Module 8.4: SRM ABM-Based Option Models

Learning objectives

- Explore measurement error between the binomial and closed-form model
- Explain how to compute Greeks based on the arithmetic Brownian motion option models
- Contrast no dividends and dividends with respect to Greek sensitivities
- Provide detailed graphical analysis of European-style and American-style options

Executive summary

This module will track closely with Module 8.3 except the models are based on arithmetic Brownian motion. Thus, you will be able to easily compare models as these two modules will be parallel in development.

Based on the notation presented in Module 5.5, we illustrate computing option Greeks within the arithmetic Brownian motion option valuation model (ABMOVM) for European-style options. We further explore differences between the Greeks based on the binomial option valuation model.

Central finance concepts

We again format this section parallel with Module 8.3 to ease comparison.

ABMOVM Greeks

We now present graphical results based on the R code provided for the Greeks as calculated by the ABMOVM. Again, we explore delta, gamma, vega, theta, and rho.

Measurement error with ABM binomial option valuation model

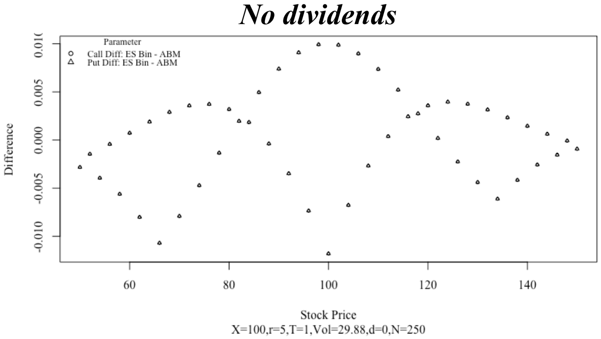
Our focus here is comparing the American-style option Greeks from the binomial option valuation model using the enhanced method and the ABMOVM. Before working our way through the Greeks, we first illustrate measurement error when comparing the binomial approach to European-style option value and the ABMOVM.

Figure 8.4.1 illustrates this error from three perspectives. As previously, ES denotes European-style and AS denotes American-style. In almost all cases, the call and put errors are indistinguishable. Panel A highlights the difference with respect to moneyness. The error in this case is almost always less than \$0.02. Panel B highlights the difference with respect to volatility. Again, the errors are very small and grow in absolute value with volatility. Panel C highlights the difference with respect to time to maturity. With longer time to maturity, the error again grows in absolute value but is very small.

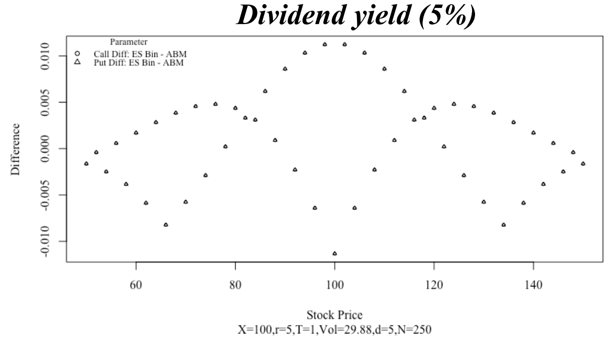
Figure 8.4.1. Measurement Error Between Binomial and ABMOVM

Panel A. Error with Respect to Stock Price

No dividends

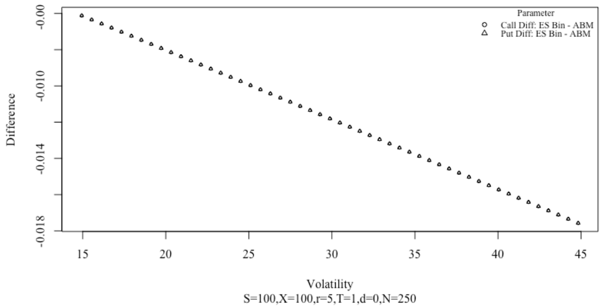


Dividend yield (5%)

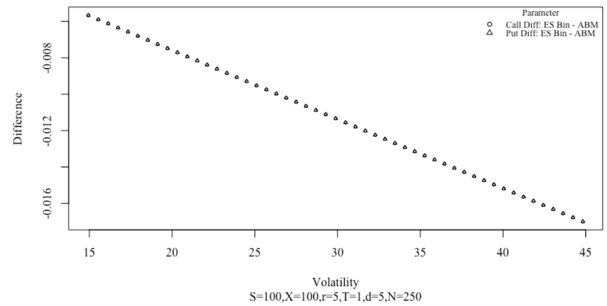


Panel B. Error with Respect to Volatility

No Dividends

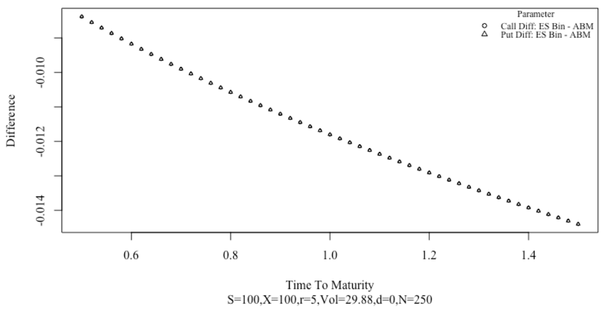


Dividend Yield (5%)

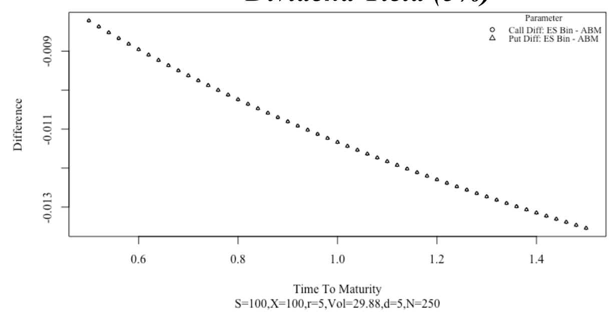


Panel C. Error with Respect to Time to Maturity

No Dividends



Dividend Yield (5%)

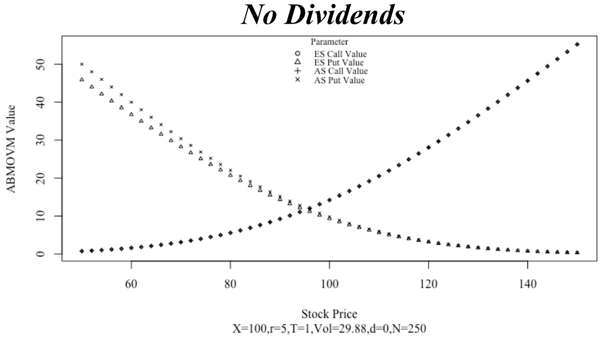


Delta

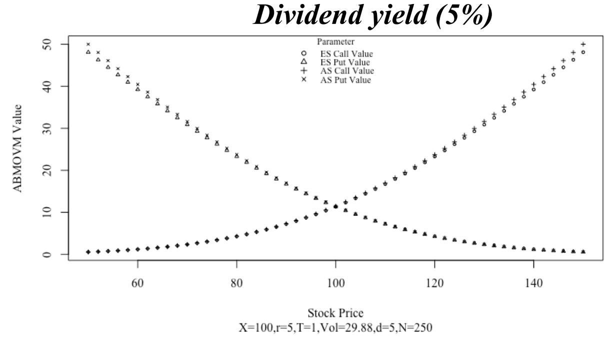
Figure 8.4.2 illustrates delta from several perspectives both without and with dividends (5%). Panel A just illustrates the relationship between the option values and the stock prices. Notice that when the interest rate equals the dividend yield the call and put equal each other at-the-money. The slopes of the lines in Panel A is delta as illustrated in Panel B. Panel B illustrates the influence of the early exercise feature on the delta for the American-style options. Panel C and D show the influence of volatility and time to maturity.

