Module 8.2: SRM ABM-Based Binomial Models

Learning objectives
- Computing call and put option Greeks using the ABM binomial option valuation approach
- Contrast European-style and American-style call and put option Greeks using the ABM binomial option valuation approach

Executive summary
Recall arithmetic Brownian motion (ABM) results in a normally distributed terminal distribution. Based on the notation presented in Module 5.3, we illustrate computing option Greeks within the ABM binomial option valuation model for both European-style and American-style options. For comparison purposes, we present this module with the same format as Module 8.1 that focuses on GBM-based binomial models.

Central finance concepts
The main idea is once we have a robust ABM-based valuation model, we are now able to explore various static risk measures. After reviewing the valuation models introduced in Module 5.3, we explore option Greeks that are simply SRMs.

ABM-based European-style binomial option valuation models
Recall the ABM-based binomial option framework is designed to converge to a *normal* distribution in the limit to be consistent with the ABMOVM. This binomial framework has several objectives:
1. Additive
2. Recombining
3. Incorporate dividends
4. Address early exercise with American-style options

Additive and recombining are incorporated using u and d parameters at each node.

There are several ABM-based multiperiod valuation models including when there are no dividends, when a dividend yield is assumed, and when discrete dividends are assumed. Further, there are several alternative ways to frame these models such as based on digital valuation models.

ABM-based American-style binomial option valuation models
For American-style options, he early exercise potential must be incorporated. As discussed below, the approach typically taken is known as backward induction. At each node, we must compare the following values, the model option value, the early exercise value, and the lower boundary condition. The existence of various forms of dividends simply changes the required formulas.

Binomial option valuation model Greeks
In the quantitative materials below, we explore delta, gamma, theta, vega, and rho, also known as the Greeks. The definitions of Greeks are independent of valuation model. Delta measures an option value's sensitivity to changes in the underlying instrument's price. Gamma measures the delta's sensitivity to changes in the underlying instrument's price. Vega (also known as kappa, lambda, and sigma) measures an option price's sensitivity to changes in the underlying asset's volatility. Theta measures an option price's sensitivity to changes in the time to maturity. Rho measures an option price's sensitivity to changes in the interest rate.

Quantitative finance materials
After a detailed review of various valuation models, we take a deep dive into SRMs related to ABM-based binomial option valuation models.

ABM European-style multiperiod option model
Recall the GBM European-style multiperiod option model results in a recombining tree in both outcomes as well as probabilities and can be expressed as

$$\begin{aligned} O_0 &= PV_r\left[E_0\left(O_T\right)\right] \\ &= PV_r\left[\sum_{j=0}^{n} \Pr\left(n,j\right) Payoff\left(\iota,n,j\right)\right] \\ &= PV_r\left\{\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left[0,\iota\left(S_0 u^j d^{n-j}-X\right)\right]\right\} \end{aligned}, \tag{8.2.1}$$

where O_0 denotes the current call or put value, ι denotes an indicator function that equals +1 if call and –1 if put, and PV_r is simply a present value factor.

Further, recall for ABM European-style multiperiod option model the probabilities are path dependent due to the geometric growth rate assumed for the underlying instrument. That is,

$$\begin{aligned} O_0 &= PV_r\left[E_0\left(O_T\right)\right] \\ &= PV_r\left[\sum_{j=0}^{n} \Pr\left(n,j\right) Payoff\left(\iota,n,j\right)\right] \\ &= PV_r\left[\sum_{j=0}^{n} \Pr\left(n,j\right)\max\left\{0,\iota_u\left[S_0+ju+\left(n-j\right)d-X\right]\right\}\right] \end{aligned}. \tag{8.2.2}$$

There are several ways to handle the computation of $\Pr(n,j)$ and recall we deployed backward recursion for both the European-style and American-style options.

ABM American-style multiperiod option model

The process for valuing American-style options is like European-style options. The only difference is that at each step, except the point in time of expiration, we consider whether early exercise is more valuable than continuation. Also, we consider whether there is a violation of lower boundary conditions. From Module 8.2.1, we repeat the conditions to evaluate at each node as we work backward through the tree.

We know that at time i for j up moves, the binomial model value (denoted with B superscript) can be expressed as

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\phi_{i,j}O_{i+1,j+1}+\left(1-\phi_{i,j}\right)O_{i+1,j}\right], \tag{8.2.3}$$

where $PV_{r,i,\Delta t}\left(\ \right)$ denotes the present value at time i for the next Δt period based on the continuously compounded rate r and as defined before $\phi_{i,j}=\dfrac{S_{i,j}\left(e^{r\Delta t}-1\right)-d}{u-d}$. With constant interest rates, we have $PV_{r,i,\Delta t}\left(1\right)=e^{-r\Delta t}$. The binomial model value, however, may be lower than the early exercise value (denoted with superscript X) that can be expressed as

$$O_{i,j}^X = \max\left[0,\iota_U\left(S_{i,j}-X\right)\right]. \tag{8.2.4}$$

Recall the lower boundary condition (denoted with superscript L) is

$$O_{i,j}^L = \max\left\{0,\iota_U\left[S_{i,j}-PV_{r,i,n-i}\left(X\right)\right]\right\}. \tag{8.2.5}$$

