Module 8.5: Static Risk Measures Geometric Brownian Motion-Based Compound Option Valuation Models

Learning objectives

- Explore static risk measures related to GBM compound options
- Illustrates the complex task of finding analytic Greeks
- Highlights ease of using numerical Greeks

Executive summary

In this module, we introduce various static risk measures related to geometric Brownian motion-based compound option valuation models (GBM-COVM). We examine the sensitivity of the compound option values and Greeks to changes in selected underlying parameters. In the quantitative finance materials section, we review the valuation model and derive several Greeks. Finally, we conclude this module with a brief contrast between numerical and analytic Greek computations.

Central finance concepts

Our focus here is on the sensitivity of the GBM-COVM Greeks to changes in selected underlying parameters.

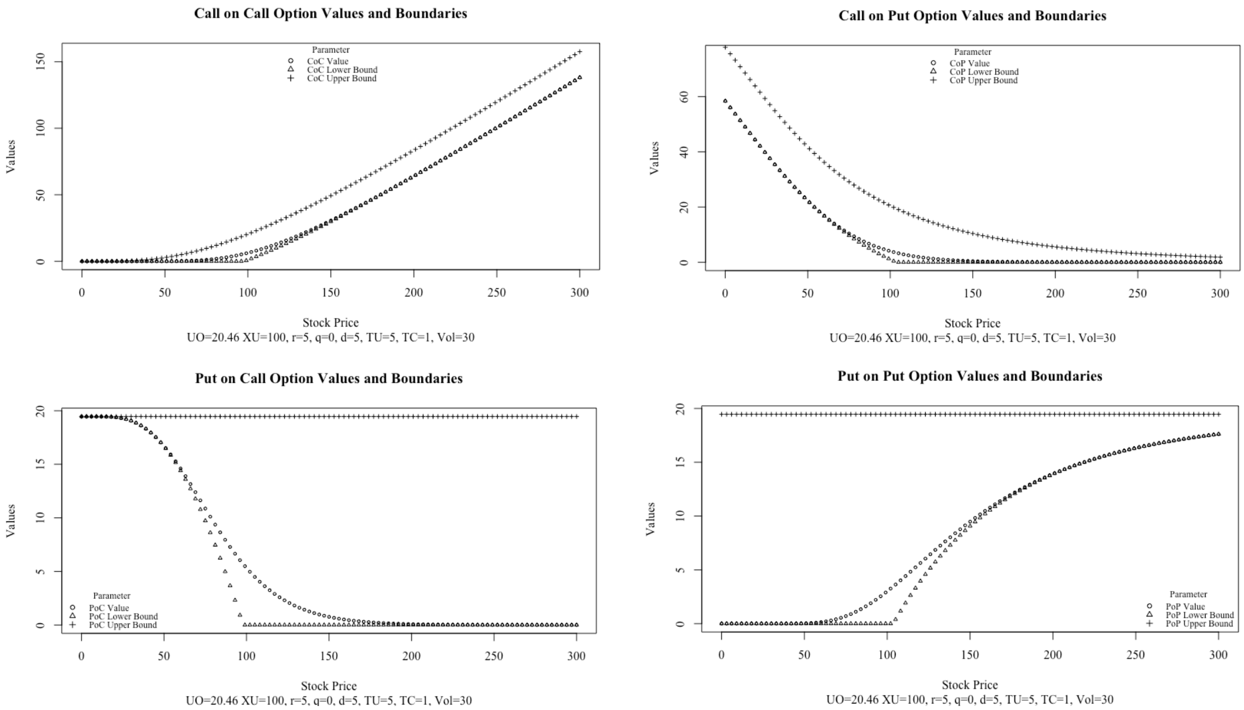
Selected Results

We provide numerous plots related to static risk measures available within the compound option framework. Specifically, we explore the sensitivities to the underlying instrument, underlying volatility, option yield, and passage of calendar time.

Sensitivity to underlying instrument

We now provide numerous plots related to compound options sensitivity to the underlying instrument value, the stock price in this example. In Figure 8.5.1 we see the call on call value rises with the stock price. As the underlying call value rises so does the call compound option on the underlying call option. The put on put value also rises with the stock price. As the underlying stock price rises, the underlying put value falls. The corresponding put compound option rises because the underlying put value falls.

Figure 8.5.1. Compound Option Value Sensitivity to the Underlying Instrument



Both the call on put and put on call decline with increases in the stock price. Note that as the stock price rises, then the underlying put value fall and hence a call on the underlying put also falls. Also, as the stock price rises, then the underlying call value rises and hence a put on the underlying call also falls.

Figure 8.5.2 presents the compound option delta sensitivity to the underlying instrument. For the call on call, the pattern of delta is like a plain vanilla call delta except the upper bound does not converge quickly to 1.0. For the call on put, the pattern of delta is like a plain vanilla put delta except the lower bound does not converge quickly to –1.0. Both the put on call and put on put have unusual delta patterns when compared to plain vanilla options. The put on call delta reflects the negative sensitivity to the underlying instrument whereas the call on put delta reflects the positive sensitivity. The put on put delta sensitivity is like a plain vanilla gamma and seems to reflect the lognormal distribution.

Figure 8.5.2. Compound Option Delta Sensitivity to the Underlying Instrument

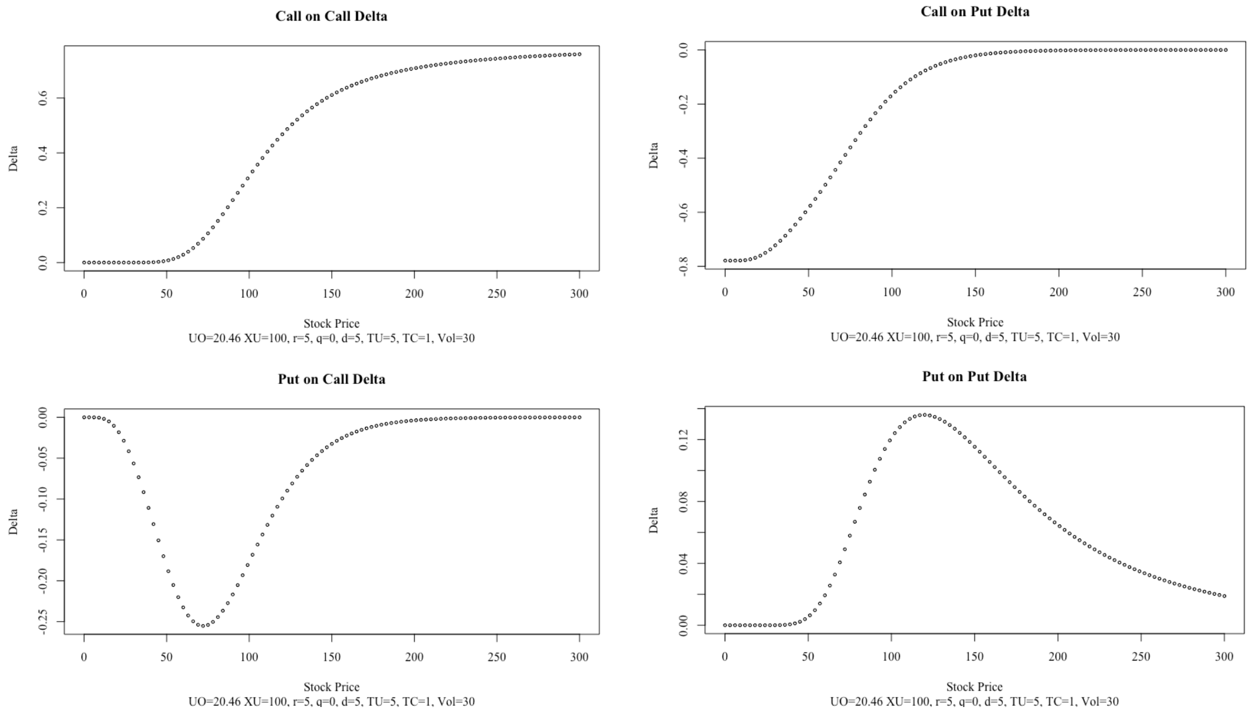


Figure 8.5.3 presents the compound option gamma sensitivity to the underlying instrument. For the call on call, the pattern of gamma is like a plain vanilla call and put gamma also reflecting the positive skew in the lognormal distribution. For the call on put, the pattern of gamma is somewhat like a plain vanilla gamma but is clearly distorted. Again, like delta, both the put on call and put on put have unusual gamma patterns when compared to plain vanilla options. The oscillations move generally in opposite directions but not identical in magnitude.

Figure 8.5.3. Compound Option Gamma Sensitivity to the Underlying Instrument

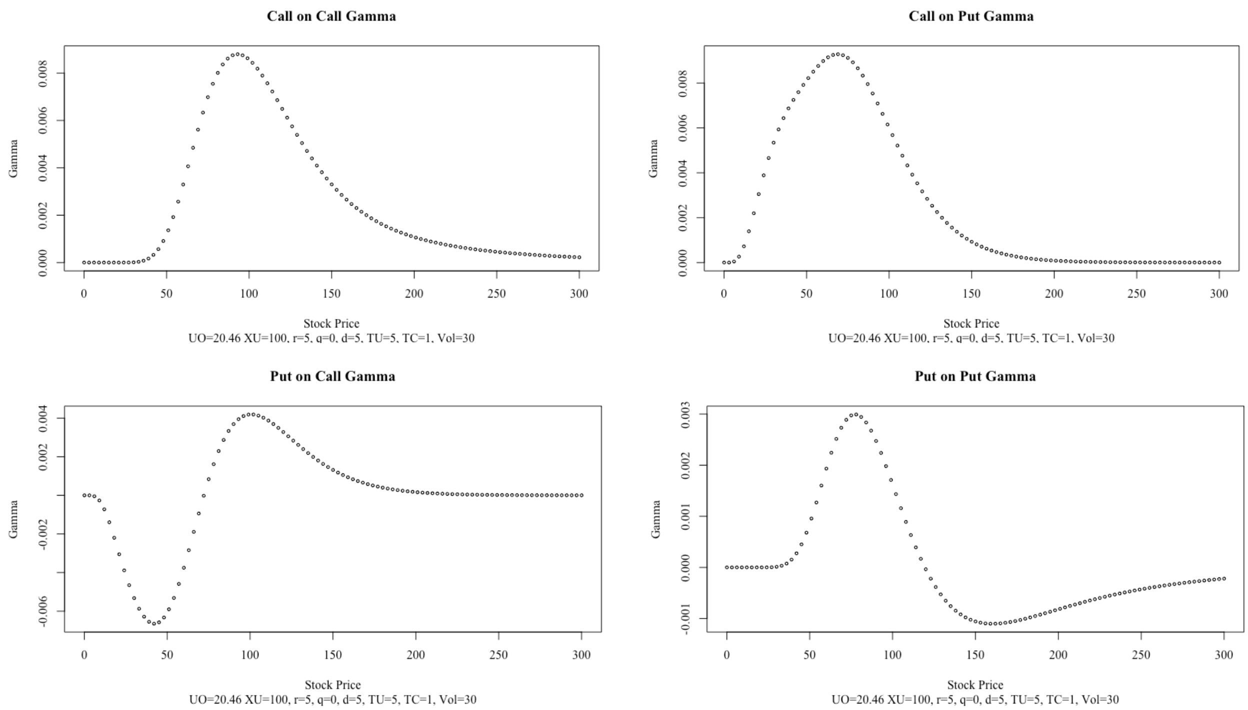


Figure 8.5.4 presents the compound option theta sensitivity to the underlying instrument. As expected, when the compound option value tends to zero, then the theta also tends to zero—low stock prices for call on calls and put on puts and high stock prices for call on puts and put on calls. For deep in-the-money compound options, the theta is reflecting the behavior of the lower boundary. For example, the underlying call option theta for deep in-the-money options is negative. Further, the compound option present value of the strike price also has a negative impact, thus the overall effect is negative for deeper in-the-money call on calls. For put on puts, the underlying put option theta for deep in-the-money options is positive. Further, the compound option present value of the strike price also has a positive impact, thus the overall effect is positive for deeper in-the-money put on puts. For call on puts, as the stock price declines the underlying put is deep in-the-money and hence has a positive theta. The influence of this positive theta more than offset the negative influence of the compound option strike price.

Figure 8.5.4. Compound Option Theta Sensitivity to the Underlying Instrument

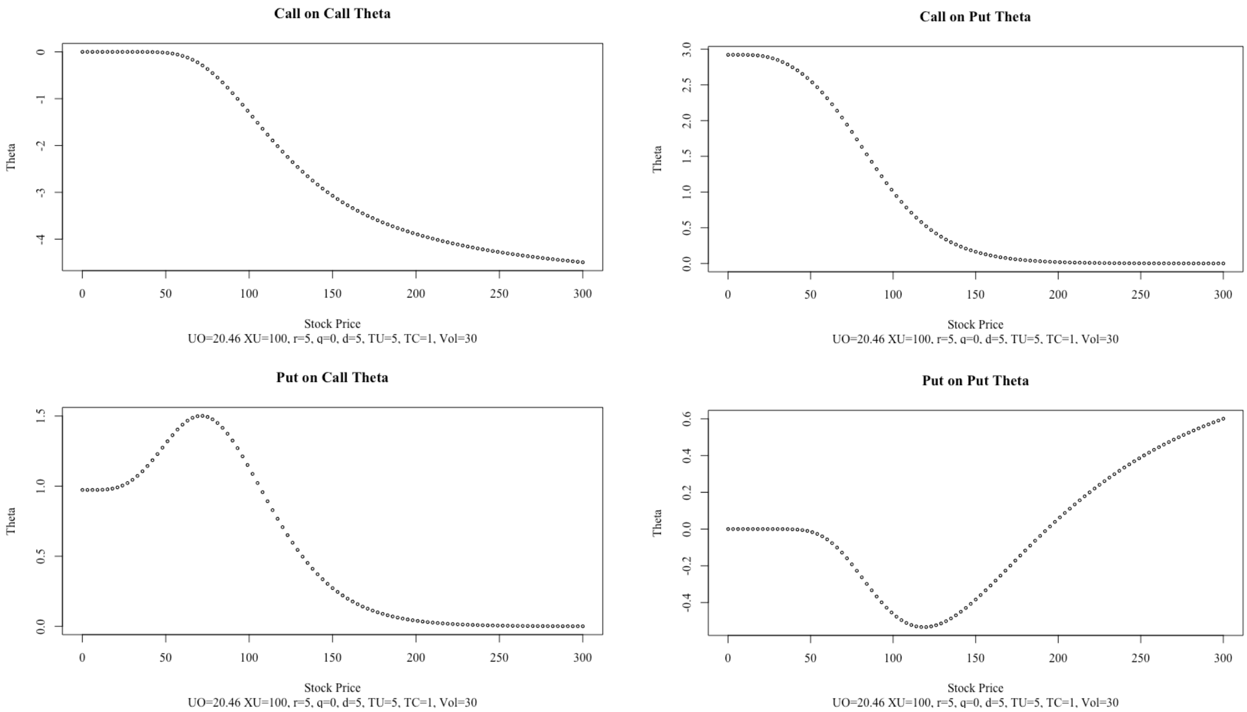


Figure 8.5.5 presents the compound option vega sensitivity to the underlying instrument. Recall, plain vanilla vegas are not sensitive to boundary conditions once the option is sufficiently deep in-the-money because volatility is assumed not to influence the underlying instrument value and volatility does not influence the present value of the underlying strike price. Similarly, at the extremes, compound option vegas converge to zero. The assumed lognormal distribution is clear in these figures, particularly call on put vegas. Recall an increase in volatility increases option values; hence put on calls and put on puts vegas are negative.