Figure 8.4.2. Call and Put Deltas Based on ABMOVM With and Without Dividends
Panel A. Option Value with Respect to Stock Price

No Dividends

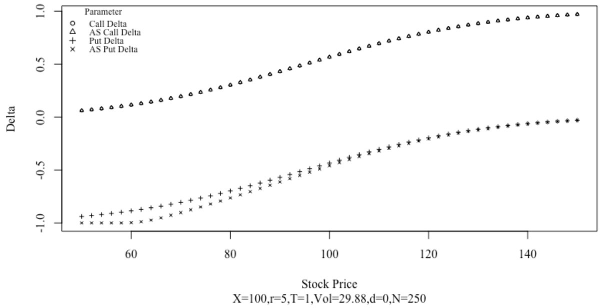


Dividend yield (5%)

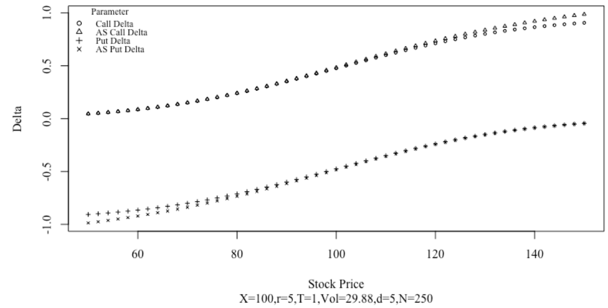


Panel B. Delta with Respect to Stock Price

No Dividends

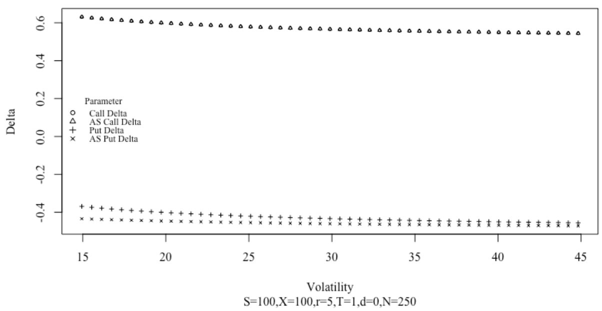


Dividend Yield (5%)

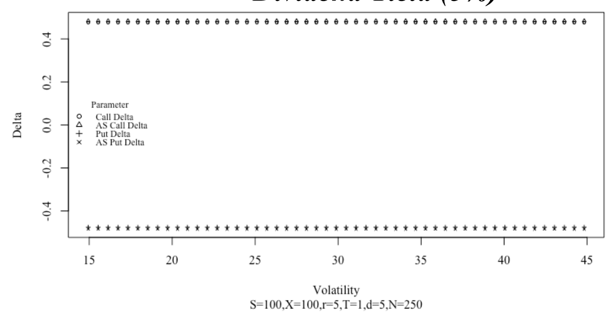


Panel C. Delta with Respect to Volatility

No Dividends

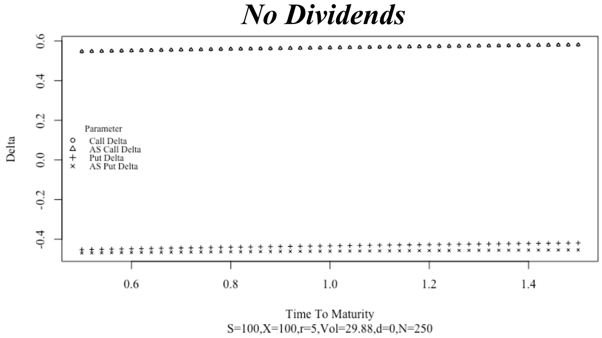


Dividend Yield (5%)

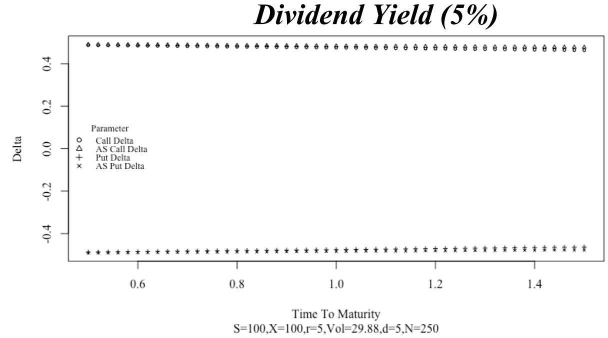


Panel D. Delta with Respect to Time to Maturity

No Dividends



Dividend Yield (5%)



These results are like the GBM-based results with noticeable but subtle differences. We now draw some insights by comparing Figure 8.4.2 with Figure 8.3.2. In Panel A with 5% dividend yield, we can see the symmetry induced by the assumed normal distribution when compared with the similar GBM result. The normal distribution is less peaked than the lognormal distribution; hence, the lower boundary conditions play a lesser role. With dividends the American-style call value is much higher than the European-style call value with GBM when compared to ABM. Further, for deep in-the-money calls, the American-style delta hits one at a much lower stock price with GBM-based models. For deep in-the-money puts, the American-style delta says at zero until a much higher stock price with GBM-based models. For low volatilities, the American-style call delta is much higher than the European-style call delta with GBM. The ABM results with dividends for American-style deltas and European-style deltas are nearly identical for both calls and puts. The results with volatility and no dividends are very similar between ABM and GBM. Similar subtle differences are also seen with changes in time to maturity.

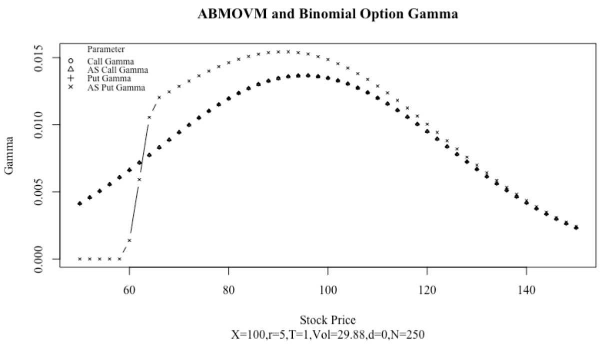
Gamma

Figure 8.4.3 highlights the role of gamma. The normal distribution assumption is clear in Panel A except when the early exercise feature impacts option gammas. Panels B and C show that the gamma increases with declining volatility and with declining time to maturity. When compared with Figure 8.3.3, the normal distribution is clearly less peaked; hence, the boundary conditions are not as influential.

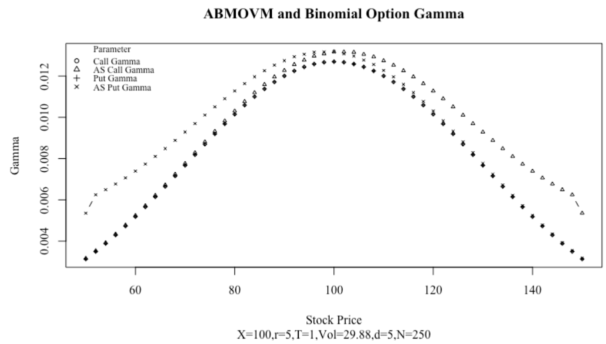
Figure 8.4.3. Call and put gamma based on ABMOVM with and without dividends

Panel A. Gamma with respect to stock price

No dividends

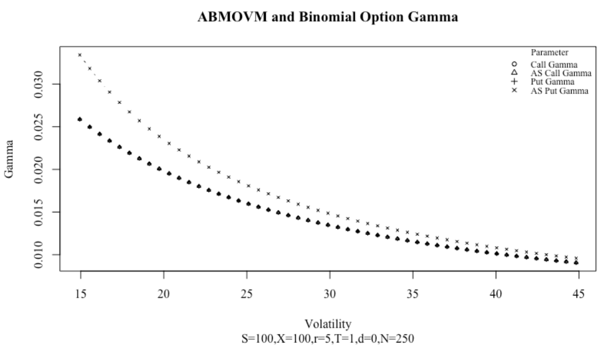


Dividend yield (5%)

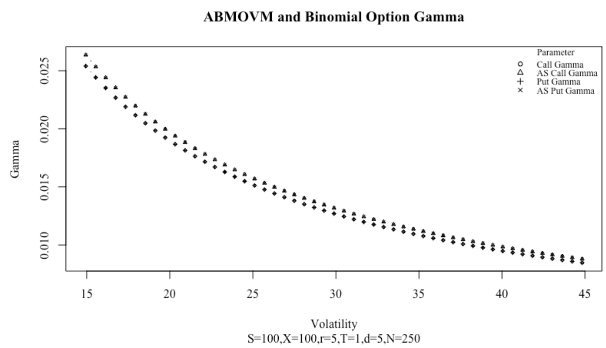


Panel B. Gamma with respect to volatility

No dividends

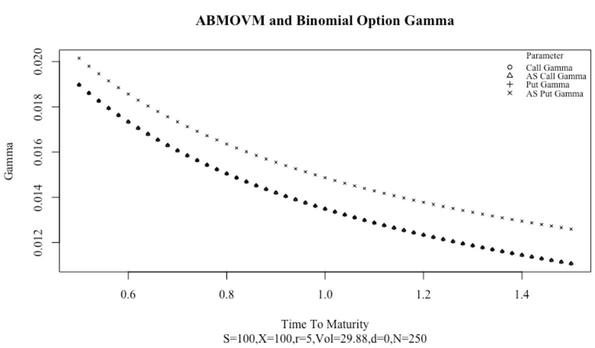


Dividend yield (5%)

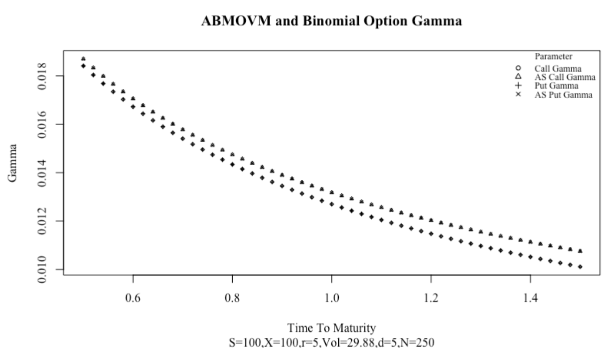


Panel C. Gamma with respect to time to maturity

No dividends



Dividend yield (5%)



Again, we see the more mass distributed wide with the normal distribution lessens to role of early exercise when compared with the GBM results.