Thus, the fair value of the American-style option at time i with j up moves is

$$O_{i,j} = \max\left(O_{i,j}^B,O_{i,j}^X,O_{i,j}^L\right). \tag{8.2.6}$$

Note assuming positive interest rates and no dividends $O_{i,j}^L \ge O_{i,j}^X$ for call options and $O_{i,j}^L \le O_{i,j}^X$ for put options. The initial option value is obtained through backward induction along the binomial lattice for the underlying instrument. Recall with European-style options, the fair value at time i with j up moves is

$$O_{i,j} = \max\left(O_{i,j}^B,O_{i,j}^L\right). \tag{8.2.7}$$

Binomial option valuation model Greeks
We follow the definitions and procedures describe in Module 8.1 closely. For convenience, we reproduce key equations.
Delta
Delta is defined as

$$\Delta_O \equiv \frac{\partial O}{\partial S}. \tag{8.2.8}$$

Within the binomial lattice, delta can be estimated in three ways,

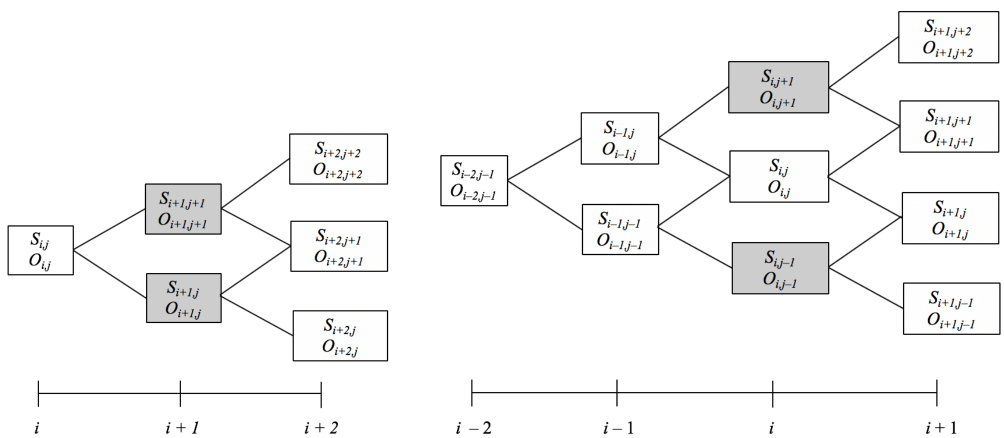
$$\Delta_{O,i,j} = \frac{O_{i+1,j+1} - O_{i+1,j}}{S_{i+1,j+1} - S_{i+1,j}}, \text{ (\textbf{Standard Binomial Method})} \tag{8.2.9}$$

$$\Delta_{O,i,j} = \frac{O_{i,j+1} - O_{i,j-1}}{S_{i,j+1} - S_{i,j-1}}, \text{ and (\textbf{Enhanced Binomial Method})} \tag{8.2.10}$$

$$\Delta_{O,i,j} = \frac{O(S+h) - O(S-h)}{2h}. \text{ (\textbf{Numerical Method})} \tag{8.2.11}$$

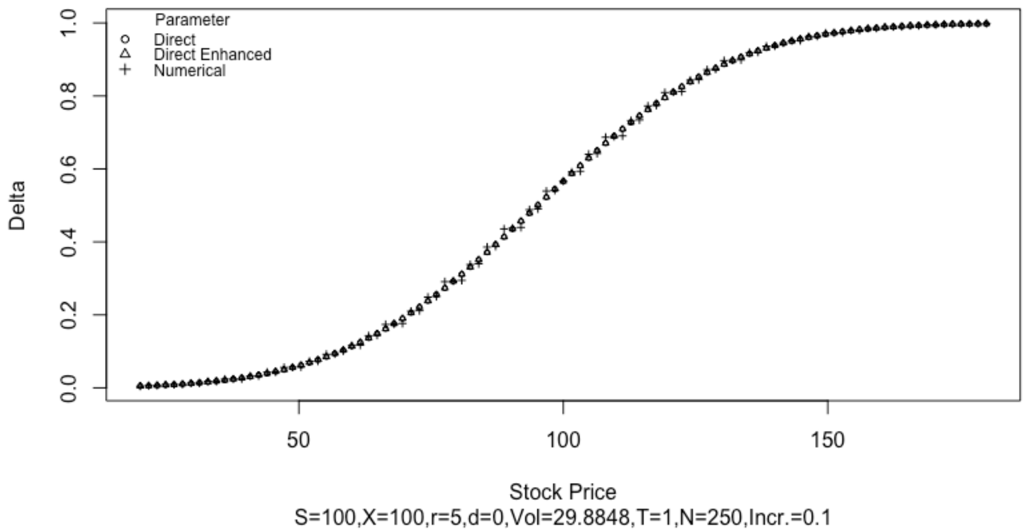
As Figure 8.2.1 illustrates, the enhanced method adds two additional time steps to align time to maturity.

Figure 8.2.1. Illustration of standard and enhanced delta within ABM-based binomial models
Standard Binomial Delta ***Enhanced Binomial Delta***



In most cases, the method of choice renders numerically similar results. Figure 8.2.2 illustrates all three methods of estimating delta. The two binomial methods are indistinguishable, and the numerical method is extremely close, but it oscillates across stock prices as seen better in Panel B.

Figure 8.2.2. Three methods to estimate ABM-based European-style binomial call delta
Panel A. Wide range of stock prices