Figure 8.5.5. Compound Option Vega Sensitivity to the Underlying Instrument

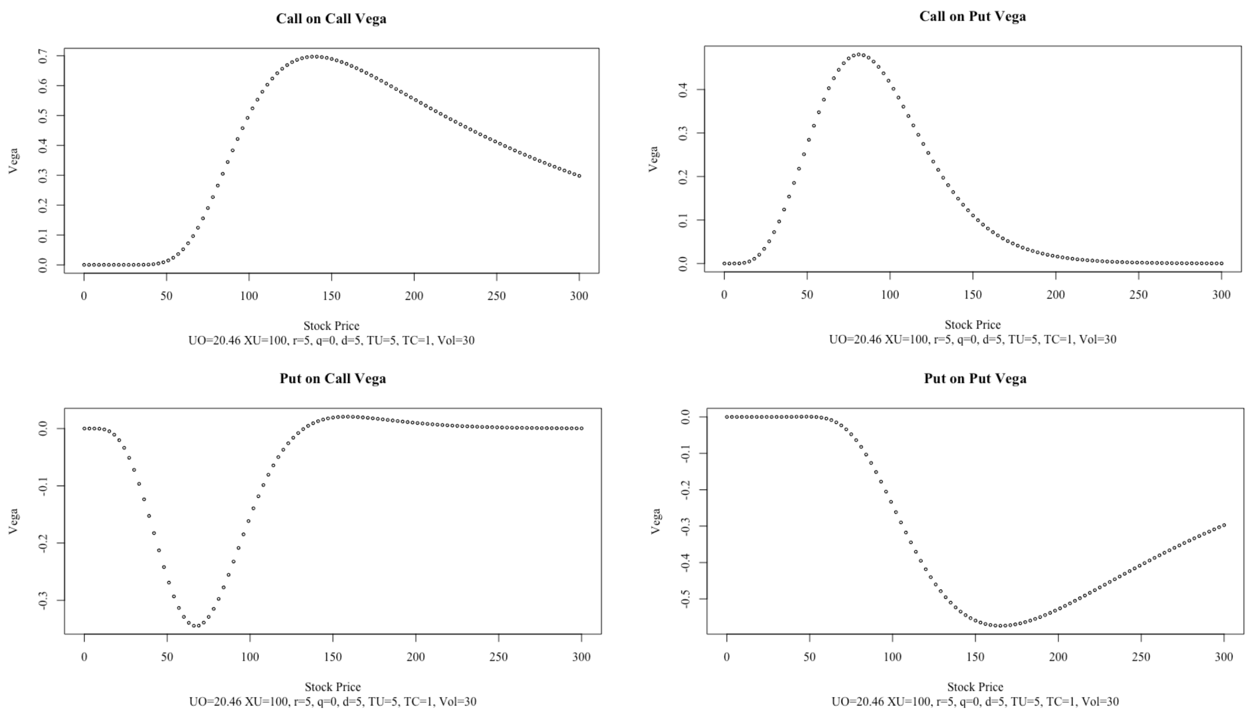
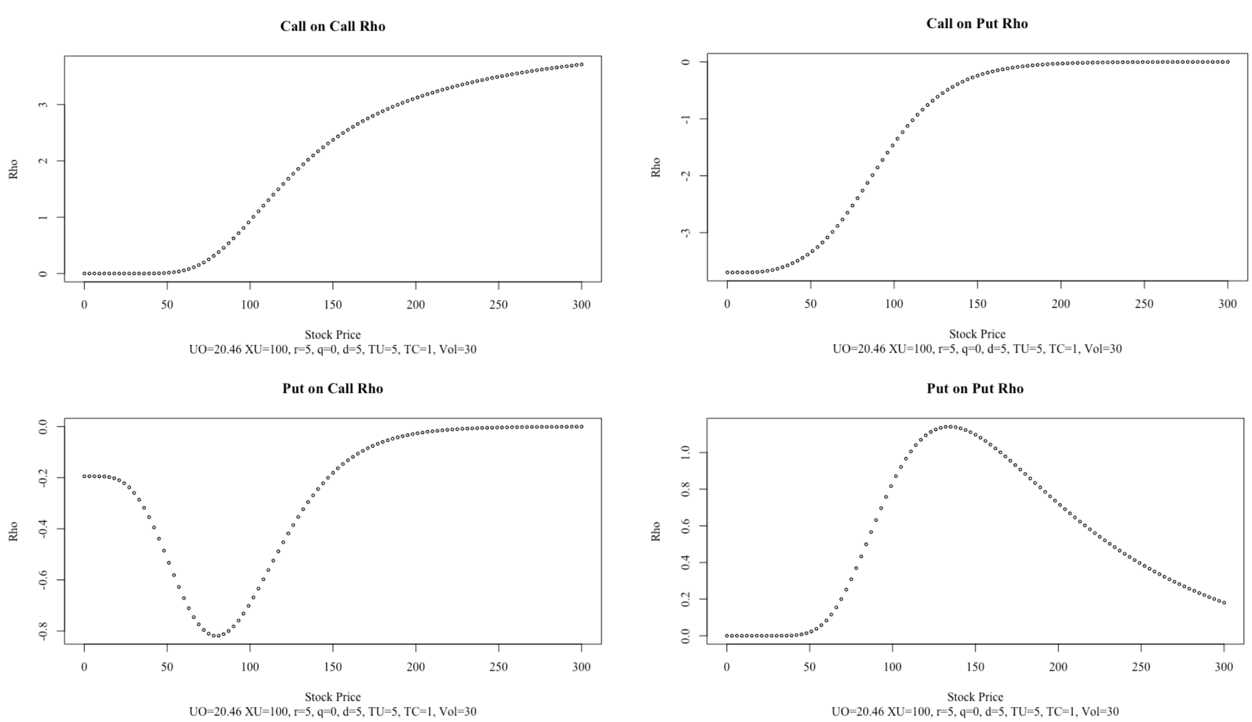


Figure 8.5.6 presents the compound option rho sensitivity to the underlying instrument. Recall, plain vanilla call rhos are positive and range from 0 to 1 whereas plain vanilla put rhos are negative and range from –1 to 0. Thus, the pattern of call on calls (call on puts) follows plain vanilla call (puts) but with greater magnitude. Put on calls and put on puts appear to reflect the lognormal distribution.

Figure 8.5.6. Compound Option Rho Sensitivity to the Underlying Instrument



Sensitivity to volatility

We now provide numerous plots related to compound options sensitivity to underlying instrument volatility. Namely, we vary the relative volatility of the underlying instrument based on geometric Brownian motion. Given the sheer number of figures presented below, we offer them without commentary.

