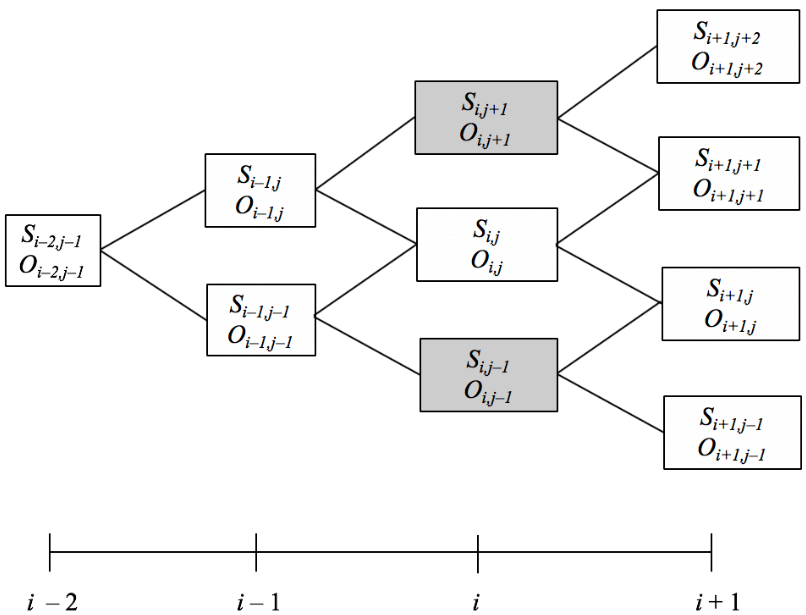
Figure 8.1.1 provides a binomial lattice illustrating the appropriate lattice inputs for the delta calculation. Clearly, this delta calculation is based on observations at time $i + 1$.

Figure 8.1.1. Illustration of standard delta within GBM-based binomial model



One alternative is to add two time steps to represent points in time ($i-2$) and ($i-1$). With these additional time steps, we can compute delta at time i. based on the two additional nodes at time i. This enhanced method resolves the timing problem and is illustrated in Figure 8.1.2.

Figure 8.1.2. Illustration of enhanced delta within GBM-based binomial model



Within the binomial lattice, this enhanced delta can be estimated as

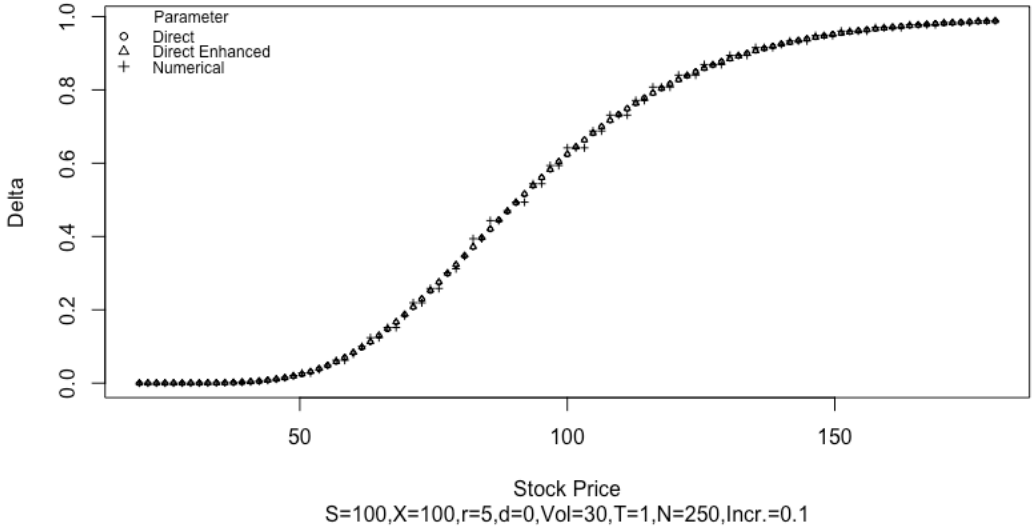
$$\Delta_{O,i,j} = \frac{O_{i,j+1} - O_{i,j-1}}{S_{i,j+1} - S_{i,j-1}}. \text{ (**Enhanced Binomial Method**)} \tag{9.1.48}$$

The final method is to simply estimate the delta using the centered differencing technique explained in Module 7.1.

$$\Delta_{O,i,j} = \frac{O(S+h) - O(S-h)}{2h}. \text{ (**Numerical Method**)} \tag{9.1.49}$$

In most cases, the method of choice renders numerically similar results. Figure 8.1.3 illustrates all three methods of estimating delta. The two binomial methods are indistinguishable, and the numerical method is extremely close, but it oscillates across stock prices as seen in Panel B.

Figure 8.1.3. Three methods to estimate GBM-based European-style binomial call delta
Panel A. Wide range of stock prices