Panel B. Narrow range of stock prices

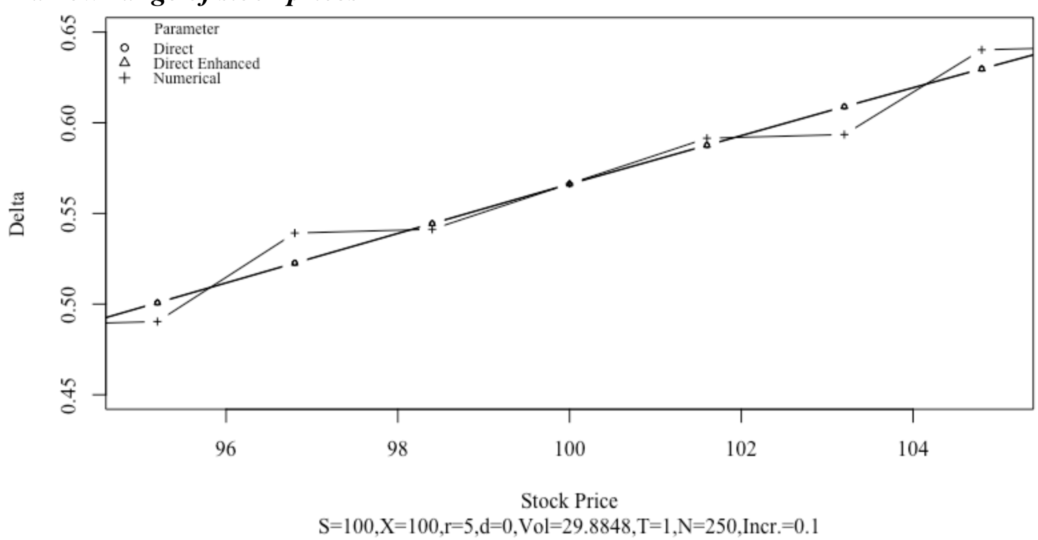
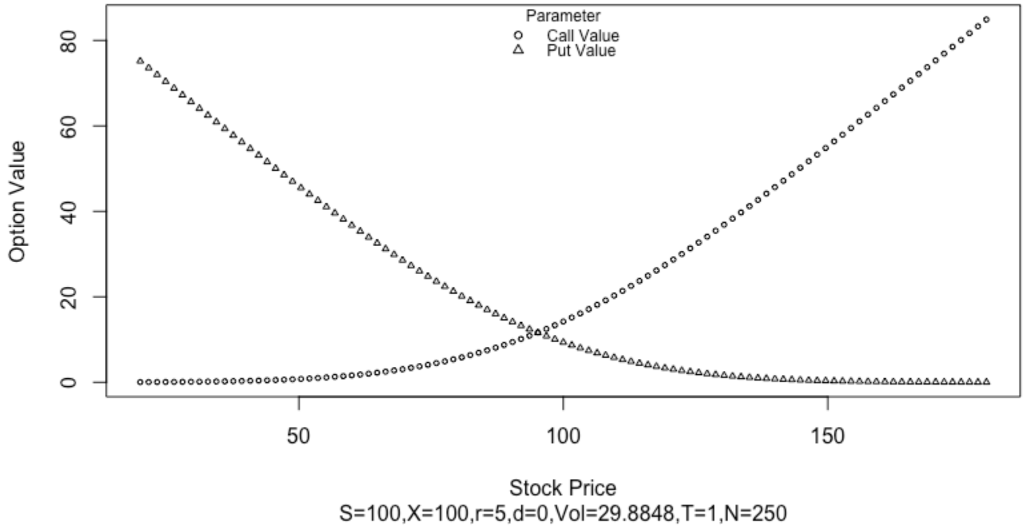


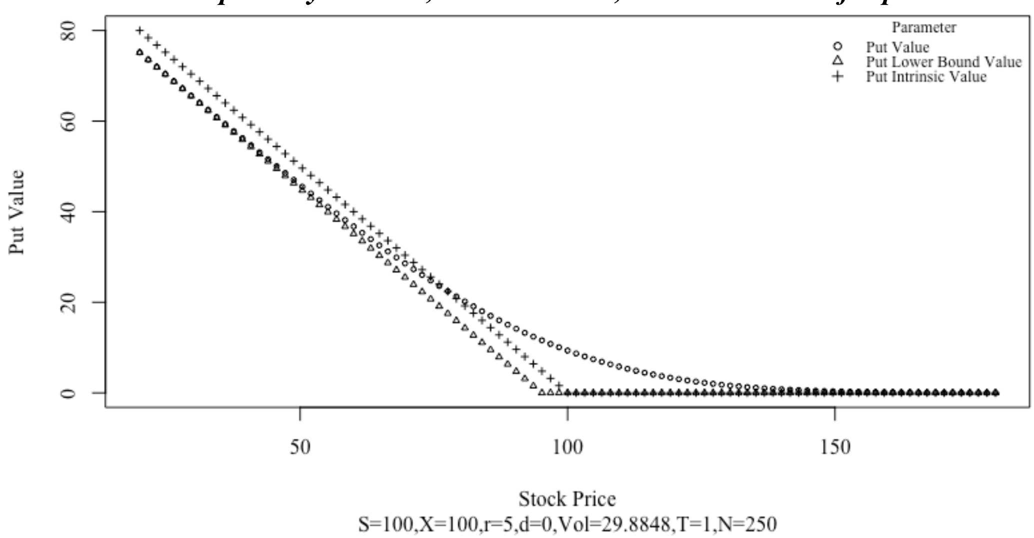
Figure 8.2.3 illustrates the stock price on the horizontal axis and the option prices on the vertical axis. The positive sloped line is the call value, and the negative sloped line is the put value.

Figure 8.2.3. ABM-based European-style binomial model for calls and puts



Recall for puts it is theoretically possible for the put option's value to fall below its intrinsic value [=max(0,$X - S_0$)] when interest rates are positive because the lower bound is max[0,$PV(X) - S_0$]. Figure 8.2.4 (Panel A) illustrates the results for the puts. Panel B illustrates the call option's value, and it will never fall below the intrinsic value [=max(0,$S_0 - X$)] because the lower bound is max[0,$S_0 - PV(X)$] is above it for positive interest rates.