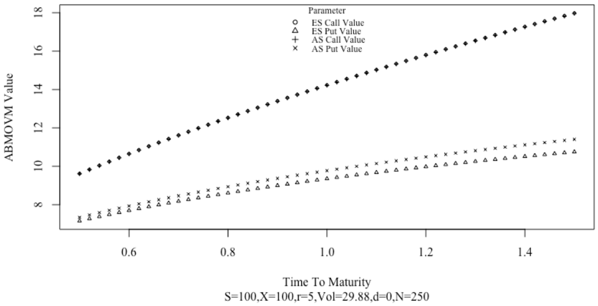
Theta

Figure 8.4.4 highlights the role of thetas. Panel A documents the well-known time value decay—option values decline with the mere passage of time. Panels B show that the theta has a non-monotonic relationship with the stock price. At-the-money options have the highest time value; hence, they have the most negative time value decay or theta. Panel C shows that theta generally declines with higher volatility. Panel D shows that theta generally increases with time to maturity. Note that ABM theta is less sensitive to changes in the underlying instrument value as volatility is absolute.

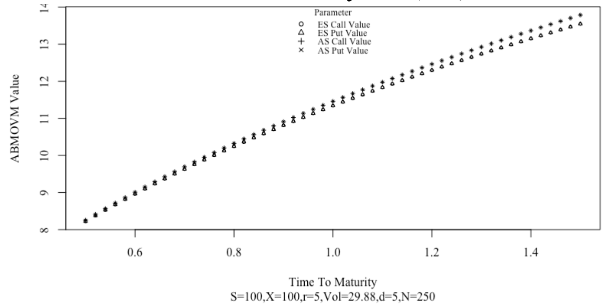
Figure 8.4.4. Call and put thetas based on ABMOVM with and without dividends

Panel A. Option value with respect to time to maturity

No dividends

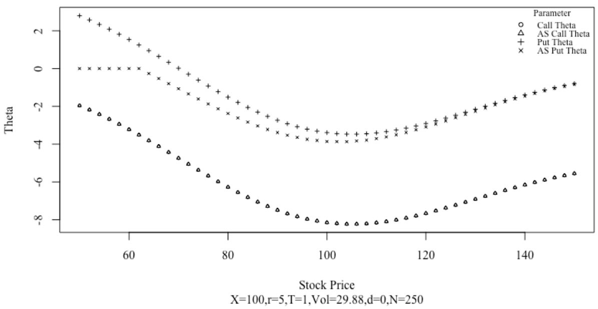


Dividend yield (5%)

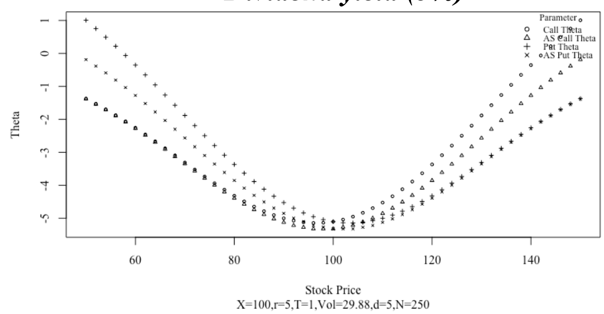


Panel B. Theta with respect to stock price

No dividends

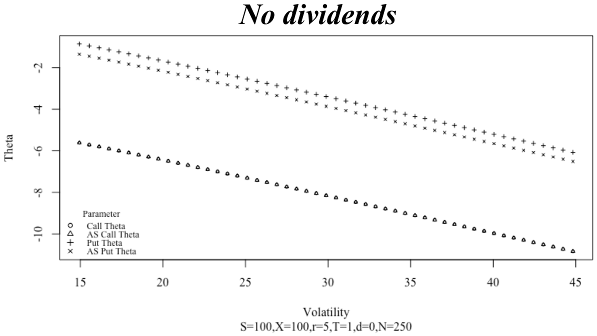


Dividend yield (5%)

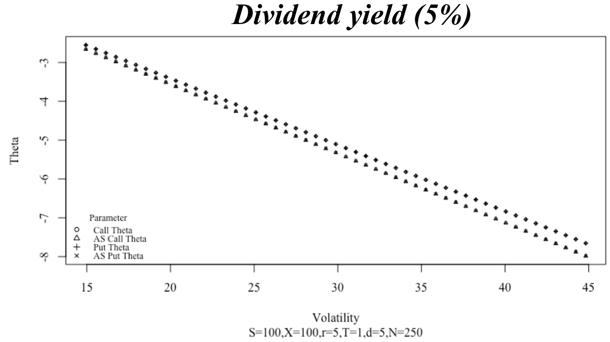


Panel C. Theta with respect to volatility

No dividends

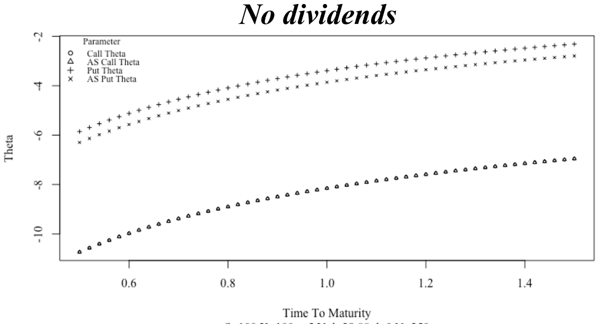


Dividend yield (5%)

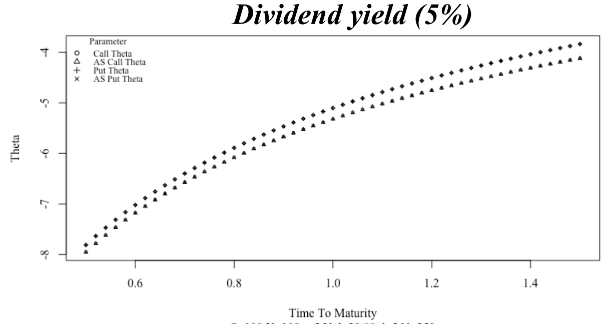


Panel D. Theta with time to maturity

No dividends



Dividend yield (5%)



ABMOVM theta is less sensitive to the stock price because volatility is absolute in nature and not dependent upon the stock price. Recall with GBM, lower the stock prices have a dampening impact on absolute volatility, a property not shared with ABM.

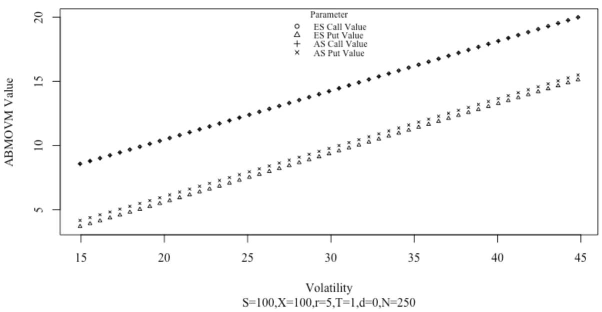
Vega

Figure 8.4.5 highlights the role of vega. Panel A shows that the option values are increasing with respect to volatility. Panel B shows that the vega reflects the lognormal distribution. Panel C shows that vega has a complex relationship with volatility. Panel D shows that vega generally increases with time to maturity.