Figure 8.5.7. Compound Option Value Sensitivity to Volatility

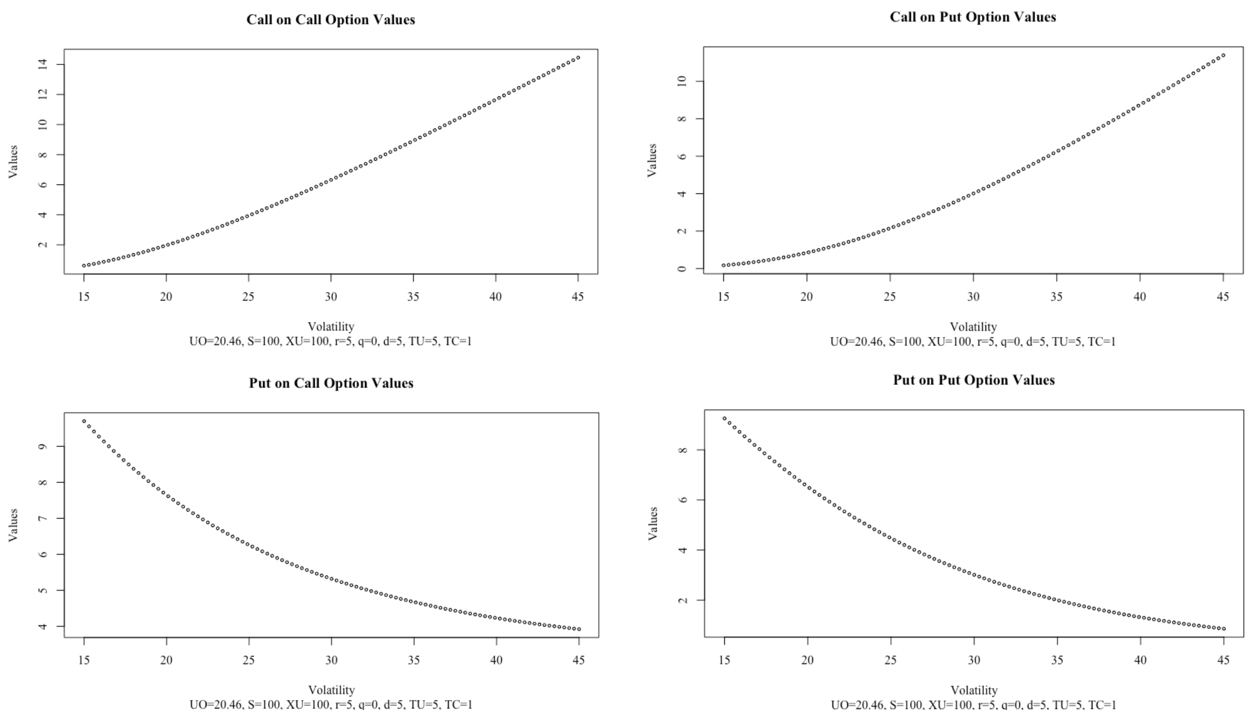


Figure 8.5.8. Compound Option Delta Sensitivity to Volatility

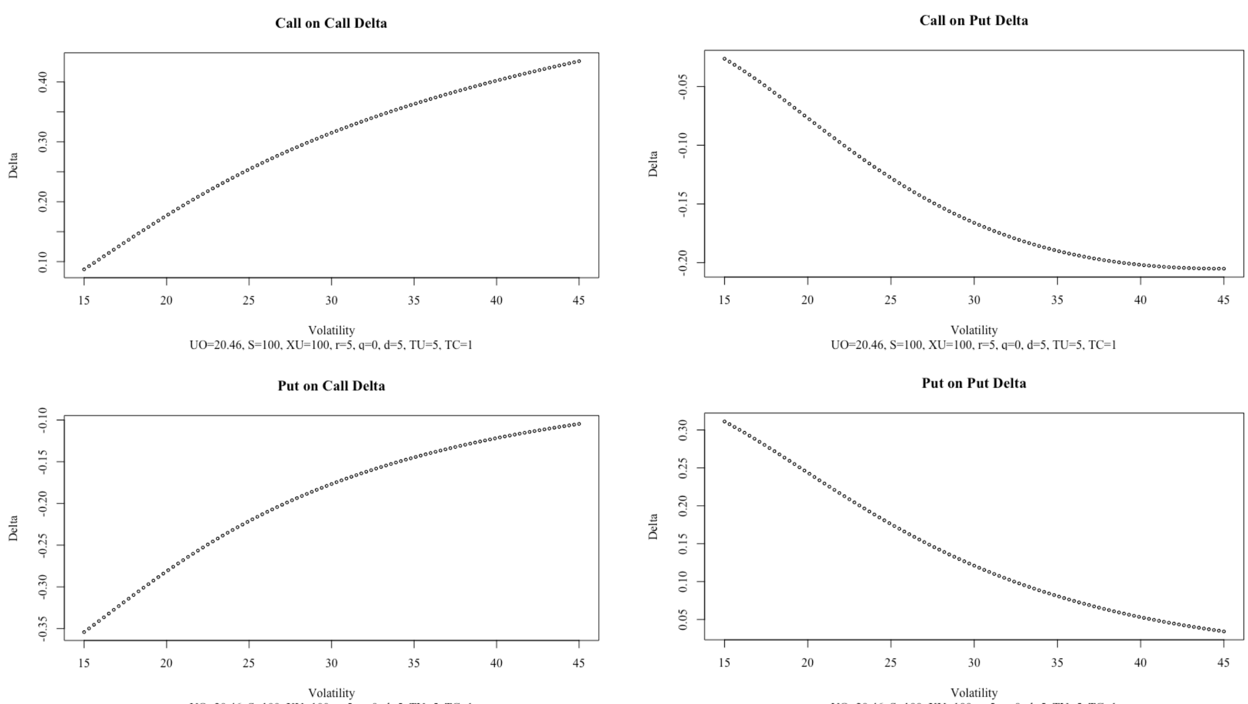


Figure 8.5.9. Compound Option Gamma Sensitivity to Volatility

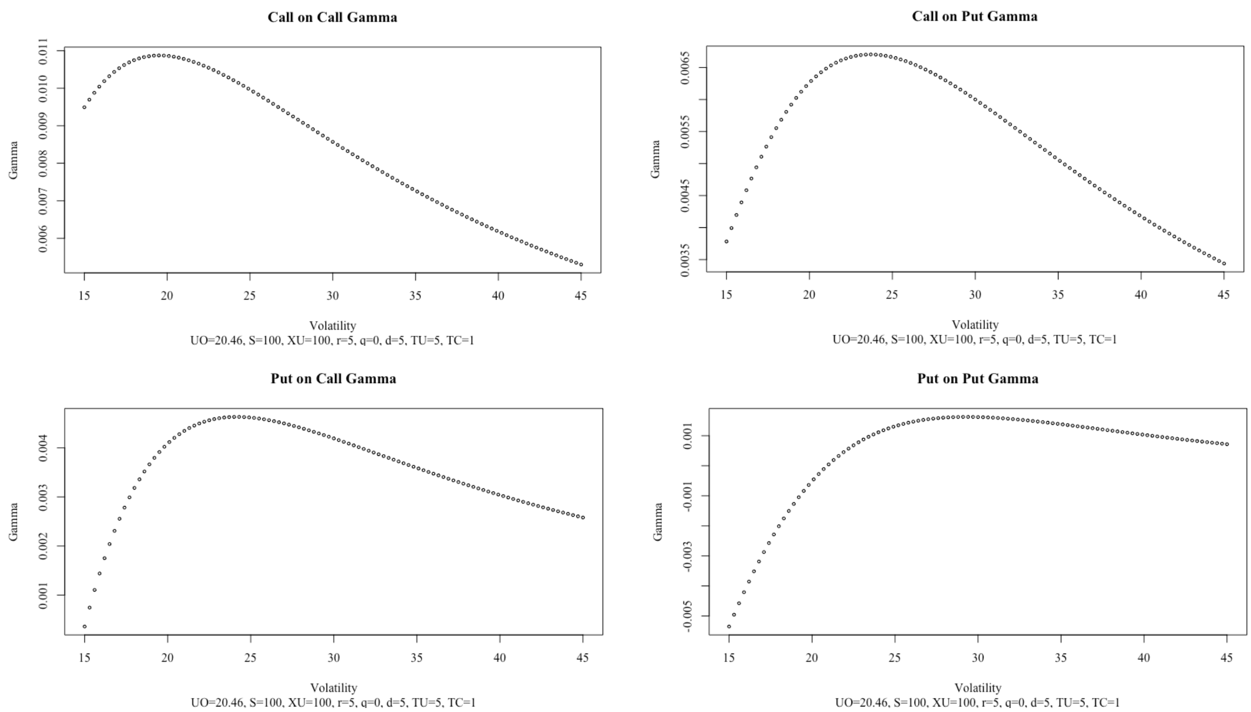


Figure 8.5.10. Compound Option Theta Sensitivity to Volatility

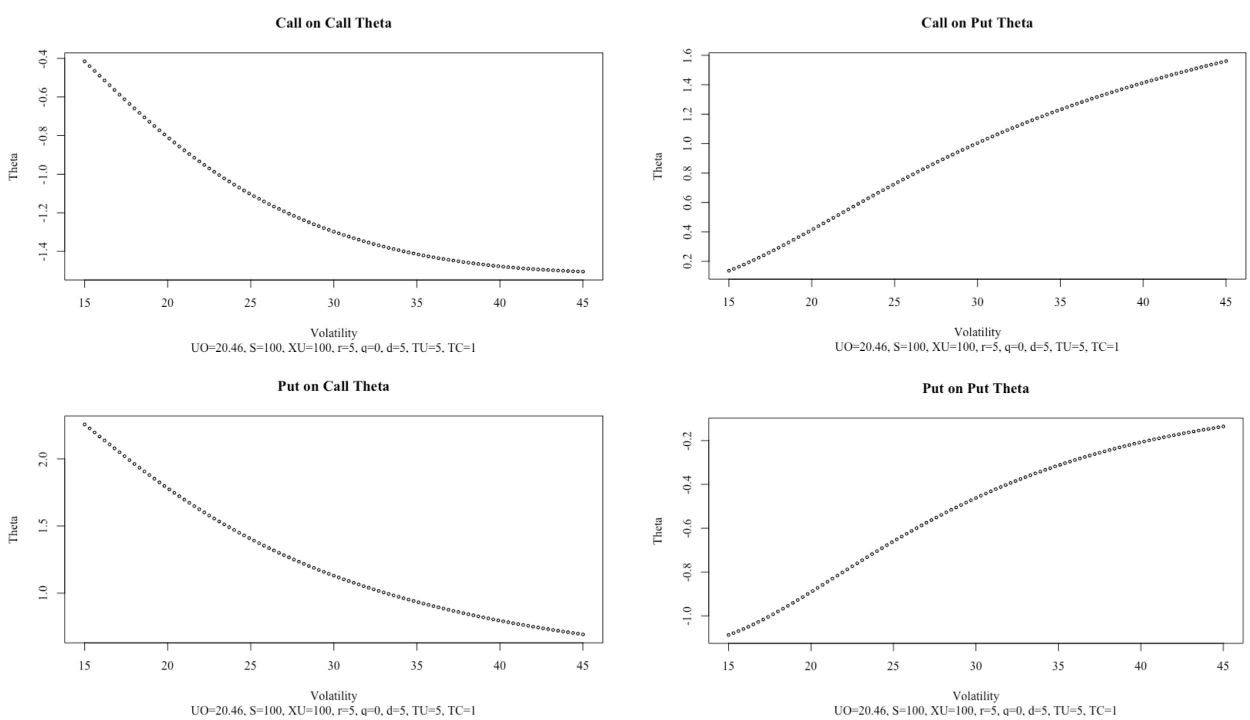


Figure 8.5.11. Compound Option Vega Sensitivity to Volatility

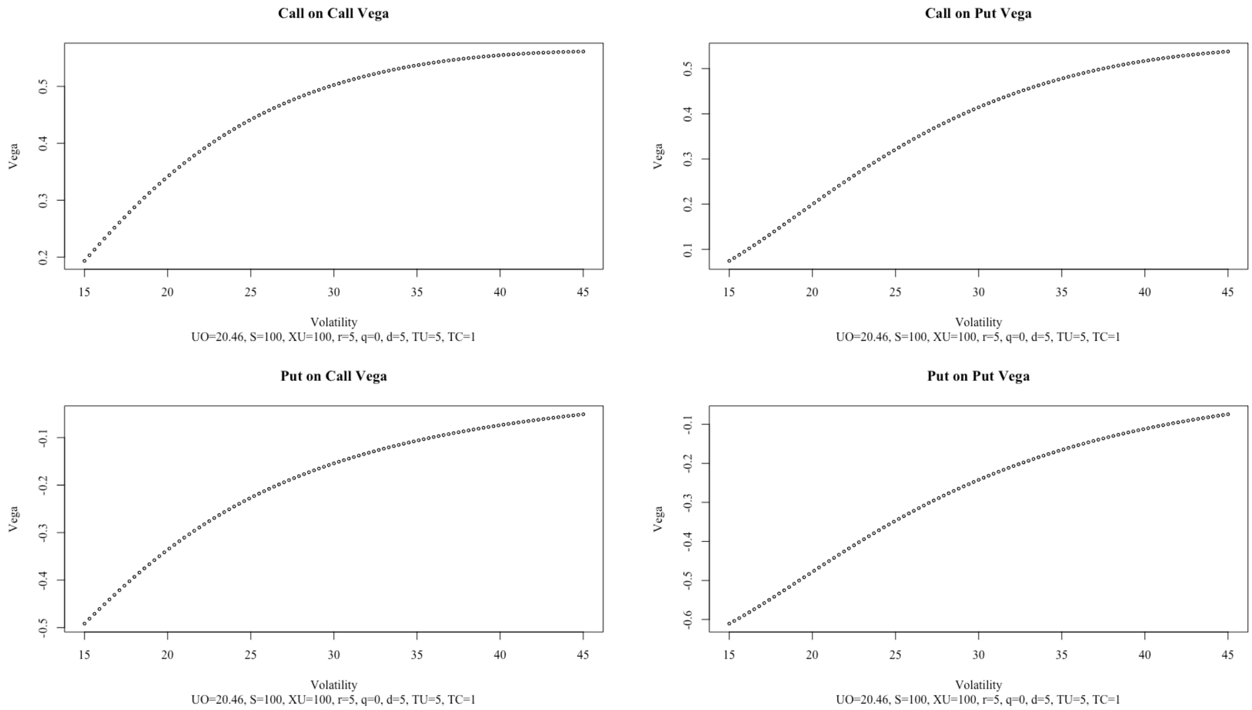
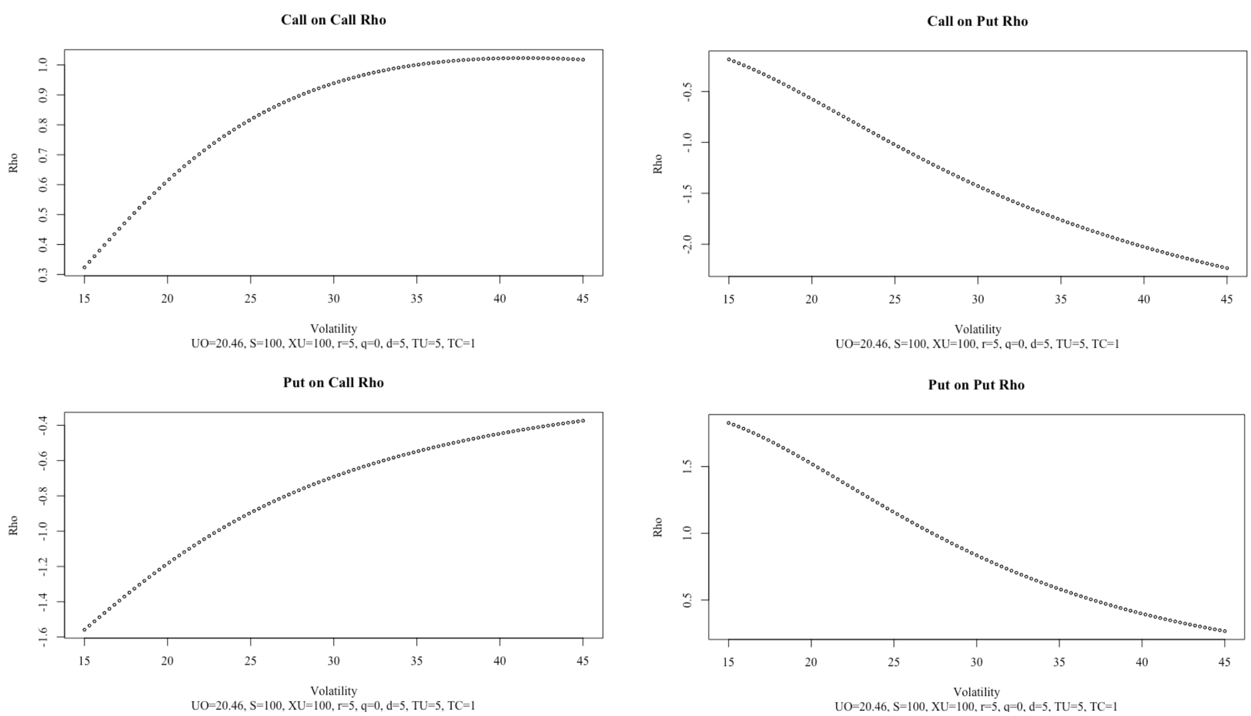


Figure 8.5.12. Compound Option Rho Sensitivity to Volatility



Sensitivity to option yield

We now provide numerous plots related to compound options sensitivity to option yield. The compound option model presented here is unique in its ability to handle cash flows related to the underlying option. Given the sheer number of figures presented below, we offer them without commentary.