Panel B. Narrow range of stock prices

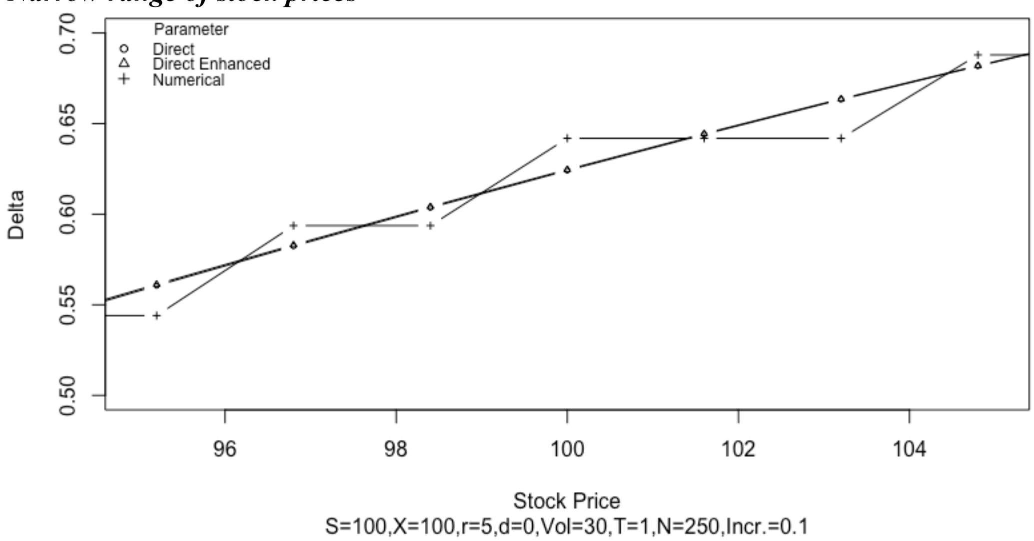
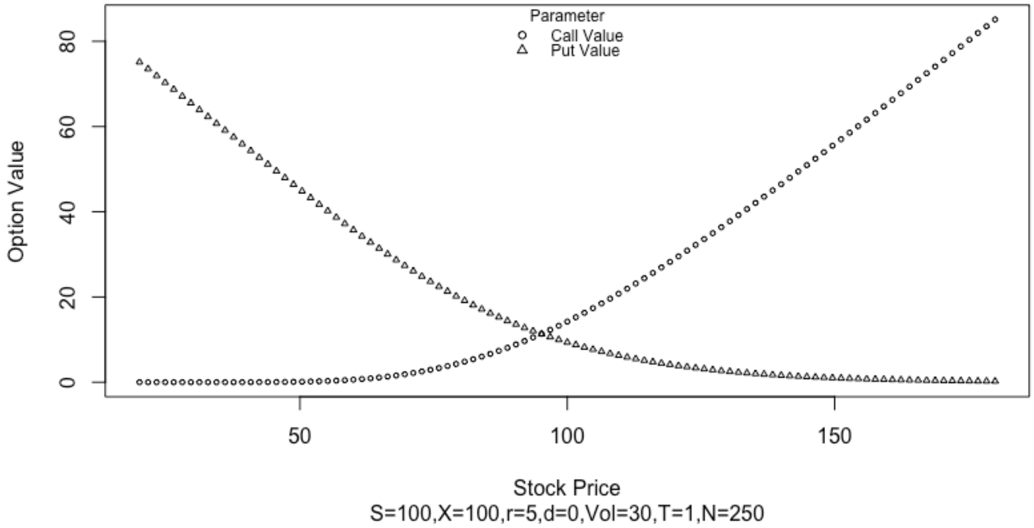


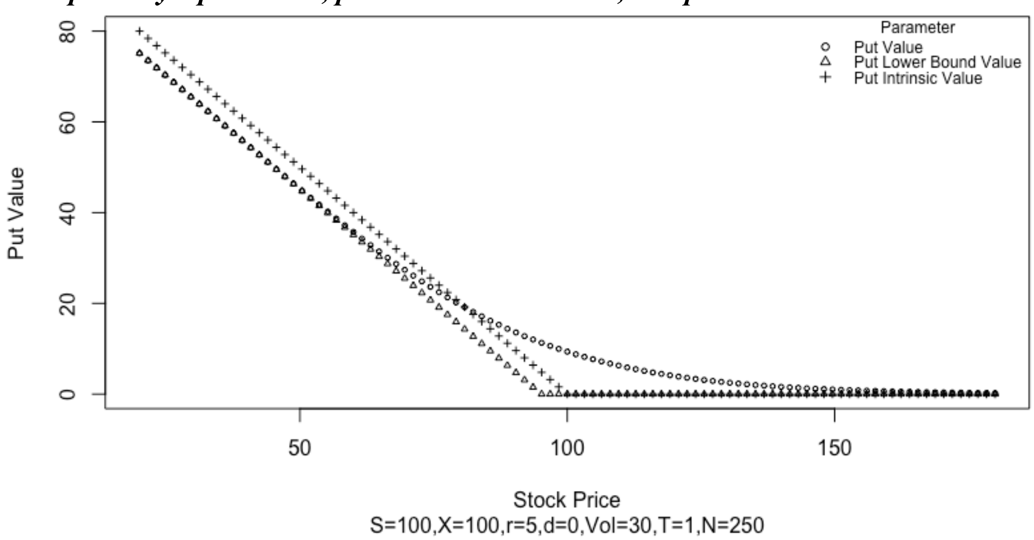
Figure 8.1.4 illustrates the stock price on the horizontal axis and the option prices on the vertical axis. The positive sloped line is the call value, and the negative sloped line is the put value.

Figure 8.1.4. Call and put values based on GBM-based European-style binomial model