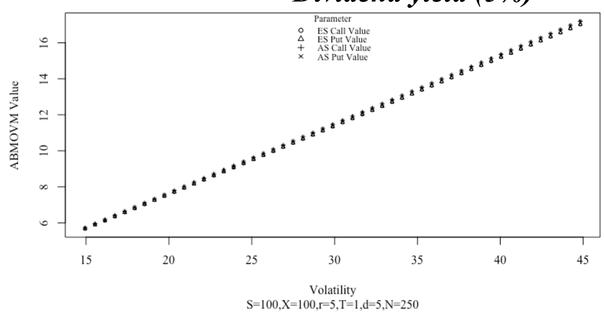
Figure 8.4.5. Call and put vegas based on ABMOVM with and without dividends

Panel A. Option value with respect to volatility

No dividends

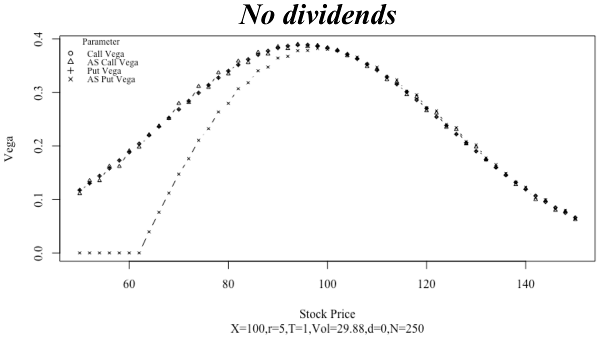


Dividend yield (5%)

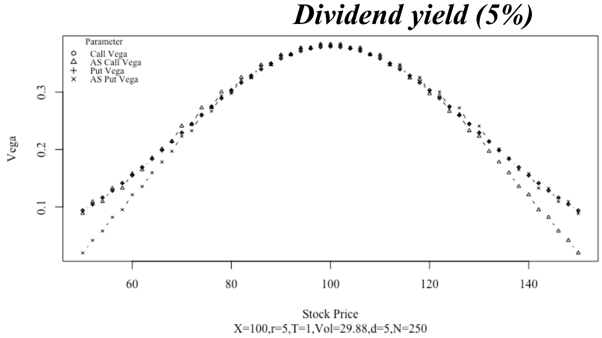


Panel B. Vega with respect to stock price

No dividends

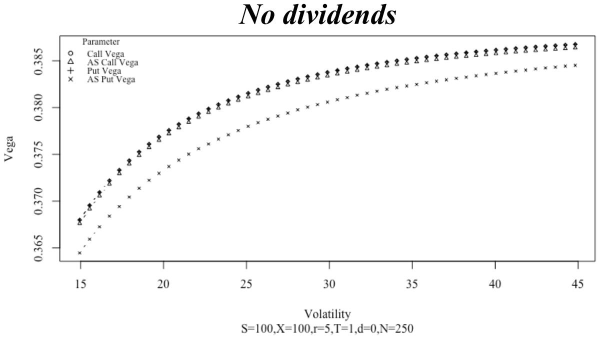


Dividend yield (5%)

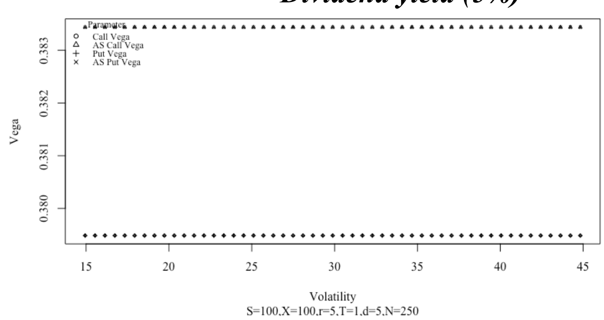


Panel C. Vega with respect to volatility

No dividends

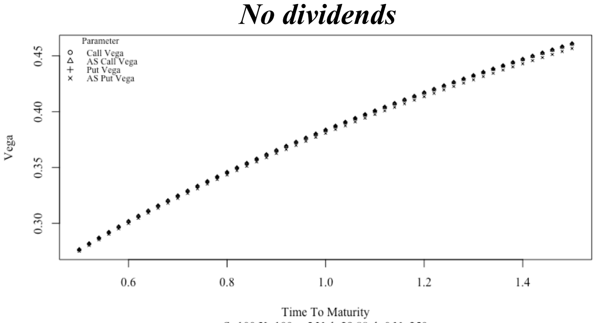


Dividend yield (5%)

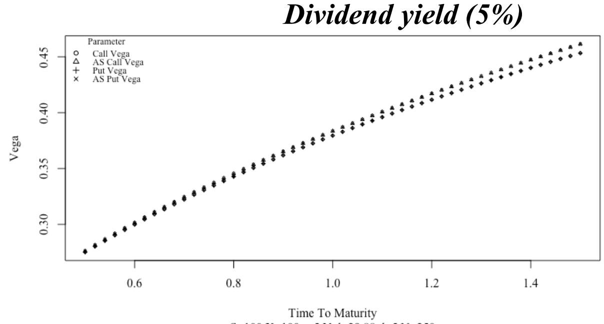


Panel D. Vega with respect to time to maturity

No dividends



Dividend yield (5%)



Once again, when compared with GBM, the boundary conditions are having a smaller influence on these sensitivities.

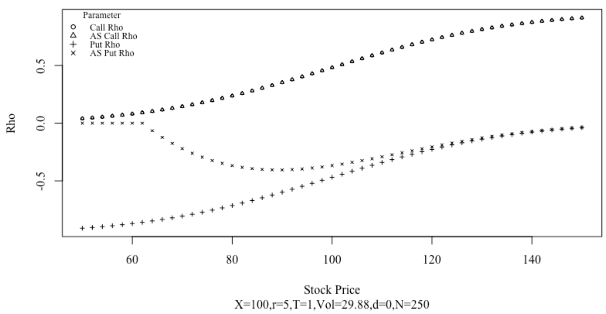
Rho

Figure 8.4.6 highlights the role of rho. Panel A shows that the rho generally increases with the stock price, except for American-style puts. Panel B shows that rho generally declines with volatility. Panel C shows that rho generally increases with time to maturity for calls and decreases for puts.

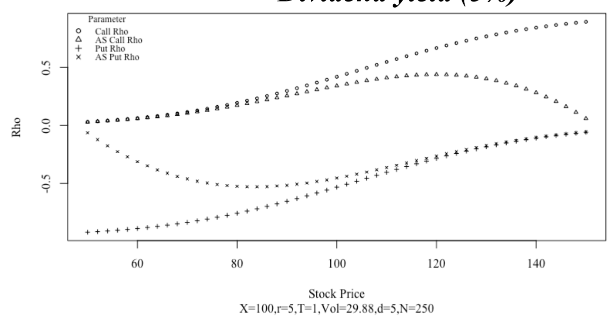
Figure 8.4.6. Call and put rhos based on ABMOVM with and without dividends

Panel A. Rho with respect to stock price

No dividends

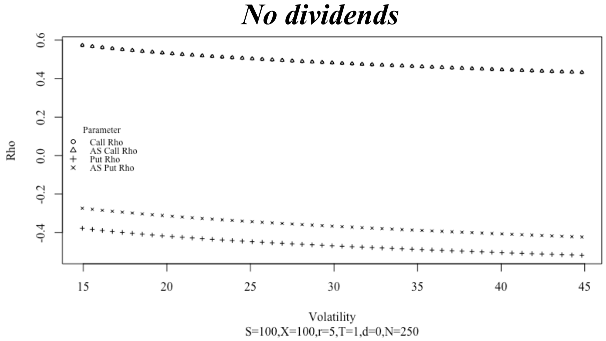


Dividend yield (5%)

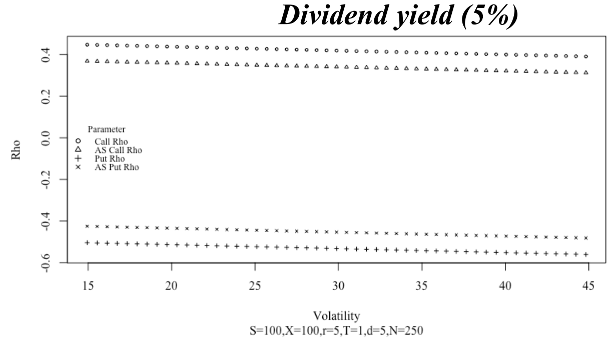


Panel B. Rho with respect to volatility

No dividends

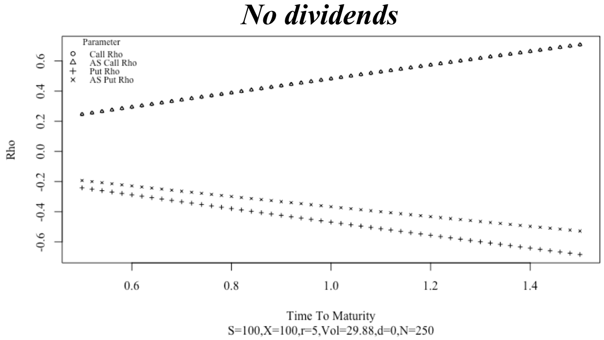


Dividend yield (5%)

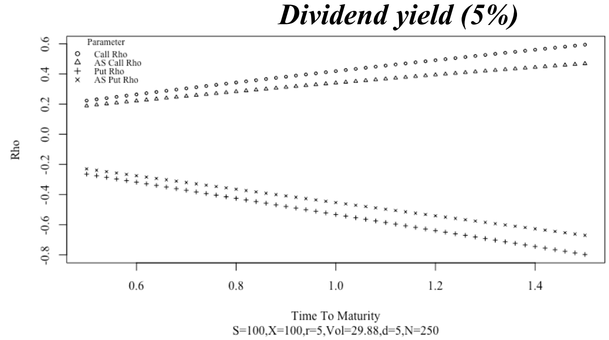


Panel C. Rho with respect to time to maturity

No dividends



Dividend yield (5%)



The influence of interest rates is like GBM except for American-style call options. With ABM, we see a much more significant decline with higher stock prices.