Figure 8.2.4. Call and put values based on the ABM-based binomial model
Panel A. ABM-based European-style values, intrinsic value, and lower bound for puts



Panel B. ABM-based European-style values, intrinsic value, and lower bound for calls

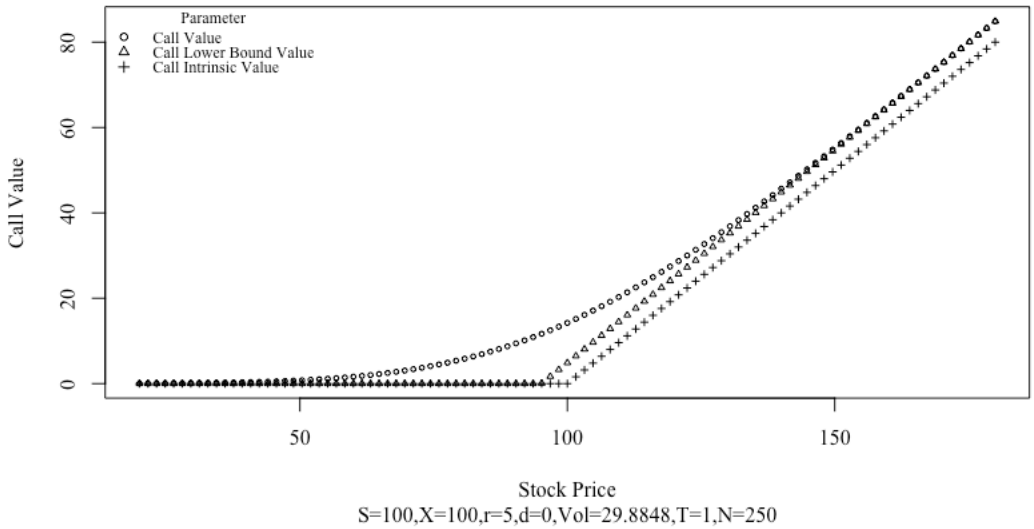
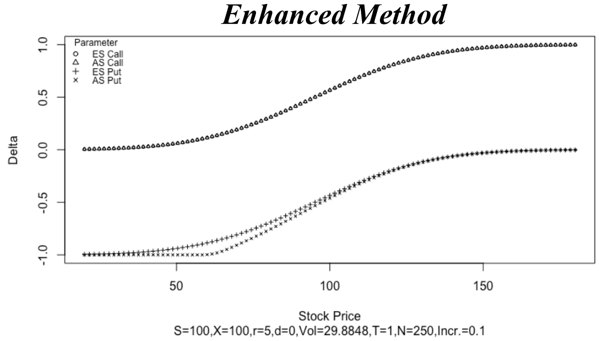


Figure 8.2.5 shows the enhanced method and numerical method for estimating deltas with the binomial model assuming no dividends for both puts and calls as well as European-style and American-style options. Notice that with 250 time steps, the numerical method lacks smoothness whereas the enhanced method is relatively smooth. For deep in-the-money American-style puts, the boundary condition results in a delta of –1.0 due to early exercise being optimal. Again, the enhanced method is virtually indistinguishable from the standard method and is not reported here.

Figure 8.2.5. ABM-based call and put deltas based on binomial model without dividends

Enhanced Method



Numerical Method

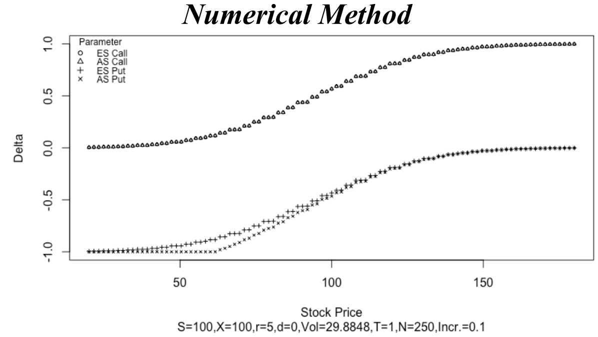
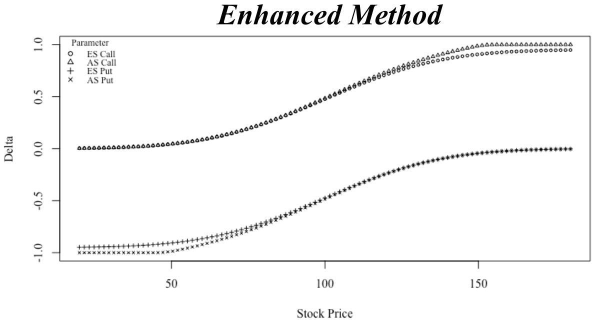


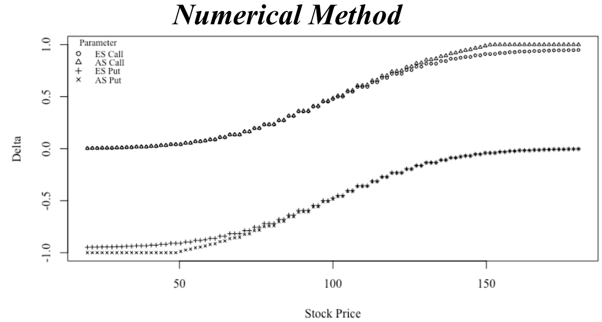
Figure 8.2.6 shows the enhanced method and numerical method for estimating deltas with the binomial model assuming a 5% dividend yield. For deep in-the-money puts and calls, the boundary condition are obtained.