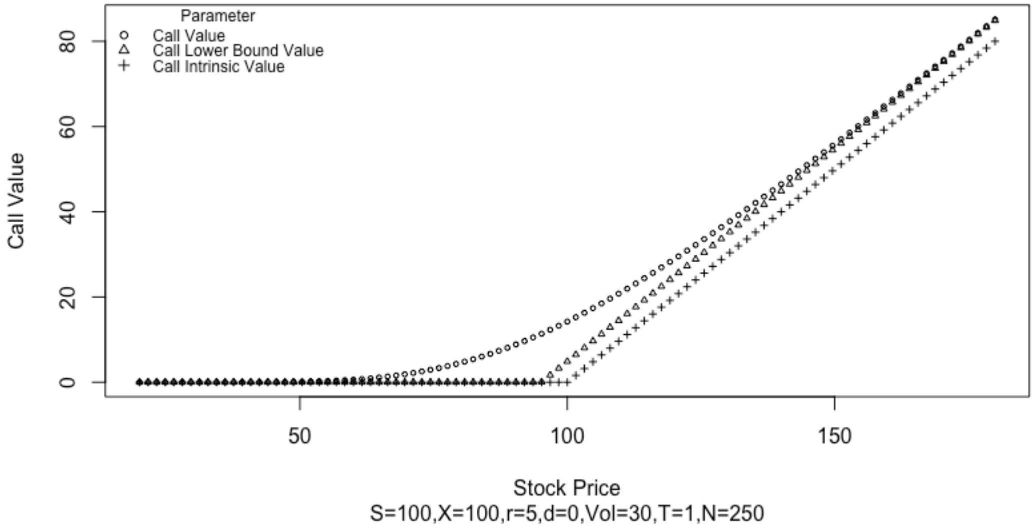


The value of the European-style put option at expiration is either $0 if it is out-of-the-money ($S_T > X$) or the intrinsic value (the dollar amount it is in-the-money) ($X - S_T$) when the stock ends up in-the-money ($S_T < X$). Prior to expiration, the option has time value as well as intrinsic value. The time value of an option depends on the relationship between the current stock price and the strike price. For puts it is theoretically possible for the put option's value to fall below its intrinsic value as illustrated in Figure 8.1.5 (Panel A). Recall the put European-style lower bound with no dividends is max[0,$PV(X) - S_0$)]. Panel B illustrates the call option's value, and it will never fall below the intrinsic value. Recall the call European-style lower bound with no dividends is max[0,$S_0 - PV(X)$].

Figure 8.1.5. Call and put values based on GBM-based binomial model
Panel A. European-style put values, put lower bound values, and put intrinsic values



Panel B. European-style call values, call lower bound values, and call intrinsic values



We now address the geometrical interpretation of delta. Delta is the change in the value of the option for a small change in the value of the stock or the first derivative of the option with respect to the stock. The relationship between the change in the option price, delta, and the change in the asset price can be expressed as approximately

$$\text{Change in Option Value} = \Delta * \text{Change in Stock Price.} \tag{9.1.50}$$

Example: Suppose the call option delta is 0.6 and the stock price increased by \$0.5, approximately how much did the call price increase?

$$\text{Change in the Call Value} = 0.6 * \$0.5 = \$0.3.$$

Rearranging the expression above, we have

$$\Delta_c = \text{Change in Option Value / Change in Stock Price.} \tag{9.1.51}$$

Example: Suppose an option price rises by \$0.3 when the stock price increases by \$0.5. What is the estimated delta?

$$\Delta_c = \$0.3 / \$0.5 = 0.6 \text{ or } 60\%. \tag{9.1.52}$$

A delta-neutral portfolio is a portfolio that has a portfolio delta of zero. A zero delta implies that the value of the portfolio does not change for infinitesimal changes in the stock price. Hence, the value of the portfolio is not affected by small changes in the stock price. Therefore, to hedge against small changes in the stock price, trades should be conducted such that the portfolio delta is zero.

Example: Suppose you had a portfolio with a delta of 24 and a particular call option has a delta of 0.6. If you desired to completely hedge this portfolio with this call option, describe the appropriate trade. You would sell 40 call options each having a delta of –0.6 (because you sold). The delta of the short 40 calls is -24 (40*(–0.6)) so the new portfolio delta is zero.

Recall put-call parity can be expressed as

$$c = Se^{-\delta T} - Xe^{-rT} + p. \tag{9.1.53}$$

Recall that delta is the first derivative of the option value with respect to the stock value. Hence, we can take the first derivative of both sides of this equation. Thus, the delta of the call is related to the delta of the put as (the discounted strike price is unaffected by changes in the stock price)

$$\Delta_c = 1 + \Delta_p, \tag{9.1.54}$$

or

$$\Delta_p = \Delta_c - 1. \tag{9.1.55}$$

For small changes in the stock price for a deep out-of-the-money call, the call price changes very little, hence the delta is close to zero. The same is true for deep out-of-the-money puts. For small changes in the stock for deep in-the-money calls, the call price changes almost dollar for dollar with the stock price, hence the delta is close to one. For deep in-the-money puts, the delta is close to –1 (as the stock goes up $1, the put value falls by almost $1). Hence, the delta of puts and calls are constrained in the following way

$$0 \le \Delta_c \le 1 \text{ and} \tag{9.1.56}$$

$$-1 \le \Delta_p \le 0. \tag{9.1.57}$$

Figure 8.1.6 shows the enhanced method and numerical method for estimating deltas with the binomial model assuming no dividends. Notice that with 250 time steps, the numerical method produces some lack of smoothness whereas the standard method is relatively smooth. For deep in-the-money puts, the boundary condition results in a delta of –1.0. The enhanced method is virtually indistinguishable from the standard method and is not reported here.

Figure 8.1.6. Call and put deltas based on GBM-based binomial model without dividends

Enhanced Method ***Numerical Method***

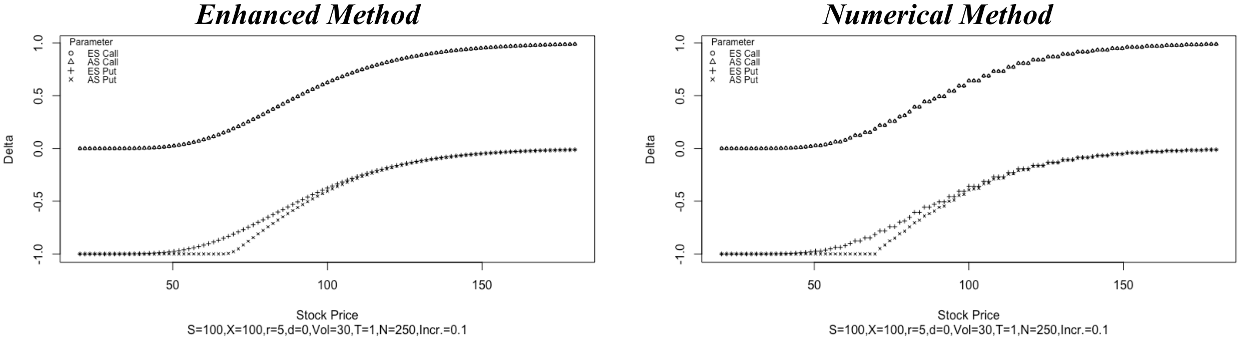
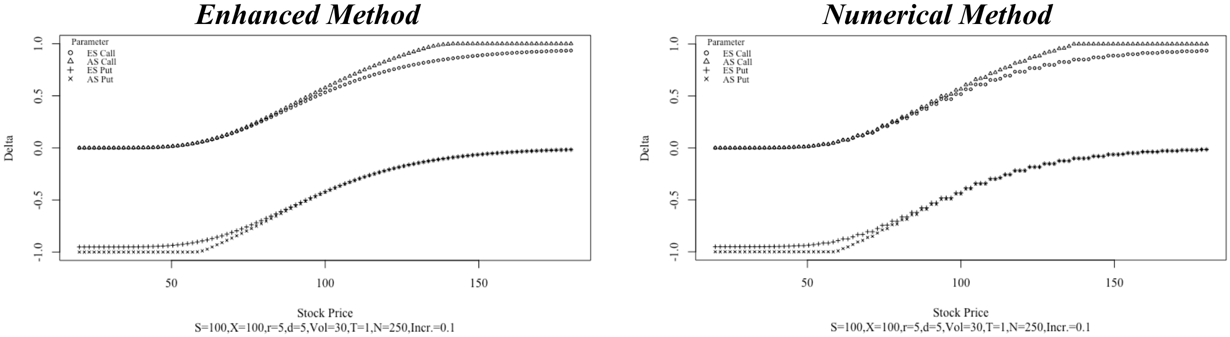