Quantitative finance materials

We now examine the technical details of ABMOVM Greeks. But first, we review the ABMOVM.

ABM option valuation model

Recall based on a set of restrictive assumptions, the ABMOVM can be expressed as

$$O\left(S_t,t;\iota_U,X,T,r,\sigma,\delta\right)=\iota_U\left(S_0B_\delta-XB_r\right)N\left(\iota_U d_n\right)+\sigma_A B_r n\left(d_n\right) \tag{8.4.1}$$

where again the indicator functions is expressed as

$$\iota_U = \begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option} \end{cases}, \tag{8.4.2}$$

$$B_r = e^{-r(T-t)},\ B_\delta = e^{-\delta(T-t)} \tag{8.4.3}$$

$$N(d) = \int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx, \text{ (area under the standard cumulative normal distribution up to d)} \tag{8.4.4}$$

$$n(d) = \frac{e^{-d^2/2}}{\sqrt{2\pi}}, \text{ (standard normal probability density function)} \tag{8.4.5}$$

$$A = \frac{B_{-2(r-\delta)} - 1}{2(r-\delta)}, \text{ (periodic adjustment to volatility)} \tag{8.4.6}$$

$$\sigma_A^2 = \sigma^2 A = \sigma^2 \frac{B_{-2(r-\delta)} - 1}{2(r-\delta)}, \text{ and (periodic adjusted volatility)} \tag{8.4.7}$$

$$d_n = \frac{S_t B_{-(r-\delta)} - X}{\sigma_A}. \text{ (quasi “z” score)} \tag{8.4.8}$$

If there is only a cash flow yield, then the call and put option equations can be expressed as

$$c_t = \left(S_t e^{-\delta(T-t)} - X e^{-r(T-t)}\right) N(d_n) + \sigma_A e^{-r(T-t)} n(d_n) \tag{8.4.9}$$

$$p_0 = \left(X e^{-r(T-t)} - S_0 e^{-\delta(T-t)}\right) N(-d_n) + \sigma_A e^{-r(T-t)} n(d_n) \tag{8.4.10}$$

ABMOVM Greeks

We now cover the Greeks as calculated by the ABMOVM. We will work through the derivations as well as illustrate graphically several results.

Selected lemmas

We provide several lemmas useful in deriving the ABM-related Greeks.

Lemma 1: Derivatives related to A

If

$$A = \sqrt{\frac{B_{-2(r-\delta)} - 1}{2(r-\delta)}}, \tag{8.4.11}$$

then

$$\frac{\partial A}{\partial t} = \frac{B_{-2(r-\delta)}}{2A} \text{ and} \tag{8.4.12}$$

$$\frac{\partial A}{\partial r} = \frac{(T-t) B_{-2(r-\delta)}}{2(r-\delta)A} - A. \tag{8.4.13}$$

Note

$$\frac{\partial A}{\partial t} = \frac{1}{2}\left[\frac{B_{-2(r-\delta)} - 1}{2(r-\delta)}\right]^{-1/2} \frac{\partial}{\partial t}\left[\frac{B_{-2(r-\delta)} - 1}{2(r-\delta)}\right] = -\frac{B_{-2(r-\delta)}}{2A} \text{ and} \tag{8.4.14}$$

$$
\begin{aligned}
\frac{\partial A}{\partial r} &= \frac{1}{2}\left[\frac{B_{-2(r-\delta)}-1}{2(r-\delta)}\right]^{-1/2}\frac{\partial}{\partial r}\left[\frac{B_{-2(r-\delta)}-1}{2(r-\delta)}\right] \\
&= \frac{1}{2A}\frac{2(T-t)B_{-2(r-\delta)}2(r-\delta)-\left(B_{-2(r-\delta)}-1\right)2}{4(r-\delta)^2} \\
&= \frac{2(T-t)B_{-2(r-\delta)}(r-\delta)-\left(B_{-2(r-\delta)}-1\right)}{4(r-\delta)^2A} \\
&= \frac{(T-t)B_{-2(r-\delta)}}{2(r-\delta)A}-A
\end{aligned}. \tag{8.4.15}
$$

Lemma 2: Derivatives related to d_n

If

$$
d_n \equiv \frac{B_{-(r-\delta)}S_t - X}{A\sigma}, \tag{8.4.16}
$$

then we have the following results

$$
\frac{\partial d_n}{\partial S} = \frac{B_{-(r-\delta)}}{A\sigma}, \tag{8.4.17}
$$

$$
\frac{\partial d_n}{\partial t} = \frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{(r-\delta)S_t}{\sigma}\right], \tag{8.4.18}
$$

$$
\frac{\partial d_n}{\partial \sigma} = -\frac{d_n}{\sigma}, \text{ and} \tag{8.4.19}
$$

$$
\frac{\partial d_n}{\partial r} = \frac{(T-t)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{(T-t)B_{-2(r-\delta)}}{2(r-\delta)A^2}+1\right]. \tag{8.4.20}
$$

Note

$$
\begin{aligned}
\frac{\partial d_n}{\partial t} &= \frac{-(r-\delta)B_{-(r-\delta)}S_tA\sigma - \left(B_{-(r-\delta)}S_t - X\right)\sigma\frac{\partial A}{\partial t}}{A^2\sigma^2} \\
&= \frac{-(r-\delta)B_{-(r-\delta)}S_tA\sigma + \left(B_{-(r-\delta)}S_t - X\right)\sigma\frac{B_{-2(r-\delta)}}{2A}}{A^2\sigma^2} \\
&= \frac{-(r-\delta)B_{-(r-\delta)}S_tA\sigma + \frac{\left(B_{-(r-\delta)}S_t - X\right)}{A\sigma}\frac{\sigma^2 B_{-2(r-\delta)}}{2}}{A^2\sigma^2} \\
&= \frac{-(r-\delta)B_{-(r-\delta)}S_tA\sigma + d_n\frac{\sigma^2 B_{-2(r-\delta)}}{2}}{A^2\sigma^2} \\
&= -\frac{(r-\delta)B_{-(r-\delta)}S_t}{A\sigma} + \frac{d_n B_{-2(r-\delta)}}{2A^2} = \frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{(r-\delta)S_t}{\sigma}\right]
\end{aligned}. \tag{8.4.21}
$$

Further,

$$\frac{\partial d_n}{\partial \sigma} = \frac{\partial}{\partial \sigma}\frac{B_{-(r-\delta)}S_t - X}{A\sigma} = -\frac{d_n}{\sigma}. \tag{8.4.22}$$

Finally,

$$\begin{aligned}
\frac{\partial d_n}{\partial r} &= \frac{\partial}{\partial r}\frac{B_{-(r-\delta)}S_t - X}{A\sigma} = \frac{(T-t)B_{-(r-\delta)}S_t A\sigma - \left(B_{-(r-\delta)}S_t - X\right)\sigma\frac{\partial A}{\partial r}}{A^2\sigma^2} \\
&= \frac{(T-t)B_{-(r-\delta)}S_t A\sigma + \left(B_{-(r-\delta)}S_t - X\right)\sigma\left[\frac{(T-t)B_{-2(r-\delta)}}{2(r-\delta)A} + A\right]}{A^2\sigma^2} \\
&= \frac{(T-t)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{(T-t)B_{-2(r-\delta)}}{2(r-\delta)A^2} + 1\right]
\end{aligned}. \tag{8.4.23}$$

Lemma 3: Derivatives related to $n(d_n)$

From the definition of the standard normal probability density function, we have

$$\frac{\partial n(d_n)}{\partial d_n} = \frac{\partial}{\partial d_n}\frac{e^{-\frac{d_n^2}{2}}}{\sqrt{2\pi}} = -d_n n(d_n). \tag{8.4.24}$$