Figure 8.2.6. ABM-based call and put deltas based on binomial model with dividends

Enhanced Method



Numerical Method



Gamma

Mathematically, gamma is defined as

$$\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2}. \tag{8.2.12}$$

Within the binomial lattice, gamma can be estimated in three ways,

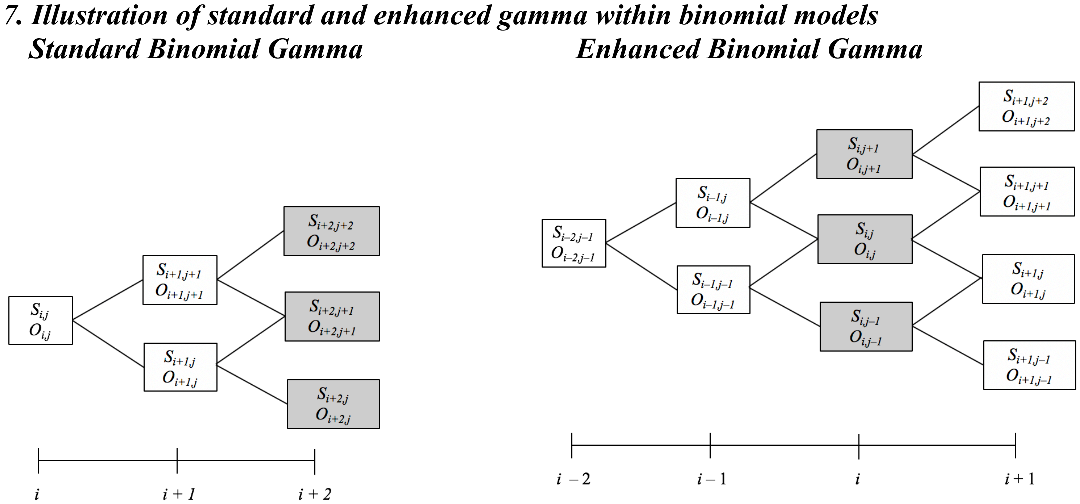
$$\Gamma_{O,i,j} = \frac{\dfrac{O_{i+2,j+2} - O_{i+2,j+1}}{S_{i+2,j+2} - S_{i+2,j+1}} - \dfrac{O_{i+2,j+1} - O_{i+2,j}}{S_{i+2,j+1} - S_{i+2,j}}}{\dfrac{S_{i+2,j+2} - S_{i+2,j}}{2}}. \text{ (\textbf{Standard Binomial Method})} \tag{8.2.13}$$

$$\Gamma_{O,i,j} = \frac{\dfrac{O_{i,j+1} - O_{i,j}}{S_{i,j+1} - S_{i,j}} - \dfrac{O_{i,j} - O_{i,j-1}}{S_{i,j} - S_{i,j-1}}}{\dfrac{S_{i,j+1} - S_{i,j-1}}{2}}. \text{ (\textbf{Enhanced Binomial Method})} \tag{8.2.14}$$

$$\Gamma_{O,i,j} = \frac{\left[O(S+h) - O(S)\right] - \left[O(S) - O(S-h)\right]}{h^2}. \text{ (\textbf{Numerical Method})} \tag{8.2.15}$$

Figure 8.2.7 provides a binomial lattice illustrating the appropriate lattice inputs for both the standard and enhanced methods for calculating gamma.

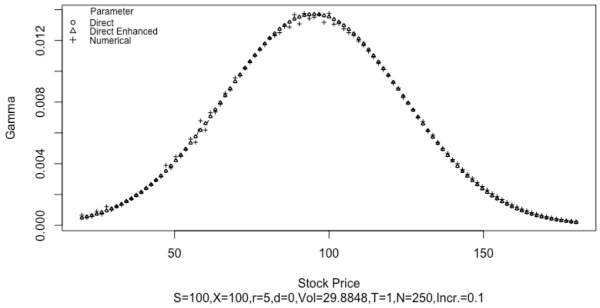
Figure 8.2.7. Illustration of standard and enhanced gamma within binomial models



In most cases, the method of choice renders numerically similar results. Figure 8.2.8 illustrates all three methods of estimating gamma. The two binomial methods are indistinguishable and the numerical method is extremely close, but it oscillates across stock prices. Further, the call and put results are identical.

Figure 8.2.8. Three methods to estimate ABM-based European-style option gamma with no dividends

Call Option



Put Options

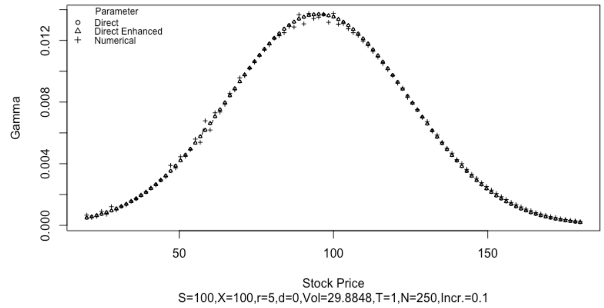
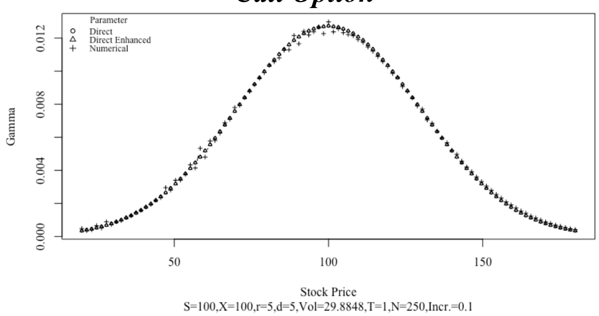


Figure 8.2.9 illustrate the three methods to estimating gamma in the presence of dividends. We assume here a 5% dividend yield.