Figure 8.1.7 shows the standard method and numerical method for estimating deltas with the binomial model assuming a 5% dividend yield. Notice again that with 250 time steps, the numerical method produces some lack of smoothness whereas the standard method is relatively smooth. For deep in-the-money puts and calls, the boundary condition are obtained. Again, the enhanced method is virtually indistinguishable from the standard method and is not reported here.

Figure 8.1.7. Call and put deltas based on GBM-based binomial model with dividends

Enhanced Method ***Numerical Method***



Gamma

Mathematically, gamma is defined as

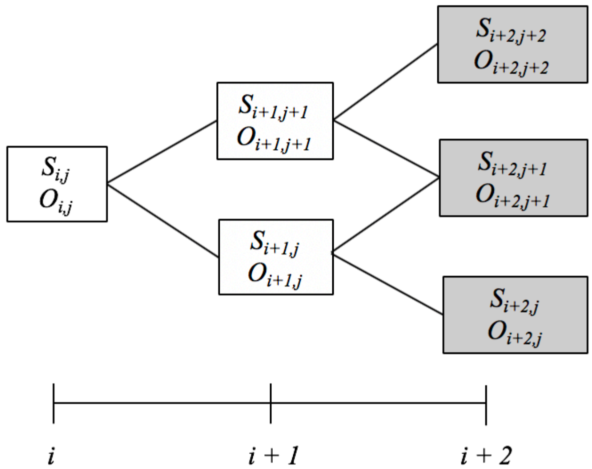
$$\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2}. \tag{9.1.58}$$

Within the binomial lattice, gamma can be estimated as

$$\Gamma_{O,i,j} = \frac{\dfrac{O_{i+2,j+2} - O_{i+2,j+1}}{S_{i+2,j+2} - S_{i+2,j+1}} - \dfrac{O_{i+2,j+1} - O_{i+2,j}}{S_{i+2,j+1} - S_{i+2,j}}}{\dfrac{S_{i+2,j+2} - S_{i+2,j}}{2}}. \text{ (\textbf{Standard Binomial Method})} \tag{9.1.59}$$

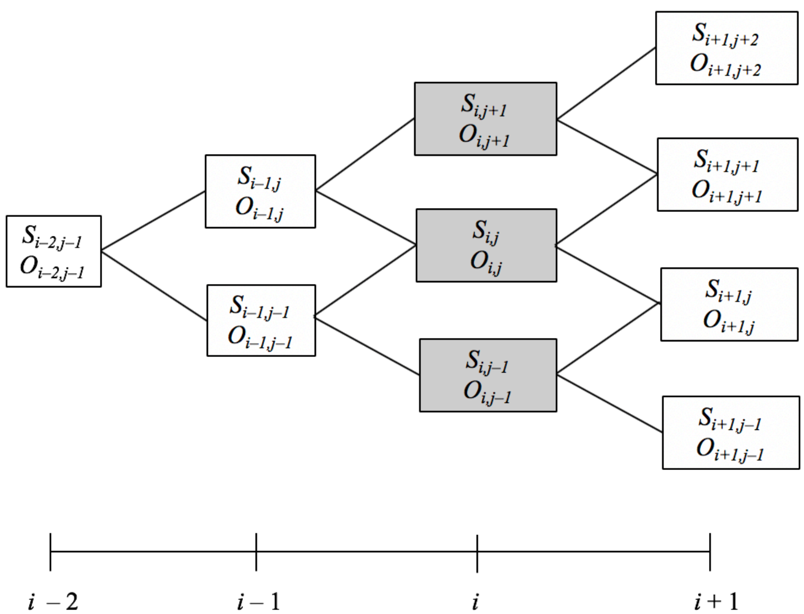
Figure 8.1.8 provides a binomial lattice illustrating the appropriate lattice inputs for the gamma calculation. Clearly, this delta calculation is based on observations at time i + 1.

Figure 8.1.8. Illustration of standard gamma within GBM-based binomial model



Again, one alternative is to add two time steps to represent points in time ($i - 2$) and ($i - 1$). With these additional time steps, we can compute delta at time *i*. based on the two additional nodes at time *i*. This enhanced method resolves the timing problem and is illustrated in Figure 8.1.9.