Figure 8.5.13. Compound Option Value Sensitivity to Option Yield

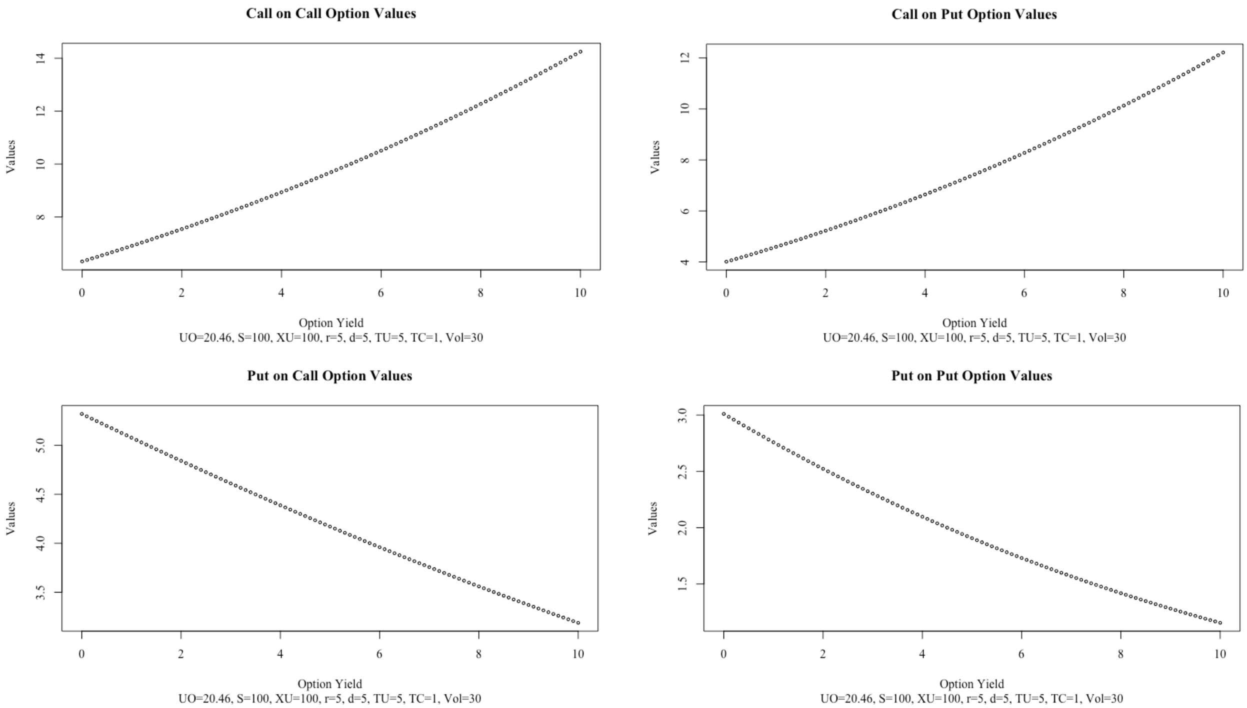


Figure 8.5.14. Compound Option Delta Sensitivity to Option Yield

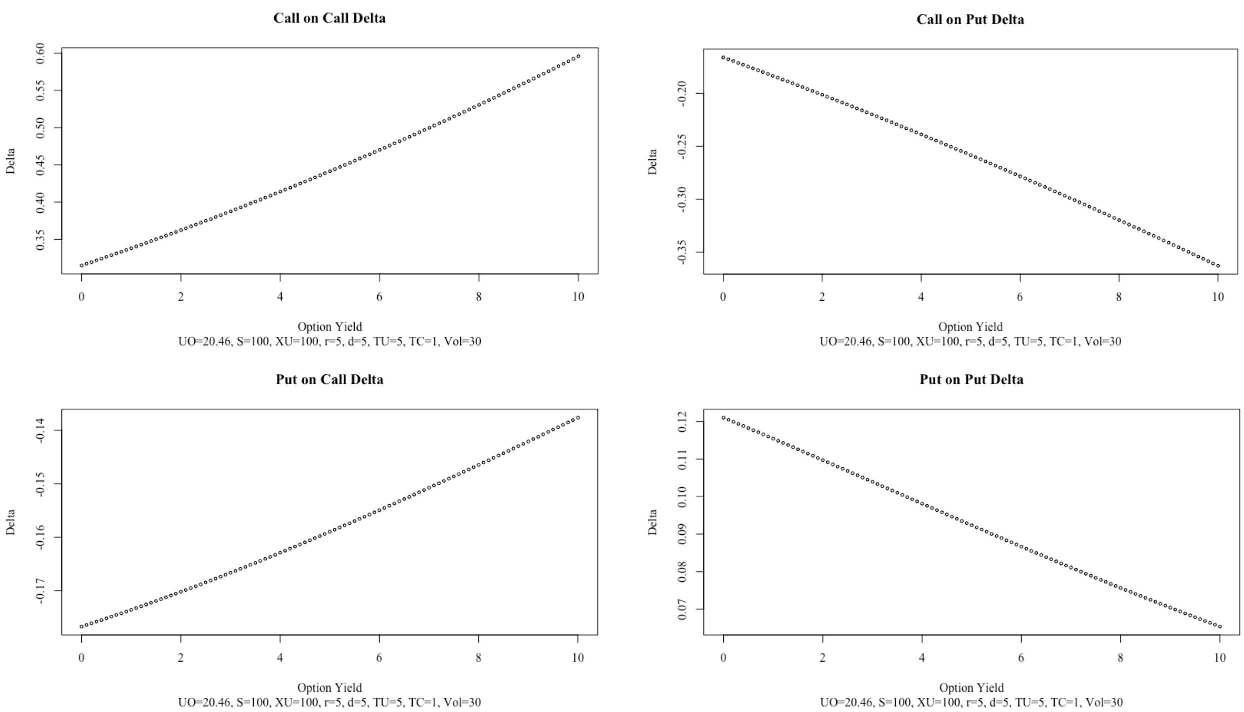


Figure 8.5.15. Compound Option Gamma Sensitivity to Option Yield

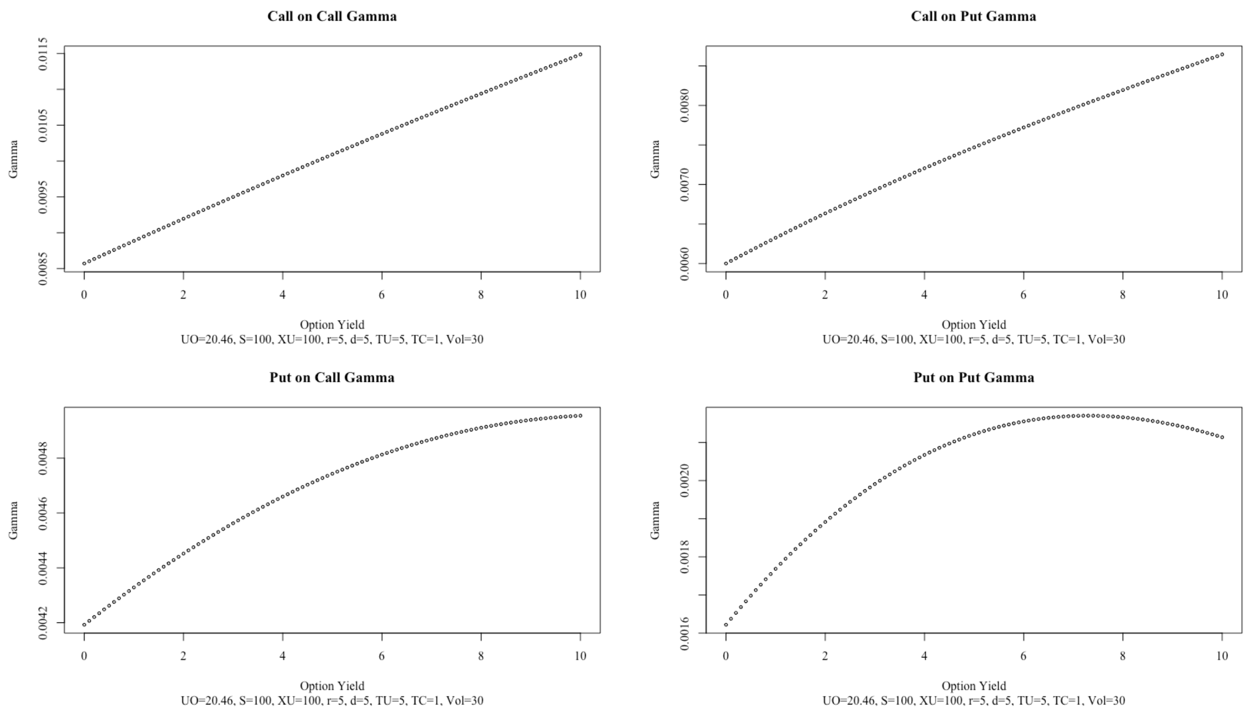


Figure 8.5.16. Compound Option Theta Sensitivity to Option Yield

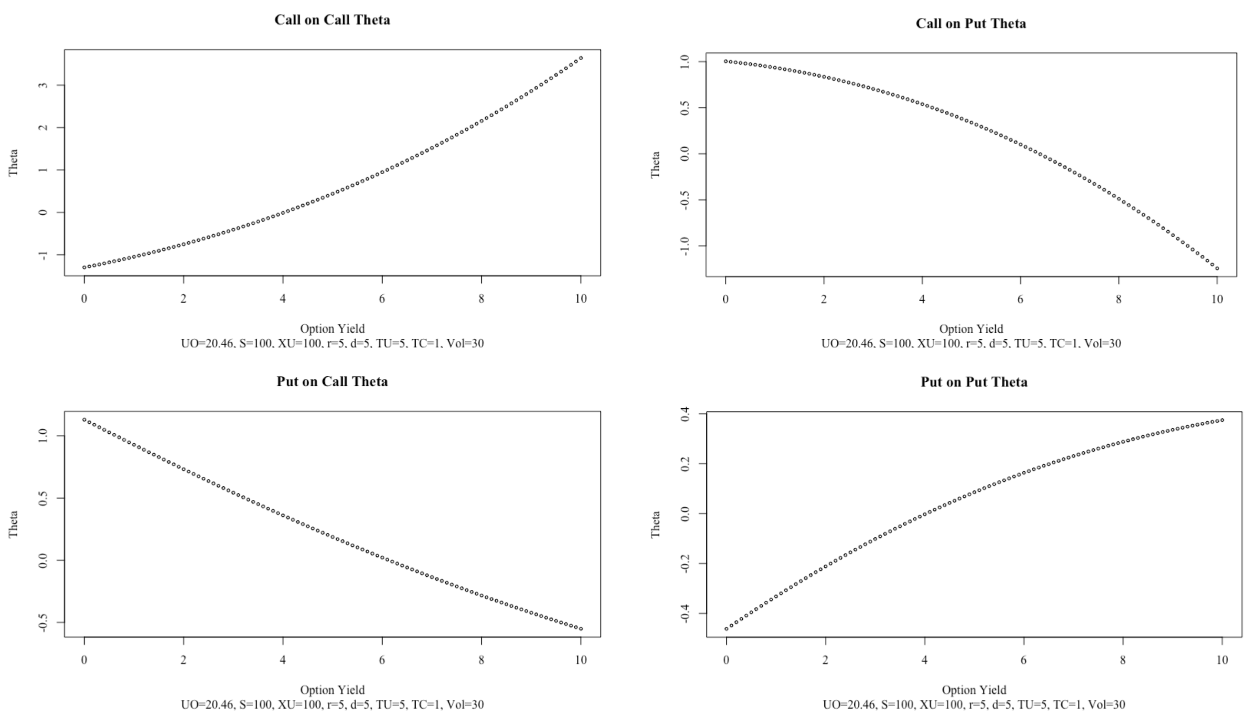


Figure 8.5.17. Compound Option Vega Sensitivity to Option Yield

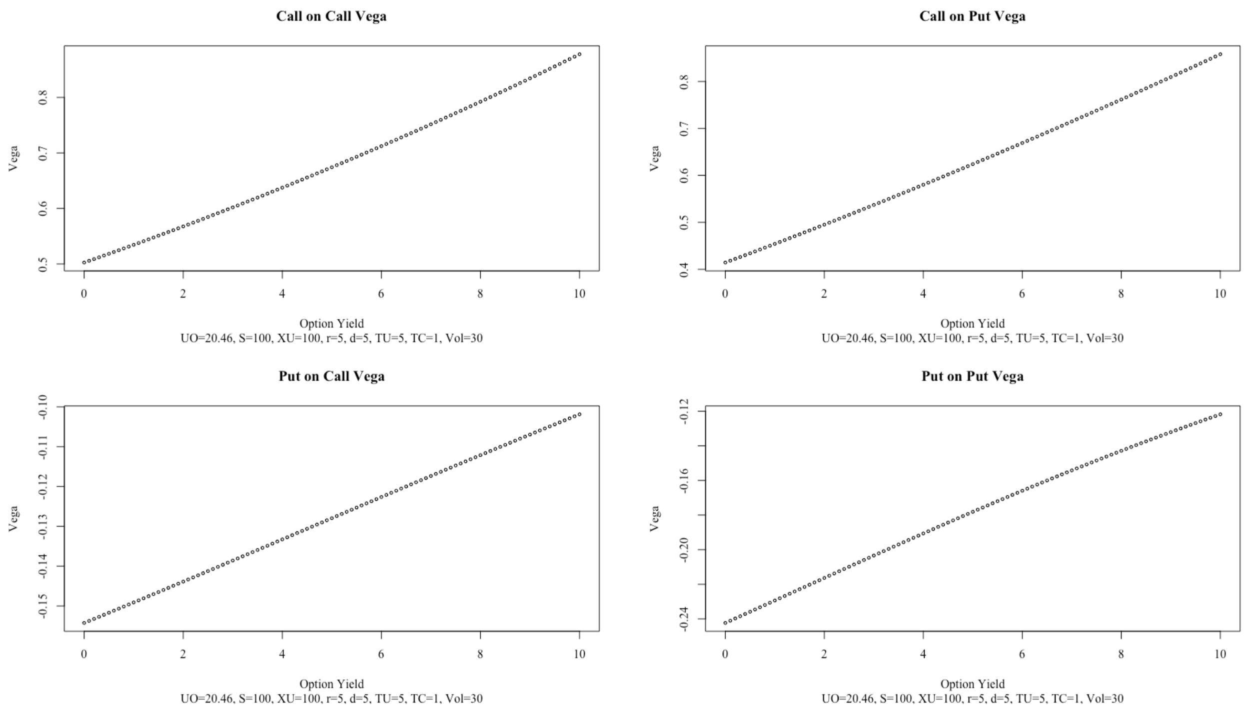
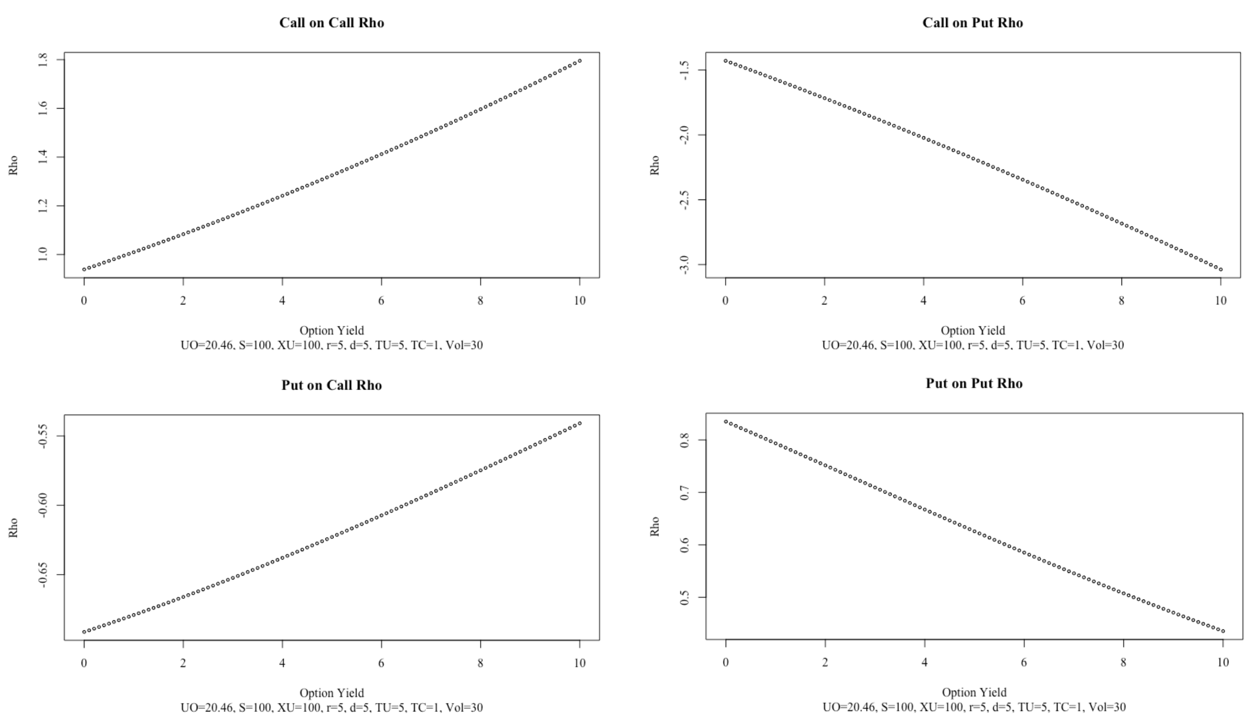


Figure 8.5.18. Compound Option Rho Sensitivity to Option Yield



Sensitivity to compound option maturity

We now provide numerous plots related to compound options sensitivity to option maturity measured by the passage of calendar time. Namely, both the compound option and underlying option maturity declines. Given the sheer number of figures presented below, we offer them without commentary.