Figure 8.2.9. Three methods to estimate ABM-based European-style gamma with dividends

Call Option



Put Options

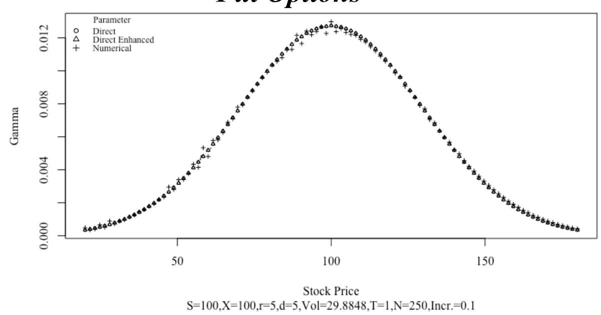
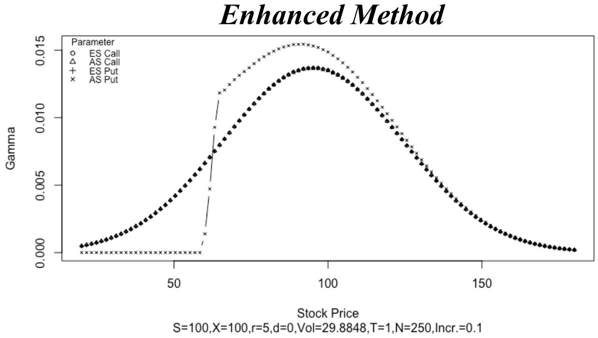


Figure 8.2.10 illustrates the difference between the European-style and American-style options without dividends. In this case, only the put option encounters the boundary condition.

Figure 8.2.10. ABM-based call and put gammas based on binomial model with no dividends

Enhanced Method



Numerical Method

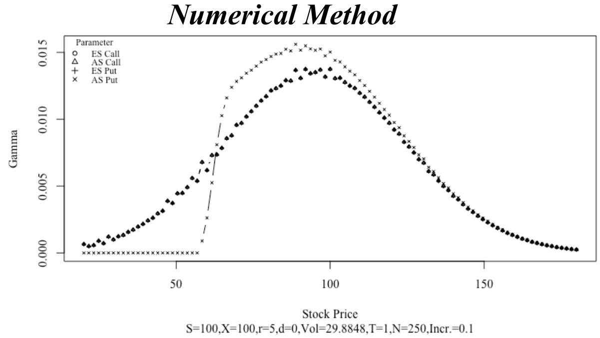
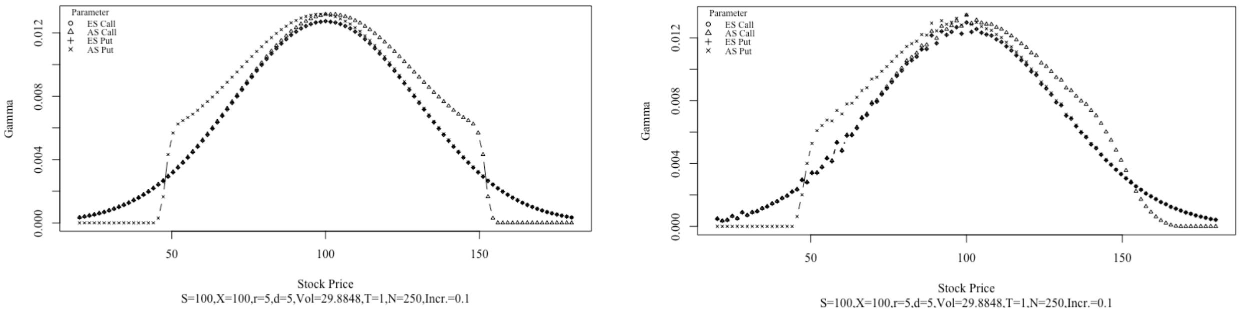


Figure 8.2.11 illustrates the difference between the European-style and American-style options with a 5% dividend yield. Both the American-style call and put encounter the boundary condition and the discontinuity points where the option valuation model encounters the lower bound.

Figure 8.2.11. Call and put gammas based on binomial model with dividends

Enhanced Method ***Numerical Method***



Theta

Mathematically, theta is defined as

$$\theta_O \equiv \frac{\partial O}{\partial t}. \tag{8.2.16}$$

Within the binomial lattice, theta can be estimated in three ways,

$$\theta_{O,i,j} = \frac{O_{i+2,j+1} - O_{i,j}}{2\Delta t}. \text{ (**Standard Binomial Method**)} \tag{8.2.17}$$

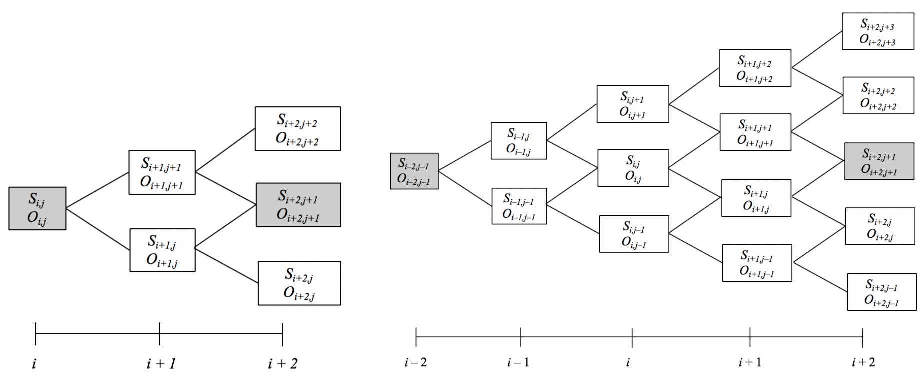
$$\theta_{O,i,j} = \frac{O_{i+2,j+1} - O_{i-2,j-1}}{4\Delta t}. \text{ (**Enhanced Binomial Method**)} \tag{8.2.18}$$

$$\theta_{O,i,j} = \frac{O(t+h) - O(t-h)}{2h}. \text{ (**Numerical Method**)} \tag{8.2.19}$$

Figure 8.2.12 provides a binomial lattice illustrating the appropriate lattice inputs for both the standard and enhanced methods for calculating theta.