Figure 8.1.9. Illustration of enhanced gamma within GBM-based binomial model



Within the binomial lattice, this enhanced gamma can be estimated as

$$\Gamma_{O,i,j} = \frac{\dfrac{O_{i,j+1} - O_{i,j}}{S_{i,j+1} - S_{i,j}} - \dfrac{O_{i,j} - O_{i,j-1}}{S_{i,j} - S_{i,j-1}}}{\dfrac{S_{i,j+1} - S_{i,j-1}}{2}}. \text{ (\textbf{Enhanced Binomial Method})} \qquad (9.1.60)$$

The final method is to simply estimate the gamma using the centered differencing technique explained in Module 7.1.

$$\Gamma_{O,i,j} = \frac{\left[O(S+h)-O(S)\right]-\left[O(S)-O(S-h)\right]}{h^2}. \text{ (**Numerical Method**)} \tag{9.1.61}$$

In most cases, the method of choice renders numerically similar results. Figure 8.1.10 illustrates all three methods of estimating gamma. The two binomial methods are indistinguishable, and the numerical method is extremely close, but it oscillates across stock prices.

Figure 8.1.10. Three methods to estimate GBM-based European-style gamma without dividends

Call Options ***Put Options***

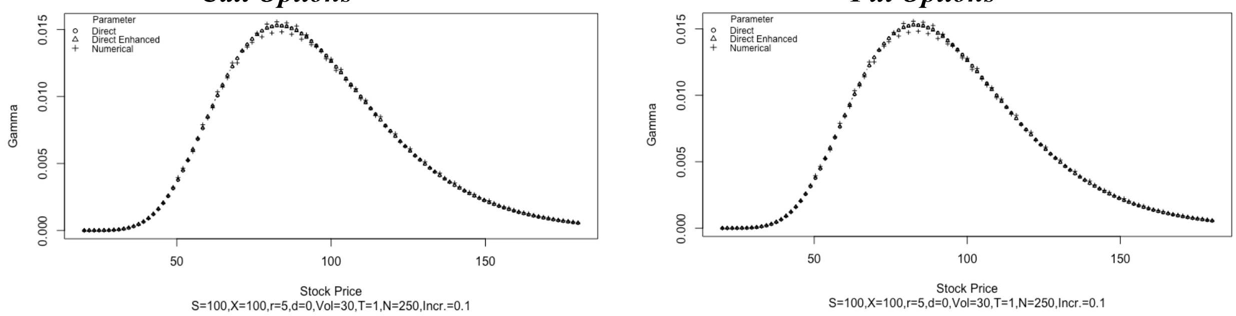


Figure 8.1.11 illustrate the three methods to estimating gamma for GBM-based European-style options in the presence of dividends. We assume here a 5% dividend yield.

Figure 8.1.11. Three methods to estimate GBM-based European-style gamma with dividends

Call Options ***Put Options***

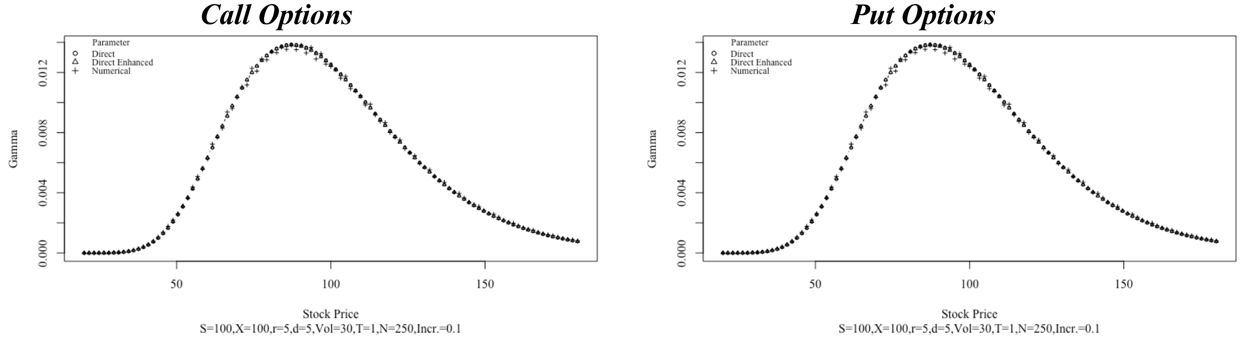


Figure 8.1.12 illustrates the difference between the European-style and American-style options without dividends. In this case, only the put option encounters the boundary condition.

Figure 8.1.12. Call and put gammas based on GBM-based binomial model with no dividends

Enhanced Method ***Numerical Method***

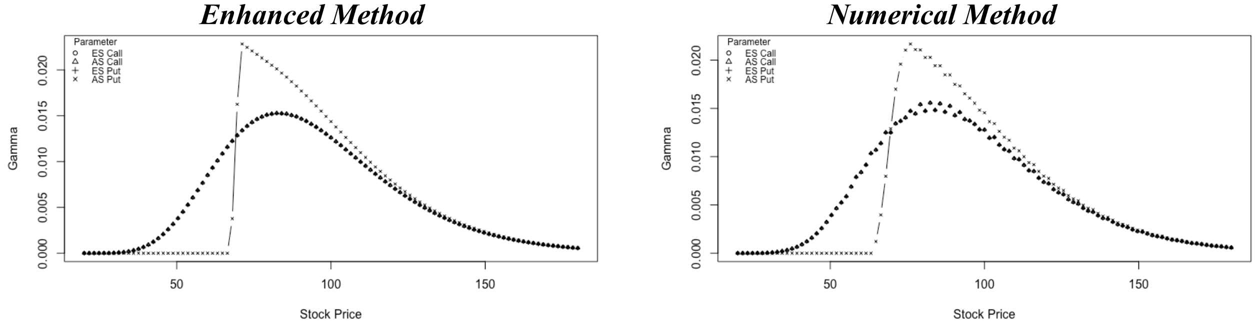
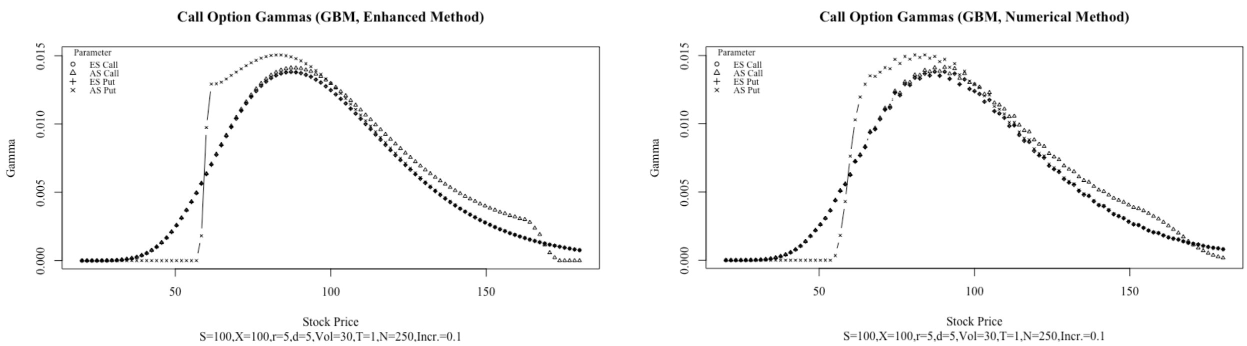