Figure 8.5.19. Compound Option Value Sensitivity to Compound Option Maturity

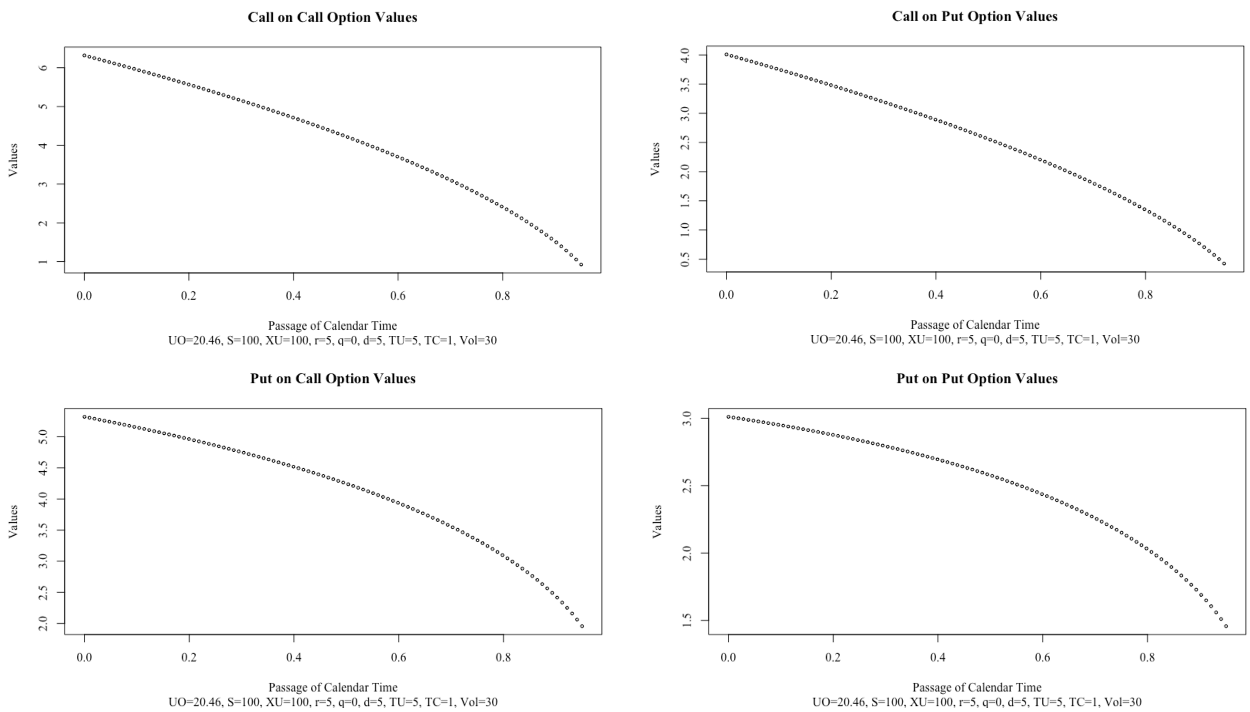


Figure 8.5.20. Compound Option Delta Sensitivity to Compound Option Maturity

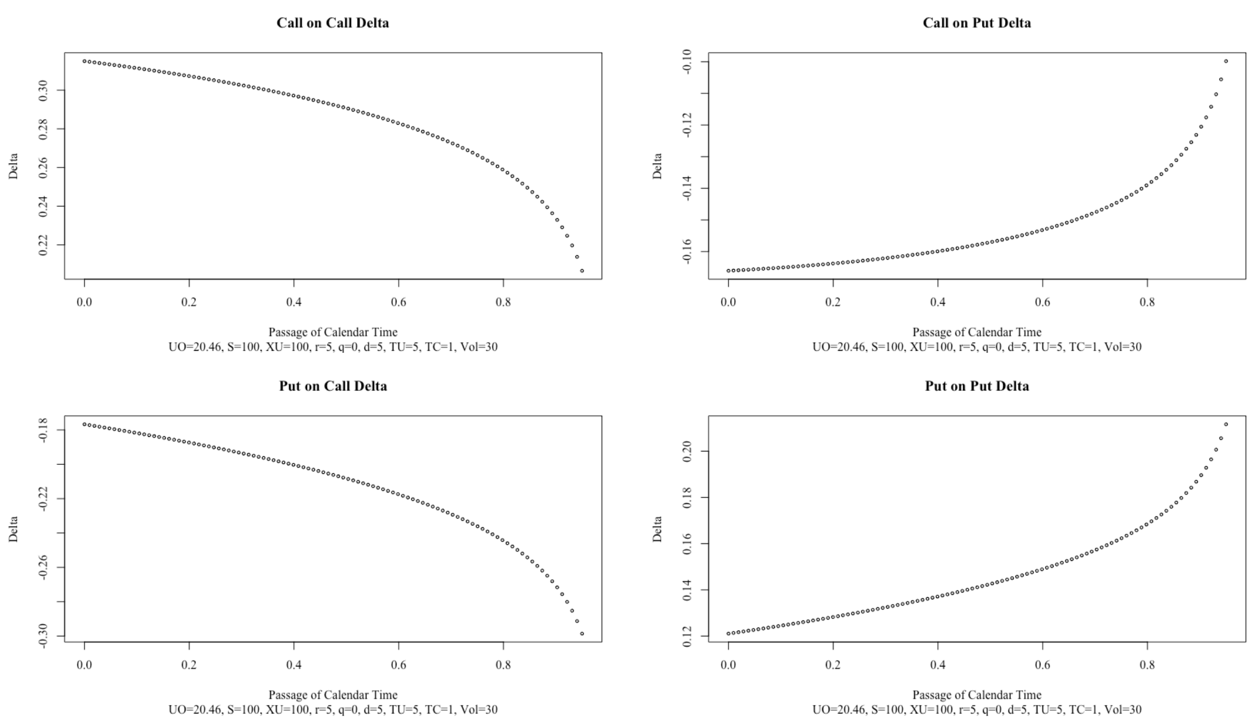


Figure 8.5.21. Compound Option Gamma Sensitivity to Compound Option Maturity

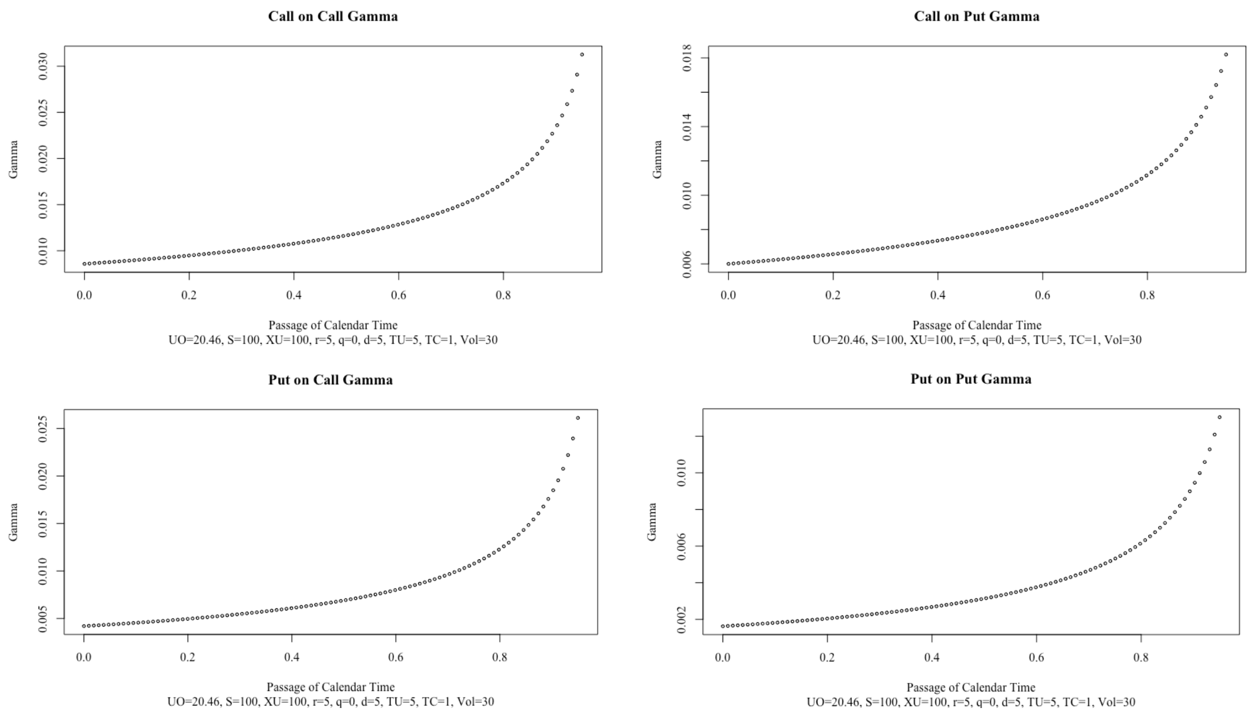


Figure 8.5.22. Compound Option Theta Sensitivity to Compound Option Maturity

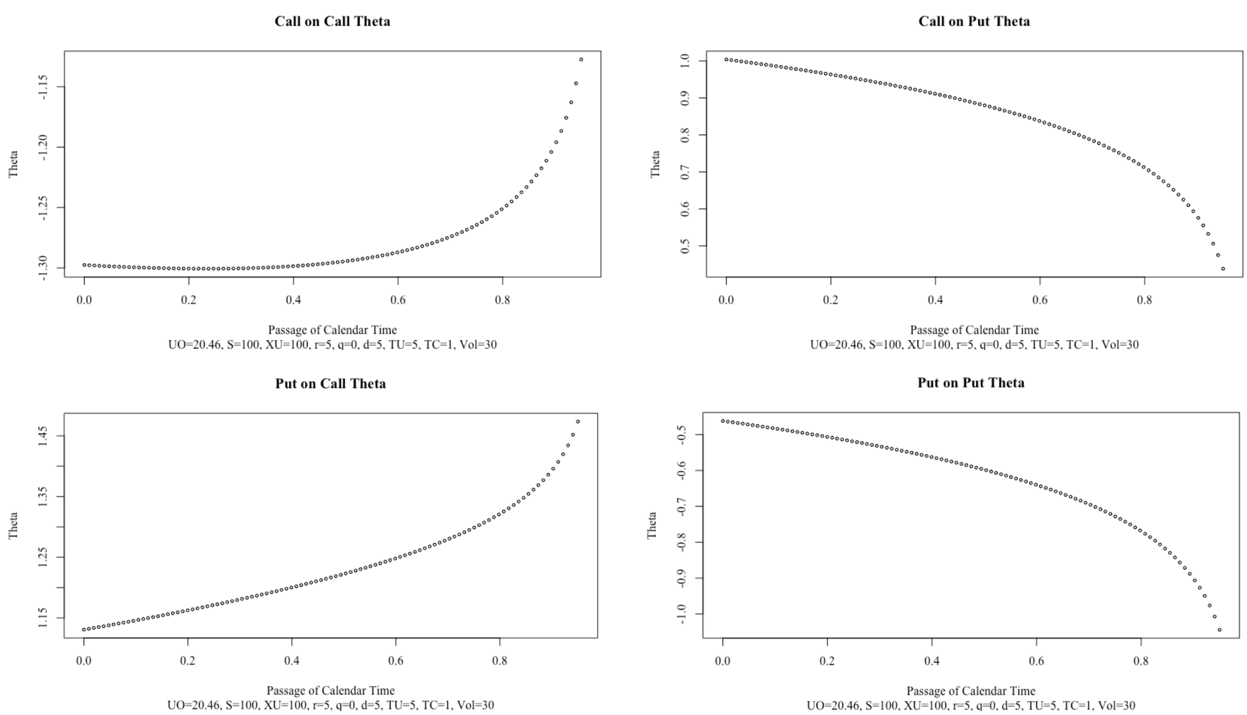


Figure 8.5.23. Compound Option Vega Sensitivity to Calendar Time

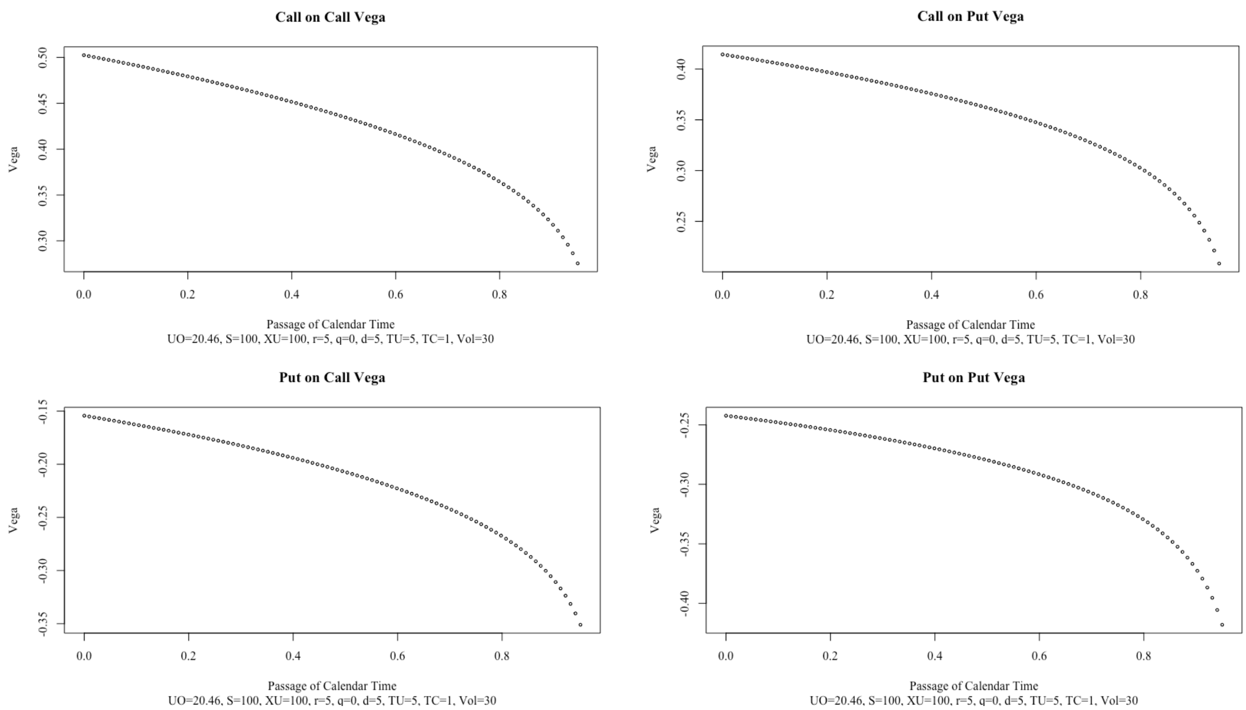
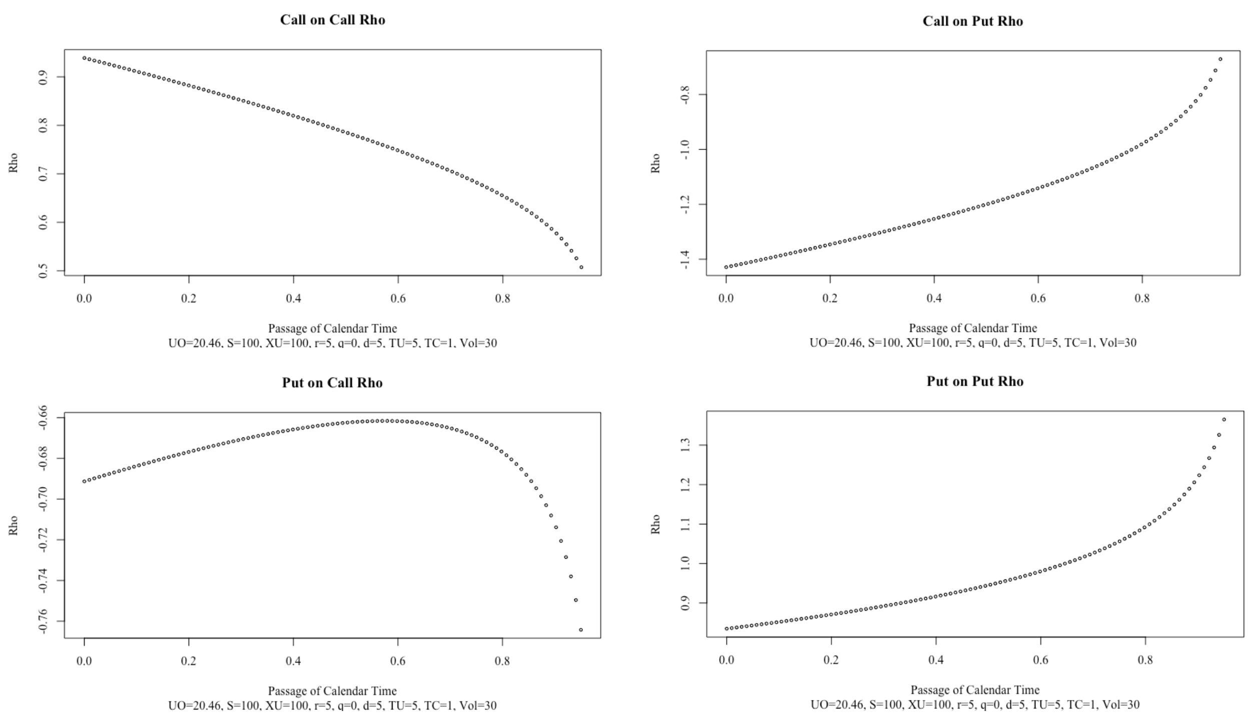


Figure 8.5.24. Compound Option Rho Sensitivity to Calendar Time



Quantitative finance materials

We now take a deep dive into the mathematical details of the GBM-COVM Greeks.

Compound option valuation model

Recall the compound option pricing model (CO) observed at time t under geometric Brownian motion based on an underlying instrument (S_t) with the compound option exercise price (X_C) expiring at time 2 (T_1) and the underlying option exercise price (X_U) expiring at time 1 ($T_2 > T_1$) can be expressed as

$$\begin{aligned}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\ &-\iota_C\iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C X_C B_{t,T_1,r} N\left(\iota_C\iota_U d_{21}\right)\end{aligned}, \tag{8.5.1}$$

where indicator functions denote

$$\iota_C = \begin{cases} +1 \text{ if compound call option} \\ -1 \text{ if compound put option} \end{cases} \text{ and} \tag{8.5.2}$$

$$\iota_U = \begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option} \end{cases}. \tag{8.5.3}$$

Recall a default-free, zero coupon, \$1 par bond be expressed as

$$B_{t,T,x} = e^{-x(T-t)}, \tag{8.5.4}$$

and the bivariate cumulative standard normal distribution

$$N_2\left(a,b;\rho\right) \equiv \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{\exp\left\{-\dfrac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}} dz_1\, dz_2 . \tag{8.5.5}$$

Using a generic time to maturity, *T*, the periodic standard deviation are

$$\sigma_{t,T} = \sigma\sqrt{T-t} . \tag{8.5.6}$$

The correlation coefficient used in the bivariate distribution is

$$\rho = \frac{\sqrt{T_1 - t}}{\sqrt{T_2 - t}}, \tag{8.5.7}$$

and thus

$$\sqrt{1-\rho^2} = \frac{\sqrt{T_2 - T_1}}{\sqrt{T_2 - t}}. \tag{8.5.8}$$

Let $S_{T_1}^*$ be defined such that underlying option is at-the-money or

$$\iota_U S_{T_1}^* B_{T_1,T_2,\delta-\hat{q}} N_1\left(\iota_U d_{1,T_1,T_2}^*\right) - \iota_U X_U B_{T_1,T_2,r-\hat{q}} N_1\left(\iota_U d_{2,T_1,T_2}^*\right) - X_C = 0, \tag{8.5.9}$$

where

$$d_{2,T_1,T_2}^* = \frac{\ln\left(\dfrac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) - \dfrac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}}, \tag{8.5.10}$$

$$d_{1,T_1,T_2}^* = \frac{\ln\left(\dfrac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) + \dfrac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}} = d_{2,T_1,T_2}^* + \sigma_{T_1,T_2}, \text{ and} \tag{8.5.11}$$

$$N_1\left(d\right)=\int_{-\infty}^{d}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx. \tag{8.5.12}$$

Let d_{ij} denote the upper bound of the bivariate normal cumulative distribution function where i = 1, 2 denotes whether the volatility term is added (i = 1) or subtracted (i = 2) and j = 1, 2 denotes whether the evaluation is S^* at T_1 (j = 1) or X_U at T_2 (j = 2). We define

$$d_{21}\equiv\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right)-\frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}, \tag{8.5.13}$$

$$d_{11}\equiv\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right)+\frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}=d_{21}+\sigma_{t,T_1}, \tag{8.5.14}$$

$$d_{22}\equiv\frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right)-\frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}, \text{ and} \tag{8.5.15}$$

$$d_{12}\equiv\frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right)+\frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}=d_{22}+\sigma_{t,T_2}. \tag{8.5.16}$$

We now turn to identify the analytic Greeks. The related proofs will be details later in this chapter should you wish to understand one method to solve for them. An effort was made to be as transparent as possible, although every step is not explained in detail.[1]

The Greeks

Recall the value of the compound option can be expressed as an indirect function of the underlying instrument S as

$$CO_I=CO_{Indirect}\left[O\left(S,t\right),t\right], \tag{8.5.17}$$