Figure 8.2.12. Illustration of standard and enhanced theta within binomial models

Standard binomial theta ***Enhanced binomial theta***

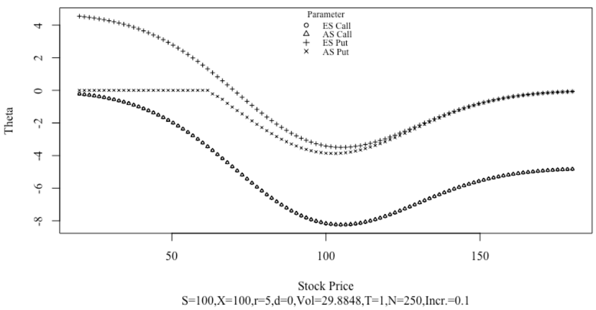


In most cases, the method of choice renders numerically similar results. Figure 8.2.13 illustrates all three methods of estimating theta without and with dividends. As seen in Panel A, without dividends the two binomial methods are indistinguishable and the numerical method is extremely close, but it oscillates across stock prices. Panel B illustrates the influence of dividends.

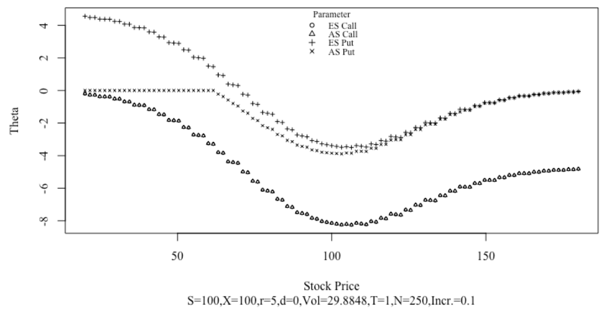
Figure 8.2.13. Illustration of theta within ABM-based binomial model

Panel A. Without dividends

Enhanced Method

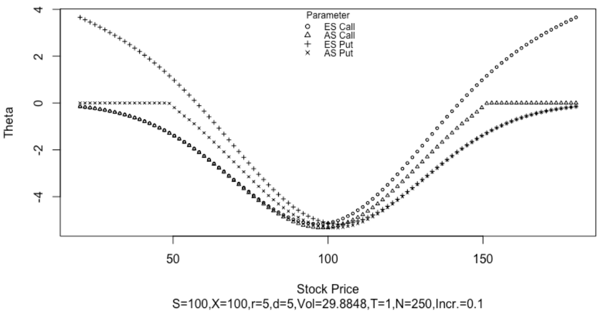


Numerical Method

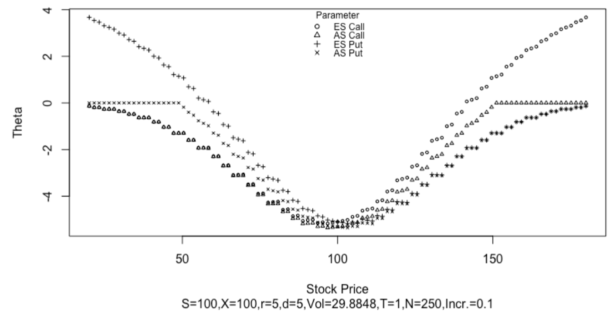


Panel B. With dividends

Enhanced Method



Numerical Method



Although the pattern is roughly similar, the existence of dividends changes the theta values for both European and American style options.

Vega

Mathematically, vega is defined as

$$\nu_O \equiv \frac{\partial O}{\partial \sigma}. \tag{8.2.20}$$

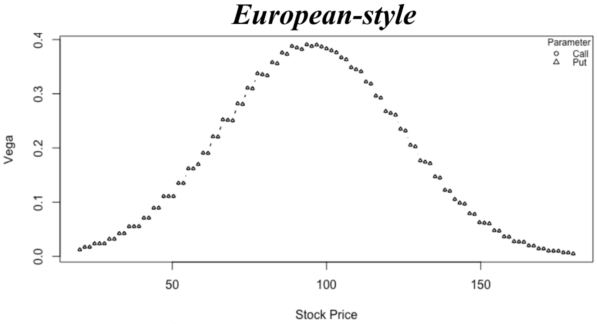
Within the binomial lattice, vega can be estimated based on the numerical method as

$$\nu_{O,i,j} = \frac{O_{\sigma+h,i,j} - O_{\sigma-h,i,j}}{2h}. \tag{8.2.21}$$

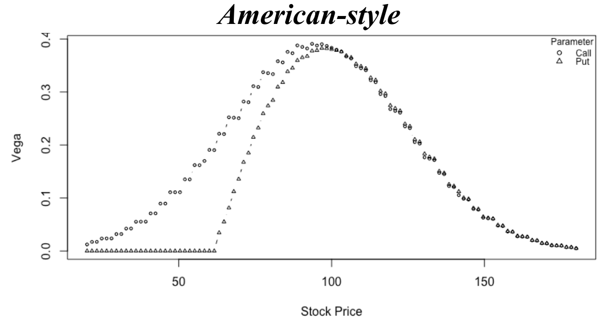
Vega is the first derivative of the option value with respect to volatility. Neither the stock nor the risk-free interest rate is assumed to be influenced by changes in the stock's volatility. Thus, volatility impacts both calls and puts the same based on put-call parity. Figure 8.2.14 illustrates this result without dividends.