Figure 8.1.13 illustrates the difference between the European-style and American-style options with a 5% dividend yield. Both the American-style call and put encounter the boundary condition and the discontinuity points where the option valuation model encounters the lower bound.

Figure 8.1.13. Call and put gammas based on GBM-based binomial model with dividends
Enhanced Method ***Numerical Method***



Theta
Mathematically, theta is defined as

$$\theta_O \equiv \frac{\partial O}{\partial t}. \tag{9.1.62}$$

Theta is particularly challenging because we seek the change in the option value for only a change in the passage of calendar time. Recall the coherent binomial lattice adopted does not place a requirement that $ud = 1$. Thus, over the lattice even by choosing the middle node, we have both the underlying and time changing.

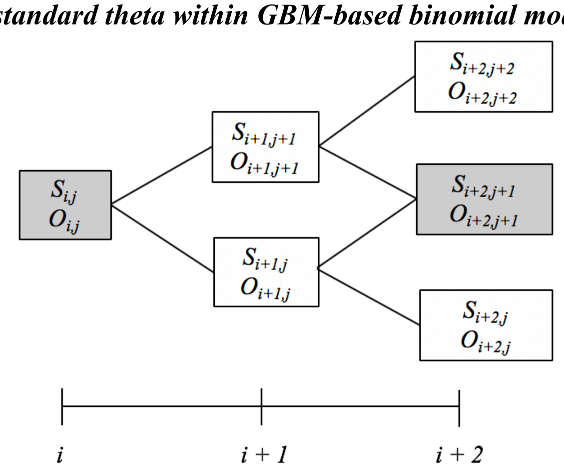
Within the binomial lattice, theta can be estimated as

$$\theta_{O,i,j} = \frac{O_{i+2,j+1} - O_{i,j}}{2\Delta t}, \text{ (\textbf{Standard Binomial Method})} \tag{9.1.63}$$

only within a lattice that forces $ud = 1$.

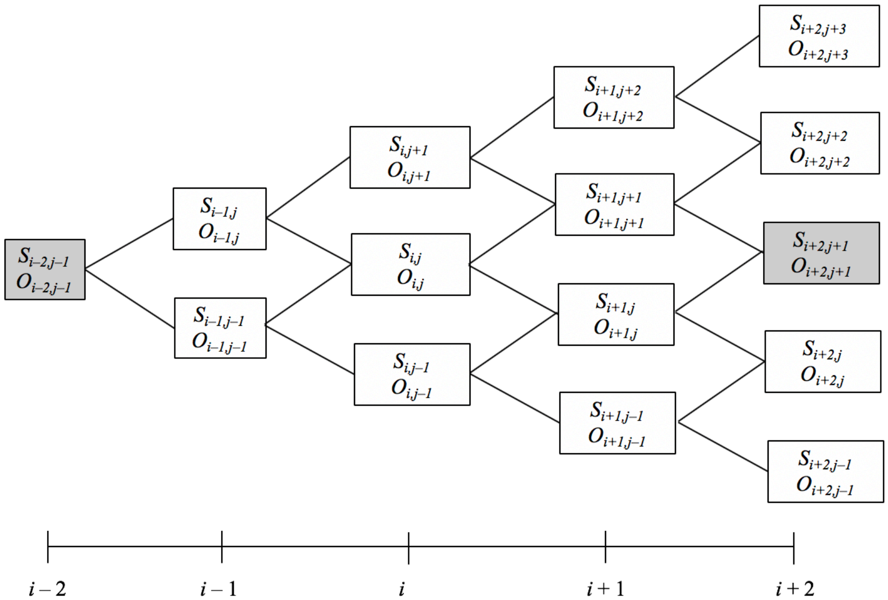
Figure 8.1.14 provides a binomial lattice illustrating the appropriate lattice inputs for the theta calculation. Clearly, this theta calculation is based on observations at time $i + 2$.

Figure 8.1.14. Illustration of standard theta within GBM-based binomial model



Again, one alternative is to add two time steps to represent points in time $(i - 2)$ and $(i - 1)$. With these additional time steps, we can compute theta at time *i*. based on the two additional nodes at time *i*. This enhanced method resolves the timing problem and is illustrated in Figure 8.1.15. Again, this results in a valid theta estimate only when $ud = 1$.

Figure 8.1.15. Illustration of enhanced theta within GBM-based binomial model



Within the binomial lattice, this enhanced theta can be estimated as

$$\theta_{O,i,j} = \frac{O_{i+2,j+1} - O_{i-2,j-1}}{4\Delta t}. \text{ (\textbf{Enhanced Binomial Method})} \qquad (9.1.64)$$

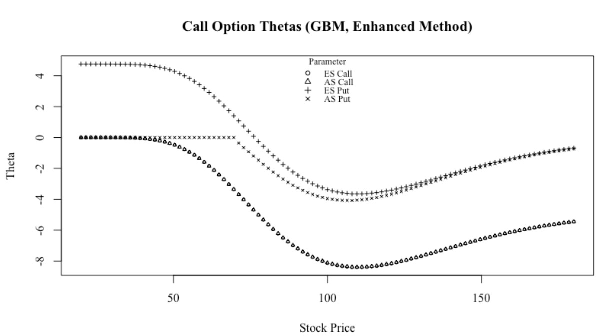
Again, the final method is to simply estimate the gamma using the centered differencing technique explained in Module 7.1.