where the underlying instrument is embedded in the underlying option $O\left(S,t\right)$. The value of the compound option can also be expressed as a direct function of the underlying instrument S as

$$CO_D=CO_{Direct}\left(S,t\right). \tag{8.5.18}$$

The underlying option is clearly a direct function of the underlying instrument and is expressed as

$$O=O\left(S,t\right). \tag{8.5.19}$$

Note the value of the compound option remains the same regardless of how it is represented or

$$CO_I\left[O\left(S,t\right),t\right]=CO_D\left(S,t\right). \tag{8.5.20}$$

We approach the Greek derivations assuming the compound option is a direct function.

Delta

The compound option delta is

$$\Delta_{CO_D}\equiv\frac{\partial CO_D\left(S,t,T_1,T_2\right)}{\partial S}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right), \tag{8.5.21}$$

[1]See Brooks (2019) for more details.

and the underlying option delta is

$$\Delta_O \equiv \frac{\partial O\left(S,t,T_2\right)}{\partial S} = \iota_U B_{t,T_2,\delta-\hat{q}} N_1\left(\iota_U d_1\right). \tag{8.5.22}$$

Gamma

The compound option gamma is

$$\begin{aligned}\Gamma_{CO_D} &= \frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2} \\ &= \frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_2}}\right],\end{aligned} \tag{8.5.23}$$

and the underlying option gamma is

$$\Gamma_O \equiv \frac{\partial^2 O\left(S,t,T_2\right)}{\partial S^2} = \frac{B_{t,T_2,\delta-\hat{q}} n_1\left(d_1\right)}{S_t \sigma_{t,T_2}}. \tag{8.5.24}$$

Theta

The compound option theta is

$$\begin{aligned}\Theta_{CO_D} &\equiv \frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right) \\ &= -\frac{\sigma^2}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\left\{N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_1}}\right]\right\}. \\ &\quad + \iota_C \iota_U \hat{q} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) \\ &\quad - \iota_C \iota_U r B_{t,T_2,r} X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C r B_{t,T_1,r} X_C N_1\left(\iota_C d_{21}\right)\end{aligned} \tag{8.5.25}$$

and the underlying option theta is

$$\begin{aligned}\Theta_O &\equiv \frac{\partial O\left(S,t,T_2\right)}{\partial t} \\ &= \iota_U\left(\delta-\hat{q}\right) S_t B_{\delta-\hat{q}} N\left(\iota_U d_1\right) - \iota_U\left(r-\hat{q}\right) X B_{r-\hat{q}} N\left(\iota_U d_2\right). \\ &\quad - \frac{\sigma^2 S_t B_{\delta-\hat{q}}}{2\sigma} n\left(d_1\right)\end{aligned} \tag{8.5.26}$$

It is important to note that we can ignore t embedded within rho. Rho is assumed constant as an input parameter once computed.

Vega

The compound option vega is

$$\frac{\partial}{\partial \sigma} CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}}\left[\begin{array}{l}\sqrt{T_1-t} N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right) n_1\left(d_{11}\right) \\ + \iota_C \sqrt{T_2-t} N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right) n_1\left(d_{12}\right)\end{array}\right], \tag{8.5.27}$$

and the underlying option vega is

$$\nu_O \equiv \frac{\partial O}{\partial \sigma} = S B_{t,T_2,\delta} n\left(d_1\right)\sqrt{T_2-t} = X B_{t,T_2,r} n\left(d_2\right)\sqrt{T_2-t}. \tag{8.5.28}$$

Rho
The compound option rho is

$$\begin{aligned}\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\left(T_2-t\right)X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\\+\iota_C B_{t,T_1,r}\left(T_1-t\right)X_C N\left(\iota_C\iota_U d_{21}\right)\end{aligned}, \tag{8.5.29}$$

and the underlying option rho is

$$\rho_O\equiv\frac{\partial O}{\partial r}=\iota_U X\left(T_2-t\right)B_{t,T_2,2}N\left(\iota_U d_2\right). \tag{8.5.30}$$

Prior to working through the details of the Greek proofs, we will rely on several lemmas.

Important lemmas
We rely on the following lemmas for deriving the Greeks. Proofs for these lemmas follow the derivations of the Greeks.

Lemma 1: Leibniz integral rule for double integral applied to bivariate normal
Assuming $N_2\left[a(x),b(x);\rho\right]$ is a standard bivariate normal cumulative distribution function, then

$$\frac{dN_2\left[a(x),b(x);\rho\right]}{dx}=\frac{dN_2\left[a(x),b(x);\rho\right]}{db(x)}\frac{db(x)}{dx}+\frac{dN_2\left[a(x),b(x);\rho\right]}{da(x)}\frac{da(x)}{dx}. \tag{8.5.31}$$

Lemma 2: Compound option partial with respect to d₂₁

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=0, \tag{8.5.32}$$

and

$$\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}}. \tag{8.5.33}$$

Lemma 3: Compound option partial with respect to d₂₂

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)=0, \tag{8.5.34}$$

and

$$\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}}. \tag{8.5.35}$$

We now turn to sketch the proofs of the Greeks.