Figure 8.2.14. Illustration of vega within binomial model without dividends

European-style



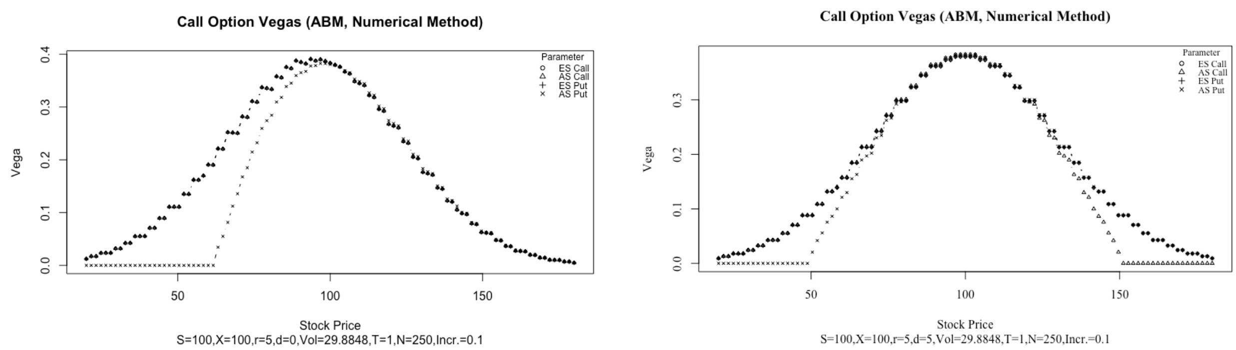
American-style



Recall for call options that are deep out-of-the-money, the call price changes very little with a small change in volatility (it does not really change the probability of the stock reaching the strike price), hence the vega is close to zero. The same is true for deep out-of-the-money puts. The relationship between the stock price and vega is illustrated in Figure 8.2.15.

Figure 8.2.15. Vega with respect to stock price within binomial model with and without dividends

No dividends ***Dividend yield (5%)***



Rho

Mathematically, rho is defined as

$$\rho_O \equiv \frac{\partial O}{\partial r}. \tag{8.2.22}$$

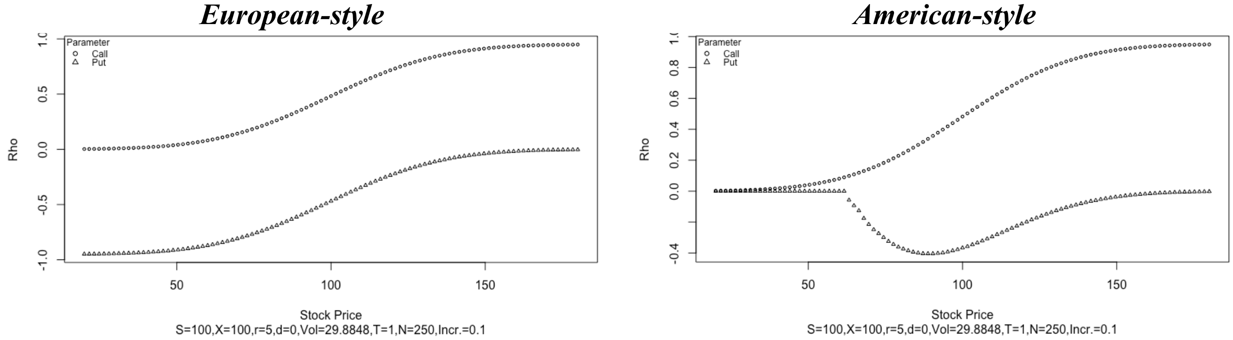
Within the binomial lattice, rho can be estimated based on the numerical method as

$$\rho_{O,i,j} = \frac{O_{r+h,i,j} - O_{r-h,i,j}}{2h}. \tag{8.2.23}$$

Rho is the first derivative of the option value with respect to the interest rate. Neither the stock nor volatility is assumed to be influenced by changes in the interest rate. Thus, the interest rate impacts calls different from puts based on put-call parity. Recall there is the present value of the exercise price in put-call parity giving an additional influence of interest rates. This result is illustrated in Figure 8.2.16.

Figure 8.2.16. Illustration of rho within binomial model without dividends

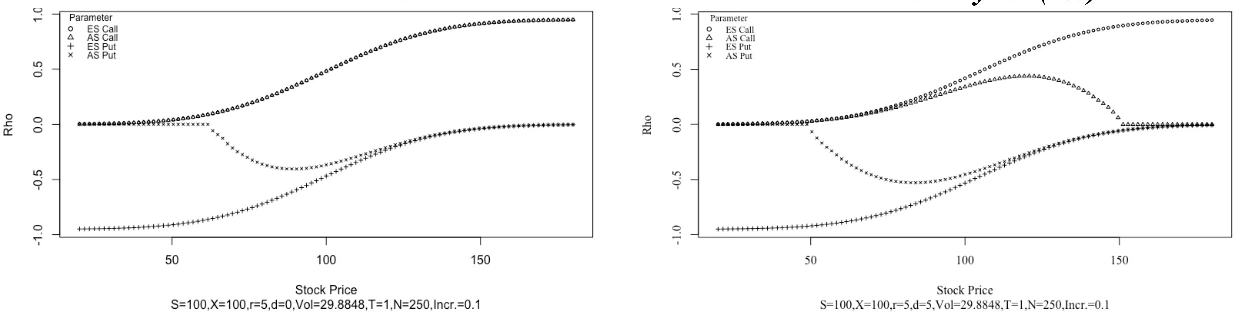
European-style ***American-style***



Recall for call options that are deep out-of-the-money options, the option price changes very little with a small change in interest rates, hence the rho is close to zero. The relationship between the stock price and vega is illustrated in Figure 8.2.17.

Figure 8.2.17. Rho with respect to stock price within binomial model with and without dividends

No dividends | *Dividend yield (5%)*



Summary

We illustrated how to compute option Greeks within the GBM binomial option valuation model for both European-style and American-style options.

In this chapter we covered one of the simplest but most important methods of valuing options: the binomial model. We showed how the model clearly illustrates the process by which a dynamically adjusted portfolio enables one to assign a value to an option that must hold to prevent arbitrage. We showed how this process works in one- and two-period models, and we also showed how the general binomial formula and Pascal's triangle illustrates the extension to a multi-period world. We illustrated how the early exercise of American options is easily accommodated within the binomial model.

References

See Module 5.3.