$$\theta_{O,i,j} = \frac{O(t+h) - O(t-h)}{2h}. \text{ (\textbf{Numerical Method})} \qquad (9.1.65)$$

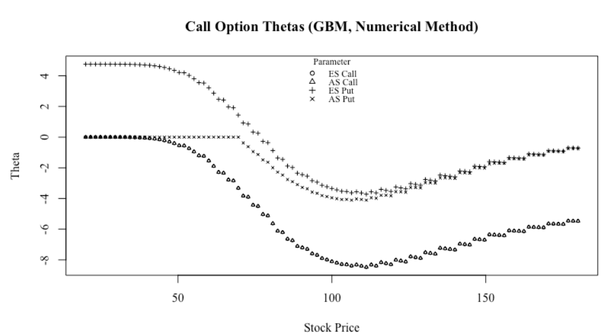
In most cases, the method of choice renders numerically similar results. Figure 8.1.16 illustrates all three methods of estimating theta without and with dividends. As seen in Panel A, without dividends the two binomial methods are indistinguishable and the numerical method is extremely close, but it oscillates across stock prices. Panel B illustrates the influence of dividends.

Figure 8.1.16. Illustration of theta within GBM-based binomial model
Panel A. Without dividends

Enhanced Method

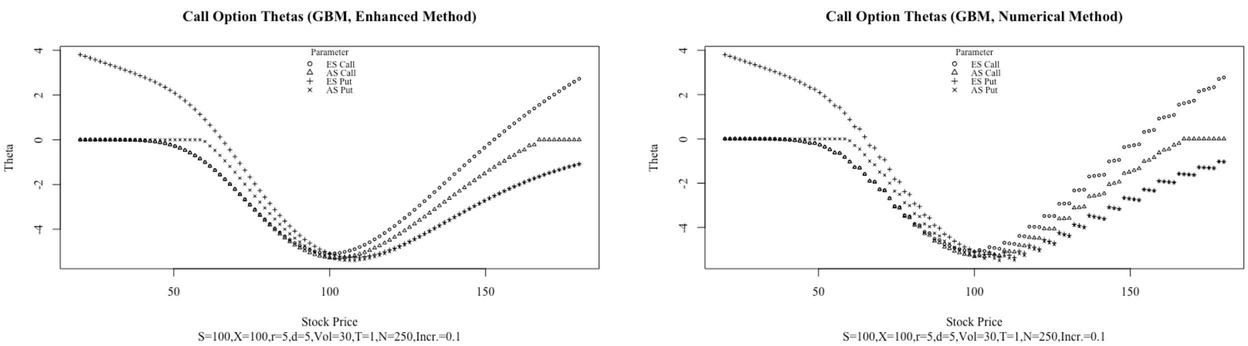


Numerical Method



Panel B. With dividends

Enhanced Method ***Numerical Method***



Although the pattern is roughly similar, the existence of dividends changes the theta values for both European and American style options.

Vega

Mathematically, vega is defined as

$$v_O \equiv \frac{\partial O}{\partial \sigma}. \tag{9.1.66}$$

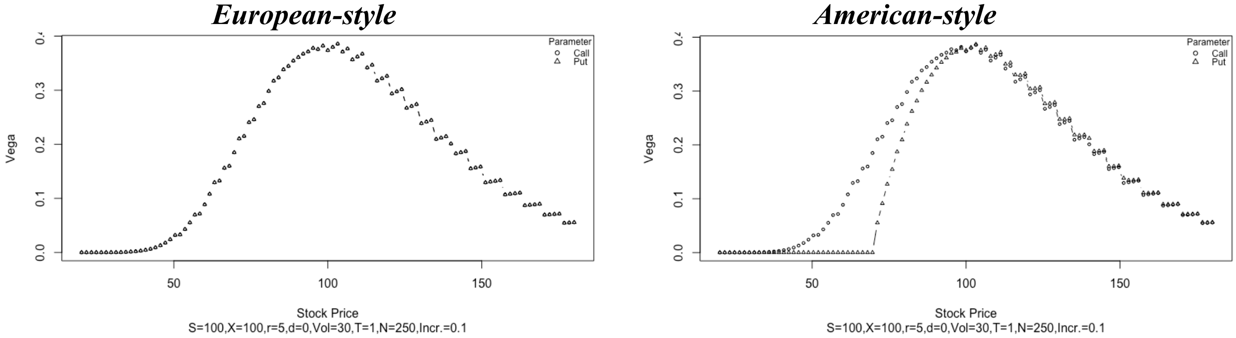
Within the binomial lattice, vega can be estimated based on the numerical method as

$$v_{O,i,j} = \frac{O_{\sigma+h,i,j} - O_{\sigma-h,i,j}}{2h}. \tag{9.1.67}$$

Vega is the first derivative of the option value with respect to volatility. Neither the stock nor the risk-free interest rate is assumed to be influenced by changes in the stock's volatility. Thus, volatility impacts both calls and puts the same based on put-call parity. Figure 8.1.17 illustrates this result without dividends.