Compound option delta proof

$$\Delta_{CO_D}=\frac{\partial CO_D\left(S,t,T_1,T_2\right)}{\partial S}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right). \tag{8.5.36}$$

Proof: Note based on Equation (8.5.1), we have

$$\begin{aligned}\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)&=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial}{\partial S}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial}{\partial S}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C X_C B_{t,T_1,r}\frac{\partial}{\partial S}N_1\left(\iota_C d_3\right)\end{aligned}. \tag{8.5.37}$$

Based on Lemma 1 (Leibniz integral rule), we have

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\rho\right]}{\partial S}\\&=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S},\end{aligned} \tag{8.5.38}$$

and

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\rho\right]}{\partial S}\\&=\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\frac{\partial d_{21}(S)}{\partial S}+\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\frac{\partial d_{22}(S)}{\partial S}.\end{aligned} \tag{8.5.39}$$

Also, from the standard normal cumulative distribution function, we note

$$\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial S}=\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial S}. \tag{8.5.40}$$

Substituting

$$\begin{aligned}&\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}\\&+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{aligned}\right\}\\&-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\frac{\partial d_{21}(S)}{\partial S}\\&+\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\frac{\partial d_{22}(S)}{\partial S}\end{aligned}\right\}\\&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial S}\end{aligned}. \tag{8.5.41}$$

Recall

$$d_{11}=d_{21}+\sigma_{t,T_1}\text{ and} \tag{8.5.42}$$

$$d_{12}=d_{22}+\sigma_{t,T_2}. \tag{8.5.43}$$

Thus

$$\frac{\partial d_{11}}{\partial S}=\frac{\partial d_{21}}{\partial S}\text{ and} \tag{8.5.44}$$

$$\frac{\partial d_{12}}{\partial S}=\frac{\partial d_{22}}{\partial S}. \tag{8.5.45}$$

Substituting these derivatives

$$
\begin{aligned}
&\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\
&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{array}{l}\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\dfrac{\partial d_{21}(S)}{\partial S}\\+\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\dfrac{\partial d_{22}(S)}{\partial S}\end{array}\right\}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\begin{array}{l}\dfrac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\dfrac{\partial d_{21}(S)}{\partial S}\\+\dfrac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\dfrac{\partial d_{22}(S)}{\partial S}\end{array}\right\}.\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial S}
\end{aligned}
\tag{8.5.46}
$$

Rearranging

$$
\begin{aligned}
&\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\
&+\frac{\partial d_{21}(S)}{\partial S}\left\{\begin{array}{l}\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\\-\iota_C\iota_U X_U B_{t,T_2,r}\dfrac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}-\iota_C X_C B_{t,T_1,r}\dfrac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\end{array}\right\}.\\
&+\frac{\partial d_{22}(S)}{\partial S}\left\{\begin{array}{l}\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\\-\iota_C\iota_U X_U B_{t,T_2,r}\dfrac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}\end{array}\right\}
\end{aligned}
\tag{8.5.47}
$$

Note based on Lemma 2,

$$
\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\text{ and}
\tag{8.5.48}
$$

$$
\begin{aligned}
&\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}.\\
&=\frac{\partial}{\partial d_{22}}CO\left(S,t,T_1,T_2\right)=0
\end{aligned}
\tag{8.5.49}
$$

And based on Lemma 3,

$$\begin{aligned}&\iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right)}{\partial d_{12}(S)}\\&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right)}{\partial d_{22}(S)}=\frac{\partial}{\partial d_{22}}CO\left(S,t,T_1,T_2\right)=0\end{aligned}. \tag{8.5.50}$$

Therefore,

$$\frac{\partial}{\partial S}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right). \tag{8.5.51}$$

Compound option gamma proof

$$\Gamma_{CO_D}=\frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2}=\frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[\begin{aligned}&N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\\&+\iota_C N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_2}}\end{aligned}\right]. \tag{8.5.52}$$

Proof: Note based on Equation (8.5.51), we have

$$\Gamma_{CO_D}=\frac{\partial^2 CO_D\left(S,t,T_1,T_2\right)}{\partial S^2}=\iota_C\iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\frac{\partial}{\partial S}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right). \tag{8.5.53}$$

Based on Lemma 1, we have

$$\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial S}\\&=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{aligned}. \tag{8.5.54}$$

Thus

$$\Gamma_{CO_D}=\iota_C\iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\left\{\begin{aligned}&\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\frac{\partial d_{11}(S)}{\partial S}\\&+\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\frac{\partial d_{12}(S)}{\partial S}\end{aligned}\right\}, \tag{8.5.55}$$

and recall

$$d_{11}=\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S_{T_1}^*}\right)+\frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}=d_{21}+\sigma_{t,T_1}, \tag{8.5.56}$$

$$\frac{\partial d_{11}}{\partial S}=\frac{1}{S_t\sigma_{t,T_1}}=\frac{\partial d_{21}}{\partial S}, \tag{8.5.57}$$

$$d_{12}=\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})} B_{T_1,T_2,-(r-\delta)}}{X_C}\right)+\frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}=d_{22}+\sigma_{t,T_2},\text{ and} \tag{8.5.58}$$

$$\frac{\partial d_{12}}{\partial S}=\frac{1}{S_t\sigma_{t,T_2}}=\frac{\partial d_{22}}{\partial S}. \tag{8.5.59}$$

Note

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial\left[\iota_C\iota_U d_{11}(S)\right]}\frac{\partial\left[\iota_C\iota_U d_{11}(S)\right]}{\partial d_{11}(S)}$$
$$=\iota_C\iota_U\frac{\partial}{\partial\left[\iota_C\iota_U d_{11}(S)\right]}\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}N_1\left[\frac{\iota_U d_{12}(S)-\iota_C\rho z_2}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1(z_2)dz_2$$
$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{12}(S)-\iota_C\rho\iota_C\iota_U d_{11}(S)}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1\left[\iota_C\iota_U d_{11}(S)\right]$$
$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{12}(S)-\rho\iota_U d_{11}(S)}{\sqrt{1-\rho^2}}\right]n_1\left[\iota_C\iota_U d_{11}(S)\right],\qquad(8.5.60)$$

and also

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial\left[\iota_U d_{12}(S)\right]}\frac{\partial\left[\iota_U d_{12}(S)\right]}{\partial d_{12}(S)}$$
$$=\iota_U\frac{\partial}{\partial\left[\iota_U d_{12}\right]}\int_{-\infty}^{\iota_U d_{12}(S)}N_1\left[\frac{\iota_C\iota_U d_{11}(S)-\iota_C\rho z_1}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1(z_1)dz_1\qquad.\qquad(8.5.61)$$
$$=\iota_U N_1\left[\frac{\iota_C\iota_U d_{11}(S)-\iota_C\rho\iota_U d_{12}(S)}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1\left[\iota_U d_{12}(S)\right]$$

Thus,

$$\Gamma_{CO_D}=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{array}{l}\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}\dfrac{\partial d_{11}(S)}{\partial S}\\+\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}\dfrac{\partial d_{12}(S)}{\partial S}\end{array}\right\}$$
$$=\iota_C\iota_U B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{array}{l}\iota_C\iota_U N_1\left[\dfrac{\iota_U d_{12}-\rho\iota_U d_{11}}{\sqrt{1-\rho^2}}\right]n_1\left[\iota_C\iota_U d_{11}\right]\dfrac{1}{S_t\sigma_{t,T_1}}\\+\iota_U N_1\left[\dfrac{\iota_C\iota_U d_{11}-\iota_C\rho\iota_U d_{12}}{\sqrt{1-(\iota_C\rho)^2}}\right]n_1\left[\iota_U d_{12}\right]\dfrac{1}{S_t\sigma_{t,T_2}}\end{array}\right\}.\qquad(8.5.62)$$

Therefore,

$$\Gamma_{CO_D}=\frac{B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}}+\iota_C N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right].\qquad(8.5.63)$$

Compound option theta proof

$$\begin{aligned}
&\Theta_{CO_D} \equiv \frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)\\
&= -\frac{\sigma^2}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \left\{ N_1\left(\iota_U \frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C \iota_U \frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_1}}\right]\right\}. \\
&+\iota_C \iota_U \hat{q} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)\\
&-\iota_C \iota_U r B_{t,T_2,r} X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C r B_{t,T_1,r} X_C N_1\left(\iota_C d_{21}\right)
\end{aligned} \tag{8.5.64}$$

Proof: There are several variables that are a function of calendar time t. Note if

$$B_{t,T,x} = e^{-x(T-t)}, \tag{8.5.65}$$

then

$$\frac{\partial}{\partial t} B_{t,T,x} = x e^{-x(T-t)} = x B_{t,T,x}. \tag{8.5.66}$$

Recall from Equation (8.5.1),

$$\begin{aligned}
CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)\\
&-\iota_C \iota_U X_U B_{t,T_2,r} N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C X_C B_{t,T_1,r} N_1\left(\iota_C d_{21}\right)
\end{aligned}. \tag{8.5.67}$$

Thus,

$$\begin{aligned}
&\frac{\partial}{\partial t} CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = \iota_C \iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \frac{\partial}{\partial t} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)\\
&+\iota_C \iota_U S_t B_{T_1,T_2,\delta} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) \frac{\partial}{\partial t} B_{t,T_1,\hat{q}}\\
&-\iota_C \iota_U X_U B_{t,T_2,r} \frac{\partial}{\partial t} N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C \iota_U X_U N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) \frac{\partial}{\partial t} B_{t,T_2,r}\\
&-\iota_C X_C B_{t,T_1,r} \frac{\partial}{\partial t} N_1\left(\iota_C d_{21}\right) - \iota_C X_C N_1\left(\iota_C d_{21}\right) \frac{\partial}{\partial t} B_{t,T_1,r}
\end{aligned}. \tag{8.5.68}$$

We now work with each partial derivative with respect to calendar time (numbered in order of appearance in the equation above). Based on Lemma 1, we have

(1) $$\frac{\partial}{\partial t} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) = \frac{\partial N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)}{\partial\left(\iota_C \iota_U d_{11}\right)} \frac{\partial\left(\iota_C \iota_U d_{11}\right)}{\partial t} + \frac{\partial N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right)}{\partial\left(\iota_U d_{12}\right)} \frac{\partial\left(\iota_U d_{12}\right)}{\partial t}. \tag{8.5.69}$$

Note

$$\begin{aligned}
\frac{\partial\left(\iota_C \iota_U d_{11}\right)}{\partial t} &= \iota_C \iota_U \frac{\partial d_{11}}{\partial t} = \iota_C \iota_U \frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}}\right) + \frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}}\right]\\
&= \iota_C \iota_U \frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t}{S^*_{T_1}}\right) + \left(r - \hat{q} + \frac{\sigma^2}{2}\right)\left(T_1 - t\right)}{\sigma\sqrt{T_1 - t}}\right]
\end{aligned}. \tag{8.5.70}$$

Let

$$A(t) \equiv \ln\left(\frac{S_t}{S_{T_1}^*}\right) + \left(r - \hat{q} + \frac{\sigma^2}{2}\right)(T_1 - t), \tag{8.5.71}$$

then

$$\frac{\partial}{\partial t} A(t) = -\left(r - \hat{q} + \frac{\sigma^2}{2}\right), \tag{8.5.72}$$

and

$$B(t) = \sigma\sqrt{T_1 - t}, \tag{8.5.73}$$

then

$$\frac{\partial}{\partial t} B(t) = \frac{\partial}{\partial t}\sigma\sqrt{T_1 - t} = \frac{\sigma}{2}(T_1 - t)^{-1/2} = \frac{\sigma}{2\sqrt{T_1 - t}}. \tag{8.5.74}$$

Therefore,

$$\begin{aligned}
\frac{\partial(\iota_C \iota_U d_{11})}{\partial t} &= \iota_C \iota_U \frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t}{S_{T_1}^*}\right) + \left(r - \hat{q} + \frac{\sigma^2}{2}\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}\right] = \iota_C \iota_U \frac{\partial}{\partial t}\left[\frac{A(t)}{B(t)}\right] \\
&= \iota_C \iota_U \left[\frac{B(t)\frac{\partial}{\partial t}A(t) - A(t)\frac{\partial}{\partial t}B(t)}{B^2(t)}\right] \\
&= \iota_C \iota_U \left[\frac{-\sigma\sqrt{T_1 - t}\left(r - \hat{q} + \frac{\sigma^2}{2}\right) + \left[\ln\left(\frac{S_t}{S_{T_1}^*}\right) + \left(r - \hat{q} + \frac{\sigma^2}{2}\right)(T_1 - t)\right]\left(\frac{\sigma}{2\sqrt{T_1 - t}}\right)}{\sigma^2(T_1 - t)}\right]
\end{aligned} \tag{8.5.75}$$

Reducing,

$$\frac{\partial(\iota_C \iota_U d_{11})}{\partial t} = \iota_C \iota_U \left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1 - t}} + \frac{d_{11}}{2(T_1 - t)}\right]. \tag{8.5.76}$$

Following a similar approach,

$$\begin{aligned}
\frac{\partial(\iota_U d_{12})}{\partial t} &= \iota_U \frac{\partial d_{12}}{\partial t} = \iota_U \frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})} B_{T_1,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma^2(T_2 - t)}{2}}{\sigma\sqrt{T_2 - t}}\right]. \\
&= \iota_C \iota_U \left[-\frac{\left(r - \hat{q} + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2 - t}} + \frac{d_{12}}{2(T_2 - t)}\right]
\end{aligned} \tag{8.5.77}$$

Therefore, (note we can cancel the indicator functions)

$$\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)=\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{11}}{2\left(T_1-t\right)}\right]$$
$$+\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{12}}{2\left(T_2-t\right)}\right]. \quad (8.5.78)$$

(2)
$$\frac{\partial B_{t,T_1,\hat{q}}}{\partial t}=\hat{q}B_{t,T_1,\hat{q}}. \quad (8.5.79)$$

(3)
$$\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)=\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial\left(\iota_C\iota_U d_{21}\right)}\frac{\partial\left(\iota_C\iota_U d_{21}\right)}{\partial t}+\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial\left(\iota_U d_{22}\right)}\frac{\partial\left(\iota_U d_{22}\right)}{\partial t}. \quad (8.5.80)$$

Note following a similar approach as above,

$$\frac{\partial\left(\iota_C\iota_U d_{21}\right)}{\partial t}=\iota_C\iota_U\frac{\partial d_{21}}{\partial t}=\iota_C\iota_U\frac{\partial}{\partial t}\left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}}\right)-\frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}}\right], \text{ and} \quad (8.5.81)$$
$$=\iota_C\iota_U\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)}+\frac{d_{21}}{2\left(T_1-t\right)}\right]$$

$$\frac{\partial\left(\iota_U d_{22}\right)}{\partial t}=\iota_U\frac{\partial d_{22}}{\partial t}=\iota_U\frac{\partial}{\partial t}\left[\frac{\ln\left(S_t B_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}\big/X_U\right)-\frac{\sigma^2_{t,T_2}}{2}}{\sigma_{t,T_2}}\right]. \quad (8.5.82)$$
$$=\iota_U\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}}{2\left(T_2-t\right)}\right]$$

Therefore, (note we can cancel the indicator functions)

$$\frac{\partial}{\partial t}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)=\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)}+\frac{d_{21}}{2\left(T_1-t\right)}\right]$$

$$+\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}}{2\left(T_2-t\right)}\right]. \qquad (8.5.83)$$

(4) $$\frac{\partial B_{t,T_2,r}}{\partial t}=rB_{t,T_2,r}. \qquad (8.5.84)$$

(5) $$\frac{\partial}{\partial t}N_1\left(\iota_C d_{21}\right)=\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial\left(\iota_C d_{21}\right)}\frac{\partial\left(\iota_C d_{21}\right)}{\partial t}=\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}}{2\left(T_1-t\right)}\right]. \qquad (8.5.85)$$

(6) $$\frac{\partial B_{t,T_1,r}}{\partial t}=rB_{t,T_1,r}. \qquad (8.5.86)$$

Substituting these results into the original equation,

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)\\
&=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{array}{l}
\dfrac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\dfrac{d_{11}}{2\left(T_1-t\right)}\right]\\
+\dfrac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\dfrac{d_{12}}{2\left(T_2-t\right)}\right]
\end{array}\right\}\\
&+\iota_C\iota_U S_t B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\hat{q}B_{t,T_1,\hat{q}}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\begin{array}{l}
\dfrac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)}+\dfrac{d_{21}}{2\left(T_1-t\right)}\right]\\
+\dfrac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\left[-\dfrac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\dfrac{d_{22}}{2\left(T_2-t\right)}\right]
\end{array}\right\}\\
&-\iota_C\iota_U X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)rB_{t,T_2,r}\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}}{2\left(T_1-t\right)}\right]\qquad . \qquad (8.5.87)\\
&-\iota_C X_C N_1\left(\iota_C d_{21}\right)rB_{t,T_1,r}
\end{aligned}
$$