Thus, we have the following results based on Lemma 2.

$$\frac{\partial n(d_n)}{\partial S} = -d_n n(d_n)\frac{B_{-(r-\delta)}}{A\sigma}. \tag{8.4.25}$$

$$\frac{\partial n(d_n)}{\partial t} = -d_n n(d_n)\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{(r-\delta)S_t}{\sigma}\right]. \tag{8.4.26}$$

$$\frac{\partial n(d_n)}{\partial \sigma} = -d_n n(d_n)\left(-\frac{d_n}{\sigma}\right). \tag{8.4.27}$$

$$\frac{\partial n(d_n)}{\partial r} = -d_n n(d_n)\left\{\frac{(T-t)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{(T-t)B_{-2(r-\delta)}}{2(r-\delta)A^2} + 1\right]\right\}. \tag{8.4.28}$$

Lemma 4: Derivatives of functions of the integrand

If

$$N\left\{f\left[d(y)\right]\right\} = \int_{-\infty}^{f[d(y)]} n(x)dx, \tag{8.4.29}$$

then based on the chain rule

$$\frac{\partial N\left\{f\left[d(y)\right]\right\}}{\partial y} = n\left\{f\left[d(y)\right]\right\}\frac{\partial f\left[d(y)\right]}{\partial d}\frac{\partial d(y)}{\partial y}. \tag{8.4.30}$$

From this lemma, we note the following results:

$$\frac{\partial N(\iota_U d_n)}{\partial S} = \iota_U n(d_n)\frac{B_{-(r-\delta)}}{A\sigma}. \tag{8.4.31}$$

$$\frac{\partial N(\iota_U d_n)}{\partial t} = \iota_U n(d_n)\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{(r-\delta)S_t}{\sigma}\right]. \tag{8.4.32}$$

$$\frac{\partial N\left(\iota_U d_n\right)}{\partial \sigma} = \iota_U n\left(d_n\right)\left(-\frac{d_n}{\sigma^2}\right). \tag{8.4.33}$$

$$\frac{\partial N\left(\iota_U d_n\right)}{\partial r} = \iota_U n\left(d_n\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right]\right\}. \tag{8.4.34}$$

Lemma 5: Derivatives of discount functions
If

$$B_x = e^{-x(T-t)},$$

then

$$\frac{\partial B_x}{\partial t} = \frac{\partial e^{-x(T-t)}}{\partial t} = -e^{-x(T-t)}\frac{\partial\left[x\left(T-t\right)\right]}{\partial t} = xB_x \text{ and}$$
$$\frac{\partial B_x}{\partial x} = \frac{\partial e^{-x(T-t)}}{\partial x} = -\left(T-t\right)e^{-x(T-t)} = -\left(T-t\right)B_x.$$

Delta
Mathematically, delta is defined as

$$\Delta_O \equiv \frac{\partial O}{\partial S} = \iota_U e^{-\delta T} N\left(\iota_U d_n\right). \tag{8.4.35}$$

Sketch of proof: Based on the lemmas above, the delta of the underlying option, $\Delta_O \equiv \frac{\partial O}{\partial S}$, can be expressed as

$$\frac{\partial O}{\partial S} = \iota_U B_\delta N\left(\iota_U d_n\right) + \iota_U B_r\left(B_{-(r-\delta)}S_t - X\right)\frac{\partial N\left(\iota_U d_n\right)}{\partial S} + \sigma B_r A\frac{\partial n\left(d_n\right)}{\partial S} \tag{8.4.36}$$

Substituting from the lemmas above, we have

$$\begin{aligned}\frac{\partial O}{\partial S} &= \iota_U B_\delta N\left(\iota_U d_n\right) + \iota_U B_r\left(B_{-(r-\delta)}S_t - X\right)\iota_U n\left(d_n\right)\frac{B_{-(r-\delta)}}{A\sigma} + \sigma B_r A\left[-d_n n\left(d_n\right)\frac{B_{-(r-\delta)}}{A\sigma}\right] \\ &= \iota_U B_\delta N\left(\iota_U d_n\right) + B_r B_{-(r-\delta)}\frac{\left(B_{-(r-\delta)}S_t - X\right)}{A\sigma}n\left(d_n\right) - B_r B_{-(r-\delta)} d_n n\left(d_n\right) \\ &= \iota_U B_\delta N\left(\iota_U d_n\right) + B_\delta d_n n\left(d_n\right) - B_\delta d_n n\left(d_n\right)\end{aligned} \tag{8.4.37}$$

Thus

$$\Delta_O \equiv \frac{\partial O}{\partial S} = \iota_U B_\delta N\left(\iota_U d_n\right). \tag{8.4.38}$$

Gamma
Mathematically, gamma is defined as

$$\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2} = \frac{B_{-(r-2\delta)}n\left(d_n\right)}{A\sigma}. \tag{8.4.39}$$

Sketch of proof: Recall based on put-call parity $\Delta_p = \Delta_c - e^{-\delta T}$ and therefore we know

$$\frac{\partial \Delta_p}{\partial S} = \frac{\partial \Delta_c}{\partial S}. \tag{8.4.40}$$

The gamma of the underlying option, $\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2}$, is

$$\frac{\partial}{\partial S}\left(\frac{\partial O}{\partial S}\right)=\frac{\partial}{\partial S}\left[\iota_U B_\delta N\left(\iota_U d_n\right)\right]=\iota_U B_\delta \frac{\partial N\left(\iota_U d_n\right)}{\partial S}=\iota_U B_\delta \iota_U n\left(d_n\right)\frac{B_{-(r-\delta)}}{A\sigma}=\frac{B_{-(r-2\delta)}n\left(d_n\right)}{A\sigma}. \quad (8.4.41)$$

Theta

Mathematically, theta can be expressed as

$$\Theta_O=\frac{\partial O}{\partial t}=\iota_U\left(\delta B_\delta S_t - rB_r X\right)N\left(\iota_U d_n\right)+\sigma r B_r A n\left(d_n\right)-\left(\frac{\sigma B_{-(r-2\delta)}}{2A}\right)n\left(d_n\right). \quad (8.4.42)$$

Sketch of proof: Given the prevalence of t, we rearrange the option valuation formula explicitly noting variables that are a function of t,

$$O=\iota_U S_t B_\delta(t)N\left[\iota_U d_n(t)\right]-\iota_U X B_r(t)N\left[\iota_U d_n(t)\right]+\sigma B_r(t)A(t)n\left[d_n(t)\right]. \quad (8.4.43)$$

The theta of the underlying option, $\Theta_O \equiv \frac{\partial O}{\partial t}$, is (based on the product rule)

$$\begin{aligned}\frac{\partial O}{\partial t}&=\iota_U S_t B_\delta(t)\frac{\partial N\left[\iota_U d_n(t)\right]}{\partial t}+\iota_U S_t N\left[\iota_U d_n(t)\right]\frac{\partial B_\delta(t)}{\partial t}\\&-\iota_U X B_r(t)\frac{\partial N\left[\iota_U d_n(t)\right]}{\partial t}-\iota_U X N\left[\iota_U d_n(t)\right]\frac{\partial B_r(t)}{\partial t}\\&+\sigma A(t)n\left[d_n(t)\right]\frac{\partial B_r(t)}{\partial t}+\sigma B_r(t)n\left[d_n(t)\right]\frac{\partial A(t)}{\partial t}+\sigma B_r(t)A(t)\frac{\partial n\left[d_n(t)\right]}{\partial t}\end{aligned}. \quad (8.4.44)$$

Rearranging based on derivatives,

$$\begin{aligned}\frac{\partial O}{\partial t}&=\iota_U\left(S_t B_\delta - X B_r\right)\frac{\partial N\left(\iota_U d_n\right)}{\partial t}+\left[\sigma A n\left(d_n\right)-\iota_U X N\left(\iota_U d_n\right)\right]\frac{\partial B_r}{\partial t}\\&+\iota_U S_t N\left(\iota_U d_n\right)\frac{\partial B_\delta}{\partial t}+\sigma B_r n\left(d_n\right)\frac{\partial A}{\partial t}+\sigma B_r A\frac{\partial n\left(d_n\right)}{\partial t}\end{aligned}, \quad (8.4.45)$$

where

$$\frac{\partial N\left(\iota_U d_n\right)}{\partial t}=\iota_U n\left(d_n\right)\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A}-\frac{(r-\delta)S_t}{\sigma}\right], \quad (8.4.46)$$

$$\frac{\partial B_x}{\partial t}=xB_x, \quad (8.4.47)$$

$$\frac{\partial A}{\partial t}=-\frac{B_{-2(r-\delta)}}{2A}, \text{ and} \quad (8.4.48)$$

$$\frac{\partial n\left(d_n\right)}{\partial t}=-d_n n\left(d_n\right)\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A}-\frac{(r-\delta)S_t}{\sigma}\right]. \quad (8.4.49)$$