Figure 8.1.17. Illustration of vega within GBM-based binomial model without dividends

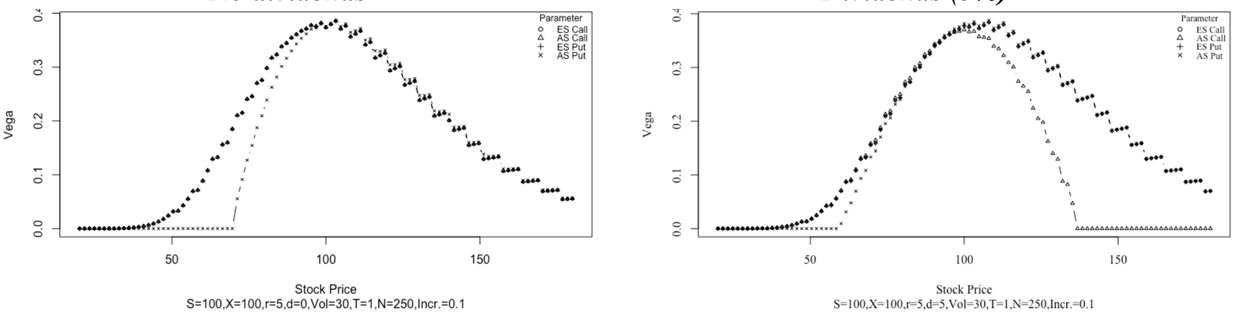
European-style ***American-style***



For call options that are deep out-of-the-money, the call price changes very little with a small change in volatility (it does not really change the probability of the stock reaching the strike price), hence the vega is close to zero. The same is true for deep out-of-the-money puts. For small changes in the volatility for deep in-the-money calls, the call price does not change much because it is already in-the-money, hence again the vega is close to zero. The relationship between the stock price and vega is illustrated in Figure 8.1.18.

Figure 8.1.18. Vega with respect to stock price within GBM-based binomial model with and without dividends

No dividends ***Dividends (5%)***



Rho

Mathematically, rho is defined as

$$\rho_O \equiv \frac{\partial O}{\partial r}. \tag{9.1.68}$$

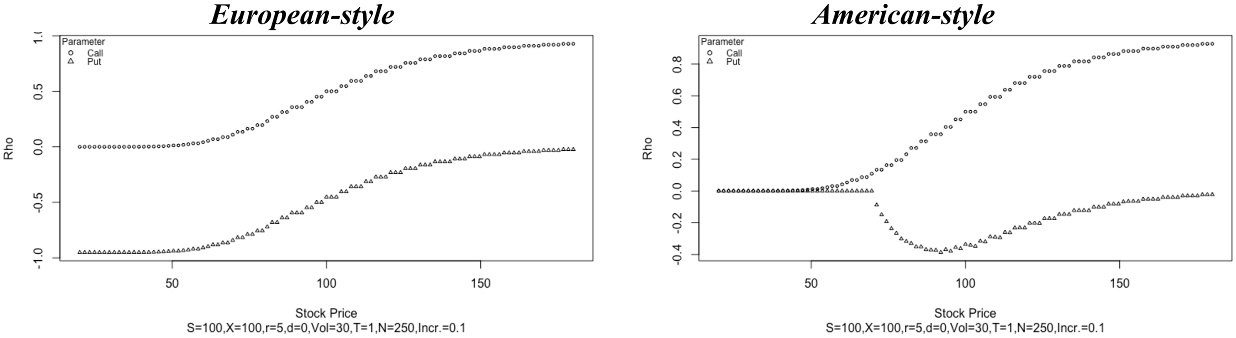
Within the binomial lattice, rho can be estimated based on the numerical method as

$$\rho_{O,i,j} = \frac{O_{r+h,i,j} - O_{r-h,i,j}}{2h}. \tag{9.1.69}$$

Rho is the first derivative of the option value with respect to the interest rate. Neither the stock nor volatility is assumed to be influenced by changes in the interest rate. Thus, the interest rate impacts calls different from puts based on put-call parity. Recall there is the present value of the exercise price in put-call parity giving an additional influence of interest rates. This result is illustrated in Figure 8.1.19.

Figure 8.1.19. Illustration of rho within GBM-based binomial model without dividends

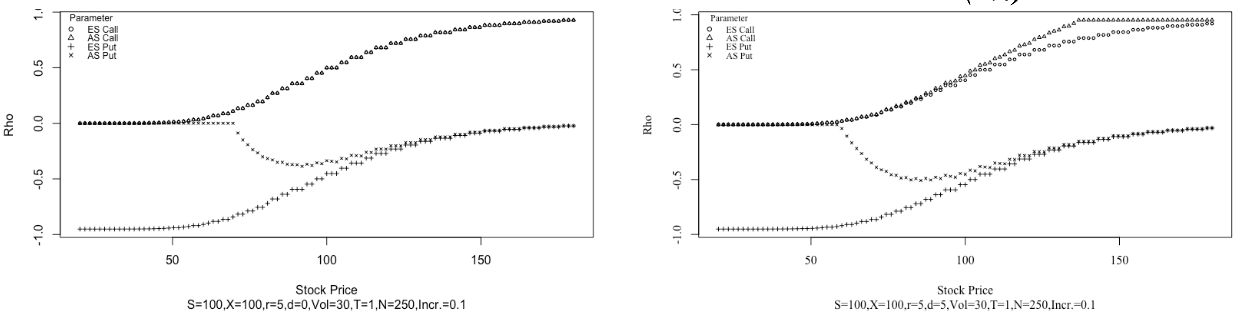
European-style ***American-style***



For call options that are deep out-of-the-money options, the option price changes very little with a small change in interest rates, hence the rho is close to zero. The same is not true for deep out-of-the-money options. For small changes in the volatility for deep in-the-money calls, the call price change is near one due to the boundary condition because it is already in-the-money. The relationship between the stock price and vega is illustrated in Figure 8.1.20.

Figure 8.1.20. Rho with respect to stock price within GBM-based binomial model with and without dividends

No dividends ***Dividends (5%)***



Summary

We illustrated how to compute option Greeks within the GBM binomial option valuation model for both European-style and American-style options.

In this chapter we covered one of the simplest but most important methods of valuing options: the binomial model. We showed how the model clearly illustrates the process by which a dynamically adjusted portfolio enables one to assign a value to an option that must hold to prevent arbitrage. We showed how this process works in one- and two-period models, and we also showed how the general binomial formula and Pascal's triangle illustrates the extension to a multi-period world. We illustrated how the early exercise of American options is easily accommodated within the binomial model.

References

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Trigeorgis, Lenos, "A Log-Transformed Binomial Numerical Analysis Method for Valuing Complex Multi-Option Investments," *Journal of Financial and Quantitative Analysis* 26(3), (September 1991), 309-326.