Substituting the derivatives,

$$\frac{\partial O}{\partial t} = \iota_U\left(S_t B_\delta - XB_r\right)\left\{\iota_U n\left(d_n\right)\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{\left(r-\delta\right)S_t}{\sigma}\right]\right\}$$
$$+\left[\sigma An\left(d_n\right) - \iota_U XN\left(\iota_U d_n\right)\right]rB_r$$
$$+\iota_U S_t N\left(\iota_U d_n\right)\delta B_\delta + \sigma B_r n\left(d_n\right)\left(-\frac{B_{-2(r-\delta)}}{2A}\right) \quad . \tag{8.4.50}$$
$$+\sigma B_r A\left\{-d_n n\left(d_n\right)\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{\left(r-\delta\right)S_t}{\sigma}\right]\right\}$$

Rearranging based on PDFs and CDFs,

$$\frac{\partial O}{\partial t} = \iota_U\left(\delta B_\delta S_t - rB_r X\right)N\left(\iota_U d_n\right)$$
$$+\begin{pmatrix} \left(S_t B_\delta - XB_r\right)\left\{\frac{B_{-(r-\delta)}}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{\left(r-\delta\right)S_t}{\sigma}\right]\right\} \\ +\sigma rB_r A + \sigma B_r\left(-\frac{B_{-2(r-\delta)}}{2A}\right) \\ -\sigma B_r B_{-(r-\delta)} d_n\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{\left(r-\delta\right)S_t}{\sigma}\right] \end{pmatrix} n\left(d_n\right). \tag{8.4.51}$$

Further reducing

$$\frac{\partial O}{\partial t} = \iota_U\left(\delta B_\delta S_t - rB_r X\right)N\left(\iota_U d_n\right) + \sigma rB_r An\left(d_n\right)$$
$$+\begin{pmatrix} B_r B_{-(r-\delta)}\frac{\left(S_t B_{-(r-\delta)} - X\right)}{A}\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{\left(r-\delta\right)S_t}{\sigma}\right] \\ +\sigma B_r\left(-\frac{B_{-2(r-\delta)}}{2A}\right) \\ -\sigma B_\delta d_n\left[\frac{d_n B_{-(r-\delta)}}{2A} - \frac{\left(r-\delta\right)S_t}{\sigma}\right] \end{pmatrix} n\left(d_n\right). \tag{8.4.52}$$

Thus,

$$\Theta_O = \frac{\partial O}{\partial t} = \iota_U\left(\delta B_\delta S_t - rB_r X\right)N\left(\iota_U d_n\right) + \sigma rB_r An\left(d_n\right) - \left(\frac{\sigma B_{-(r-2\delta)}}{2A}\right)n\left(d_n\right). \tag{8.4.53}$$

Vega

Mathematically, vega is defined as

$$\upsilon_O = \frac{\partial O}{\partial \sigma} = B_r An\left(d_n\right). \tag{8.4.54}$$

Sketch of proof: From put-call parity ($c = Se^{-\delta T} - Xe^{-rT} + p$), we know $\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma}$. Thus, the vega of the underlying option, $\upsilon_O \equiv \frac{\partial O}{\partial \sigma}$, can be expressed as

$$\frac{\partial O}{\partial \sigma} = \iota_U B_r \left(B_{-(r-\delta)} S_t - X \right) \frac{\partial N\left(\iota_U d_n\right)}{\partial \sigma} + B_r A n\left(d_n\right) + \sigma B_r A \frac{\partial n\left(d_n\right)}{\partial \sigma}. \tag{8.4.55}$$

From the lemmas given above, we note

$$\frac{\partial N\left(\iota_U d_n\right)}{\partial \sigma} = \iota_U n\left(d_n\right)\left(-\frac{d_n}{\sigma^2}\right) \text{ and} \tag{8.4.56}$$

$$\frac{\partial n\left(d_n\right)}{\partial \sigma} = -d_n n\left(d_n\right)\left(-\frac{d_n}{\sigma}\right) = \frac{d_n^2 n\left(d_n\right)}{\sigma}. \tag{8.4.57}$$

Thus,

$$\frac{\partial O}{\partial \sigma} = \iota_U B_r \left(B_{-(r-\delta)} S_t - X \right)\left[\iota_U n\left(d_n\right)\left(-\frac{d_n}{\sigma^2}\right)\right] + B_r A n\left(d_n\right) + \sigma B_r A \left[\frac{d_n^2 n\left(d_n\right)}{\sigma}\right]. \tag{8.4.58}$$

Reducing,

$$\frac{\partial O}{\partial \sigma} = \left[-B_r \frac{\left(B_{-(r-\delta)} S_t - X\right)}{\sigma}\left(\frac{d_n}{\sigma}\right) + B_r A + B_r A d_n^2 \right] n\left(d_n\right). \tag{8.4.59}$$

Further reducing,

$$\frac{\partial O}{\partial \sigma} = \left[-B_r A d_n^2 + B_r A + B_r A d_n^2 \right] n\left(d_n\right) = B_r A n\left(d_n\right). \tag{8.4.60}$$

Thus,

$$\upsilon_O = \frac{\partial O}{\partial \sigma} = B_r A n\left(d_n\right). \tag{8.4.61}$$

Rho
Mathematically, rho is defined as

$$\rho_O = \frac{\partial O}{\partial r} = \left(T-t\right) B_r \iota_U X N\left(\iota_U d_n\right) + B_r \sigma A \left\{ \left[\frac{\left(T-t\right) B_{-2(r-\delta)}}{2\left(r-\delta\right) A^2} - 1 \right] - \left(T-t\right) \right\} n\left(d_n\right). \tag{8.4.62}$$

Sketch of proof: Highlighting r, we note

$$O = \iota_U B_\delta S_t N\left[\iota_U d_n\left(r\right)\right] - \iota_U X B_r\left(r\right) N\left[\iota_U d_n\left(r\right)\right] + \sigma B_r\left(r\right) A\left(r\right) n\left[d_n\left(r\right)\right]. \tag{8.4.63}$$

The rho of the underlying option, $\rho_O \equiv \frac{\partial O}{\partial r}$, is

$$\begin{aligned} \frac{\partial O}{\partial r} = {} & \iota_U B_\delta S_t \frac{\partial N\left[\iota_U d_n\left(r\right)\right]}{\partial r} - \iota_U X N\left[\iota_U d_n\left(r\right)\right] \frac{\partial B_r\left(r\right)}{\partial r} - \iota_U X B_r\left(r\right) \frac{\partial N\left[\iota_U d_n\left(r\right)\right]}{\partial r} \\ & + \sigma A\left(r\right) n\left[d_n\left(r\right)\right] \frac{\partial B_r\left(r\right)}{\partial r} + \sigma B_r\left(r\right) n\left[d_n\left(r\right)\right] \frac{\partial A\left(r\right)}{\partial r} + \sigma B_r\left(r\right) A\left(r\right) \frac{\partial n\left[d_n\left(r\right)\right]}{\partial r} \end{aligned}. \tag{8.4.64}$$

Rearranging based on derivatives,

$$\begin{aligned}\frac{\partial O}{\partial r} &= \left(\iota_U B_\delta S_t - \iota_U X B_r\right)\frac{\partial N\left(\iota_U d_n\right)}{\partial r} - \left[\iota_U X N\left(\iota_U d_n\right) - \sigma A n\left(d_n\right)\right]\frac{\partial B_r}{\partial r} \\ &+ \sigma B_r n\left(d_n\right)\frac{\partial A}{\partial r} + \sigma B_r A \frac{\partial n\left(d_n\right)}{\partial r}\end{aligned}, \tag{8.4.65}$$

where

$$\frac{\partial N\left(\iota_U d_n\right)}{\partial r} = \iota_U n\left(d_n\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2} + 1\right]\right\}, \tag{8.4.66}$$

$$\frac{\partial B_r}{\partial r} = -\left(T-t\right)B_r, \tag{8.4.67}$$

$$\frac{\partial A}{\partial r} = \frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A} - A, \text{ and} \tag{8.4.68}$$

$$\frac{\partial n\left(d_n\right)}{\partial r} = -d_n n\left(d_n\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2} + 1\right]\right\}. \tag{8.4.69}$$

Substituting,

$$\begin{aligned}\frac{\partial O}{\partial r} &= \left(\iota_U B_\delta S_t - \iota_U X B_r\right)\iota_U n\left(d_n\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2} + 1\right]\right\} \\ &- \left[\iota_U X N\left(\iota_U d_n\right) - \sigma A n\left(d_n\right)\right]\left[-\left(T-t\right)B_r\right] \\ &+ \sigma B_r n\left(d_n\right)\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A} - A\right] \\ &+ \sigma B_r A\left(-d_n n\left(d_n\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2} + 1\right]\right\}\right)\end{aligned}. \tag{8.4.70}$$

Rearranging,

$$\begin{aligned}\frac{\partial O}{\partial r} &= \left(T-t\right)B_r \iota_U X N\left(\iota_U d_n\right) \\ &+ \left(\begin{aligned}&\left(B_\delta S_t - X B_r\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2} + 1\right]\right\} \\ &-\left(T-t\right)B_r \sigma A \\ &+\sigma B_r\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A} - A\right] \\ &-\sigma B_r A d_n\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2} + 1\right]\right\}\end{aligned}\right)n\left(d_n\right)\end{aligned} \tag{8.4.71}$$

Focusing on the second major term,

$$\begin{pmatrix} \left(B_\delta S_t - XB_r\right)\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right]\right\} - \left(T-t\right)B_r\sigma A \\ +\sigma B_r\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A} - A\right] - \sigma B_r A d_n\left\{\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} + d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right]\right\} \end{pmatrix}$$

$$\begin{aligned} &B_r\left(T-t\right)B_{-(r-\delta)}S_t\frac{\left(B_{-(r-\delta)}S_t - X\right)}{A\sigma} \\ = \; &+B_r\left(B_{-(r-\delta)}S_t - X\right)d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right] - \left(T-t\right)B_r\sigma A \\ &+\sigma B_r A\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}-1\right] - \sigma B_r A d_n\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} - \sigma B_r A d_n^2\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right] \end{aligned} \qquad .(8.4.72)$$

Thus,

$$\begin{aligned} &B_r\left(T-t\right)B_{-(r-\delta)}S_t\frac{\left(B_{-(r-\delta)}S_t - X\right)}{A\sigma} \\ &+B_r\left(B_{-(r-\delta)}S_t - X\right)d_n\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right] - \left(T-t\right)B_r\sigma A \\ &+\sigma B_r A\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}-1\right] - \sigma B_r A d_n\frac{\left(T-t\right)B_{-(r-\delta)}S_t}{A\sigma} \\ &-\sigma B_r A d_n^2\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right] \\ &= \begin{matrix} \left(T-t\right)B_\delta d_n S_t + \sigma B_r A d_n^2\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right] - \left(T-t\right)B_r\sigma A \\ +\sigma B_r A\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}-1\right] - \left(T-t\right)B_\delta d_n S_t - \sigma B_r A d_n^2\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}+1\right] \end{matrix} \\ &= -\left(T-t\right)B_r\sigma A + \sigma B_r A\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}-1\right] \\ &= B_r\sigma A\left\{\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}-1\right] - \left(T-t\right)\right\} \end{aligned} \qquad . \qquad (8.4.73)$$

Reducing,

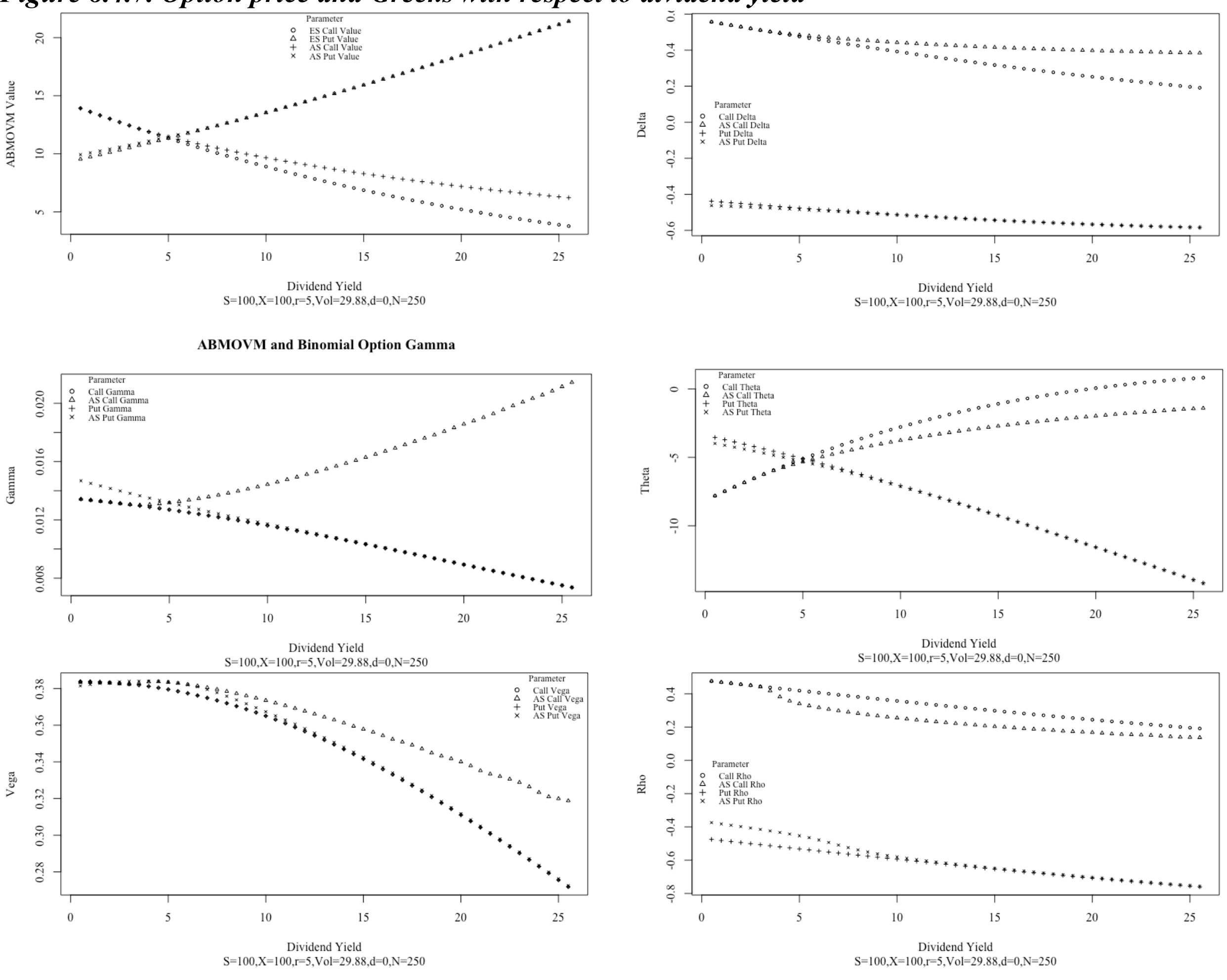
$$\frac{\partial O}{\partial r} = \left(T-t\right)B_r\iota_U XN\left(\iota_U d_n\right) + B_r\sigma A\left\{\left[\frac{\left(T-t\right)B_{-2(r-\delta)}}{2\left(r-\delta\right)A^2}-1\right] - \left(T-t\right)\right\}n\left(d_n\right). \qquad (8.4.74)$$

Therefore,

$$\rho_O = \frac{\partial O}{\partial r} = \left(T-t\right) B_r \iota_U X N\left(\iota_U d_n\right) + B_r \sigma A \left\{ \left[\frac{\left(T-t\right) B_{-2(r-\delta)}}{2\left(r-\delta\right) A^2} - 1 \right] - \left(T-t\right) \right\} n\left(d_n\right). \qquad (8.4.75)$$

Selected other insights based on Greeks
We now examine a few other aspects related to the GBMOVM. Specifically, we explore the sensitivity of the Greeks to dividend yield as well as a few extended Greeks. Finally, we explore the use of selected Greeks to estimate option price changes.
Sensitivity to dividend yield
Figure 8.4.7 provides several graphs related to dividend yield. The dividend yield on the horizontal axis and the option prices or Greeks are on the vertical axis. For option values, the negative sloped line is the call and the positive sloped line is the put. Note that when the interest rate equals the dividend yield, the value of the call equals the value of the put. Each Greek is sensitive to dividend yield.

Figure 8.4.7. Option price and Greeks with respect to dividend yield



Note that the results presented here are very similar to GBMOVM.

Summary
Again, this module tracked closely with Module 8.3 except the models are based on arithmetic Brownian motion. Thus, you were able to easily compare models as these two modules will be parallel in development.

We illustrated computing option Greeks within the arithmetic Brownian motion option valuation model (ABMOVM) for European-style options. We further explore differences between the Greeks based on the binomial option valuation model.

References

See Module 5.4.