Rearranging to exploit Lemmas 2 and 3,

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)\\
&=\iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{11}}{2\left(T_1-t\right)}\right]\\
&+\iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{12}}{2\left(T_2-t\right)}\right]\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)}+\frac{d_{21}}{2\left(T_1-t\right)}\right]\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}}{2\left(T_2-t\right)}\right]\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}}{2\left(T_1-t\right)}\right]\\
&+\iota_C\iota_U S_t B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\hat{q}B_{t,T_1,\hat{q}}\\
&-\iota_C\iota_U X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)rB_{t,T_2,r}\\
&-\iota_C X_C N_1\left(\iota_C d_{21}\right)rB_{t,T_1,r}
\end{aligned}
\quad . \qquad (8.5.88)
$$

Substituting for d_{11} and d_{12} and related partials, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)\\
&=\left\{
\begin{aligned}
&\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}+\sigma\sqrt{T_1-t}}{2\left(T_1-t\right)}\right]\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\left(T_1-t\right)}+\frac{d_{21}}{2\left(T_1-t\right)}\right]\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left(\iota_C d_{21}\right)}{\partial d_{21}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}}{2\left(T_1-t\right)}\right]
\end{aligned}
\right\}\\
&+\left\{
\begin{aligned}
&\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}+\sigma\sqrt{T_2-t}}{2\left(T_2-t\right)}\right]\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\left[-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}}{2\left(T_2-t\right)}\right]
\end{aligned}
\right\}. \qquad (8.5.89)\\
&+\iota_C\iota_U S_t B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\hat{q}B_{t,T_1,\hat{q}}\\
&-\iota_C\iota_U X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)rB_{t,T_2,r}\\
&-\iota_C X_C N_1\left(\iota_C d_{21}\right)rB_{t,T_1,r}
\end{aligned}
$$

Note that

$$
-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}+\sigma\sqrt{T_1-t}}{2\left(T_1-t\right)}=-\frac{\left(r-\hat{q}-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_1-t}}+\frac{d_{21}}{2\left(T_1-t\right)}-\frac{\sigma}{2\sqrt{T_1-t}} \qquad (8.5.90)
$$

and

$$
-\frac{\left(r-\hat{q}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}+\sigma\sqrt{T_2-t}}{2\left(T_2-t\right)}=-\frac{\left(r-\hat{q}-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T_2-t}}+\frac{d_{22}}{2\left(T_2-t\right)}-\frac{\sigma}{2\sqrt{T_2-t}}. \qquad (8.5.91)
$$

Substituting these results, based on Lemmas 2 and 3, the remaining terms are (substituting back for d_{21} and d_{22})

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=-\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\left(\frac{\sigma}{2\sqrt{T_1-t}}\right)\\
&-\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\left(\frac{\sigma}{2\sqrt{T_2-t}}\right)\\
&\iota_C\iota_U\hat{q}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\
&-\iota_C\iota_U rB_{t,T_2,r}X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C rB_{t,T_1,r}X_C N_1\left(\iota_C d_{21}\right)
\end{aligned}
\quad . \tag{8.5.92}
$$

Recall

$$
\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}=\iota_C\iota_U N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right), \tag{8.5.93}
$$

and also

$$
\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}=\iota_U N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right). \tag{8.5.94}
$$

Thus,

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=-S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\left(\frac{\sigma}{2\sqrt{T_1-t}}\right)\\
&-\iota_C S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\left(\frac{\sigma}{2\sqrt{T_2-t}}\right)\\
&\iota_C\iota_U\hat{q}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\
&-\iota_C\iota_U rB_{t,T_2,r}X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C rB_{t,T_1,r}X_C N_1\left(\iota_C d_{21}\right)
\end{aligned}
\quad . \tag{8.5.95}
$$

And finally,

$$
\begin{aligned}
&\frac{\partial}{\partial t}CO_D\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=-\frac{\sigma^2}{2}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left\{\begin{aligned}&N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{11}\right)}{\sigma_{t,T_1}}\right]\\&-\iota_C N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1\left(d_{12}\right)}{\sigma_{t,T_1}}\right]\end{aligned}\right\}\\
&+\iota_C\iota_U\hat{q}S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)\\
&-\iota_C\iota_U rB_{t,T_2,r}X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C rB_{t,T_1,r}X_C N_1\left(\iota_C d_{21}\right)
\end{aligned}
\quad . \tag{8.5.96}
$$

Compound option vega proof

$$
\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\begin{bmatrix}\sqrt{T_1-t}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\\+\iota_C\sqrt{T_2-t}N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\end{bmatrix}. \tag{8.5.97}
$$

Proof: There are several variables that are a function of volatility, ⊠. Highlighting this dependency, we have

$$\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial\sigma}N_2\left[\iota_C\iota_U d_{11}\left(\sigma\right),\iota_U d_{12}\left(\sigma\right);\iota_C\rho\right]$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial\sigma}N_2\left[\iota_C\iota_U d_{21}\left(\sigma\right),\iota_U d_{22}\left(\sigma\right);\iota_C\rho\right]-\iota_C X_C B_{t,T_1,r}\frac{\partial}{\partial\sigma}N\left[\iota_C\iota_U d_{21}\left(\sigma\right)\right], \quad (8.5.98)$$

Based on lemma 1, we have

$$\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left\{\begin{array}{l}\frac{\partial N_2\left[\iota_C\iota_U d_{11}\left(\sigma\right),\iota_U d_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{11}}\frac{\partial d_{11}}{\partial\sigma}\\+\frac{\partial N_2\left[\iota_C\iota_U d_{11}\left(\sigma\right),\iota_U d_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{12}}\frac{\partial d_{12}}{\partial\sigma}\end{array}\right\}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\left\{\begin{array}{l}\frac{\partial N_2\left[\iota_C\iota_U d_{21}\left(\sigma\right),\iota_U d_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}\\+\frac{\partial N_2\left[\iota_C\iota_U d_{21}\left(\sigma\right),\iota_U d_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{22}}\frac{\partial d_{22}}{\partial\sigma}\end{array}\right\}, \quad (8.5.99)$$
$$-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left[\iota_C\iota_U d_{21}\left(\sigma\right)\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}$$

Rearranging to exploit Lemmas 2 and 3, we have

$$\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11}\left(\sigma\right),\iota_U d_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{11}}\frac{\partial d_{11}}{\partial\sigma}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{21}\left(\sigma\right),\iota_U d_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}$$
$$-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left[\iota_C\iota_U d_{21}\left(\sigma\right)\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma}, \quad (8.5.100)$$
$$+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11}\left(\sigma\right),\iota_U d_{12}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{12}}\frac{\partial d_{12}}{\partial\sigma}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{21}\left(\sigma\right),\iota_U d_{22}\left(\sigma\right);\iota_C\rho\right]}{\partial d_{22}}\frac{\partial d_{22}}{\partial\sigma}$$

Based on Lemma 2

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{21}}\frac{\partial d_{11}}{\partial\sigma} \\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma} \\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left[\iota_C\iota_U d_{21}\right]}{\partial d_{21}}\frac{\partial d_{21}}{\partial\sigma} \\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{22}}\frac{\partial d_{12}}{\partial\sigma} \\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right]}{\partial d_{22}}\frac{\partial d_{22}}{\partial\sigma}
\end{aligned}
\quad , \qquad (8.5.101)
$$

Note

$$
\frac{\partial d_{11}}{\partial\sigma} = \frac{\partial\left(d_{21}+\sigma\sqrt{T_1-t}\right)}{\partial\sigma} = \frac{\partial d_{21}}{\partial\sigma}+\sqrt{T_1-t} \text{ and} \qquad (8.5.102)
$$

$$
\frac{\partial d_{12}}{\partial\sigma} = \frac{\partial\left(d_{22}+\sigma\sqrt{T_2-t}\right)}{\partial\sigma} = \frac{\partial d_{22}}{\partial\sigma}+\sqrt{T_2-t}\,. \qquad (8.5.103)
$$

Substituting for these partials and rearranging, we have

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \frac{\partial d_{21}}{\partial\sigma}\left\{
\begin{aligned}
&\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{21}} \\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right]}{\partial d_{21}} \\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left[\iota_C\iota_U d_{21}\right]}{\partial d_{21}}
\end{aligned}
\right\} \\
&+\frac{\partial d_{22}}{\partial\sigma}\left\{
\begin{aligned}
&\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{22}} \\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right]}{\partial d_{22}}
\end{aligned}
\right\} \\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{21}}\sqrt{T_1-t} \\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{22}}\sqrt{T_2-t}
\end{aligned}
\quad . \qquad (8.5.104)
$$

Again, based on Lemma 2 and 3, we have

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{11}}\sqrt{T_1-t} \\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{12}}\sqrt{T_2-t}
\end{aligned}
\quad . \qquad (8.5.105)
$$

Note

$$
\begin{aligned}
\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{11}} &= \frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial\left[\iota_C\iota_U d_{11}\right]}\frac{\partial\left[\iota_C\iota_U d_{11}\right]}{\partial d_{11}} \\
&= \iota_C\iota_U \frac{\partial}{\partial\left[\iota_C\iota_U d_{11}\right]}\int_{-\infty}^{\iota_C\iota_U d_{11}} N_1\left[\frac{\iota_U d_{12}-\iota_C\rho z_2}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_2\right)dz_2 \\
&= \iota_C\iota_U N_1\left[\frac{\iota_U d_{12}-\iota_C\rho\iota_C\iota_U d_{11}}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(\iota_C\iota_U d_{11}\right) \\
&= \iota_C\iota_U N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)
\end{aligned}, \qquad (8.5.106)
$$

and also

$$
\begin{aligned}
\frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial d_{12}} &= \frac{\partial N_2\left[\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right]}{\partial\left[\iota_U d_{12}\right]}\frac{\partial\left[\iota_U d_{12}\right]}{\partial d_{12}} \\
&= \iota_U \frac{\partial}{\partial\left[\iota_U d_{12}\right]}\int_{-\infty}^{\iota_U d_{12}} N_1\left[\frac{\iota_C\iota_U d_{11}-\iota_C\rho z_1}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_1\right)dz_1 \\
&= \iota_U N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)
\end{aligned}. \qquad (8.5.107)
$$

Substituting,

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\iota_C\iota_U N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right)\right]\sqrt{T_1-t} \\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\iota_U N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\right]\sqrt{T_2-t}
\end{aligned}. \qquad (8.5.108)
$$

Therefore, the compound option vega can be expressed as

$$
\frac{\partial}{\partial\sigma}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\begin{bmatrix}\sqrt{T_1-t}N_1\left(\iota_U\frac{d_{12}-\rho d_{11}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{11}\right) \\ +\iota_C\sqrt{T_2-t}N_1\left(\iota_C\iota_U\frac{d_{11}-\rho d_{12}}{\sqrt{1-\rho^2}}\right)n_1\left(d_{12}\right)\end{bmatrix}. \qquad (8.5.109)
$$

Compound option rho proof

$$
\begin{aligned}
\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\left(T_2-t\right)X_U N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right) \\
&+\iota_C B_{t,T_1,r}\left(T_1-t\right)X_C N\left(\iota_C\iota_U d_{21}\right)
\end{aligned}. \qquad (8.5.110)
$$

Proof: There are several variables that are a function of the interest rate, *r*.

$$
\begin{aligned}
&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial r}N_2\left[\iota_C\iota_U d_{11}\left(r\right),\iota_U d_{12}\left(r\right);\iota_C\rho\right]\\
&-\iota_C\iota_U X_U B_{t,T_2,r}\left(r\right)B_{T_1,T_2,-\hat{q}}\frac{\partial}{\partial r}N_2\left[\iota_C\iota_U d_{21}\left(r\right),\iota_U d_{22}\left(r\right);\iota_C\rho\right]\\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left[\iota_C\iota_U d_{21}\left(r\right),\iota_U d_{22}\left(r\right);\iota_C\rho\right]\frac{\partial}{\partial r}B_{t,T_2,r}\left(r\right)\\
&-\iota_C X_C B_{t,T_1,r}\left(r\right)\frac{\partial}{\partial r}N\left[\iota_C\iota_U d_{21}\left(r\right)\right]\\
&-\iota_C X_C N\left[\iota_C\iota_U d_{21}\left(r\right)\right]\frac{\partial}{\partial r}B_{t,T_1,r}\left(r\right)
\end{aligned}
\quad, \qquad (8.5.111)
$$

From lemma 1, we note

$$
\begin{aligned}
&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\left[\begin{array}{l}\dfrac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\dfrac{\partial d_{11}}{\partial r}\\+\dfrac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\dfrac{\partial d_{12}}{\partial r}\end{array}\right]\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\left[\begin{array}{l}\dfrac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\dfrac{\partial d_{21}}{\partial r}\\+\dfrac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\dfrac{\partial d_{22}}{\partial r}\end{array}\right]\\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}
\end{aligned}
\quad, \qquad (8.5.112)
$$

Rearranging to exploit Lemmas 2 and 3, we have

$$
\begin{aligned}
&\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{11}}\frac{\partial d_{11}}{\partial r}\\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{12}}\frac{\partial d_{12}}{\partial r}\\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r}\\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r}\\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}
\end{aligned}
\quad, \qquad (8.5.113)
$$

Based on Lemma 2

$$\begin{aligned}
\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r} \\
&+\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r} \\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r} \\
&-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}\frac{\partial d_{22}}{\partial r} \\
&-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}\frac{\partial d_{21}}{\partial r} \\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}
\end{aligned}, \qquad (8.5.114)$$

Note

$$\frac{\partial d_{11}}{\partial r}=\frac{\partial\left(d_{21}+\sigma\sqrt{T_1-t}\right)}{\partial r}=\frac{\partial d_{21}}{\partial r} \text{ and} \qquad (8.5.115)$$

$$\frac{\partial d_{12}}{\partial r}=\frac{\partial\left(d_{22}+\sigma\sqrt{T_2-t}\right)}{\partial r}=\frac{\partial d_{22}}{\partial r}. \qquad (8.5.116)$$

Substituting for these partials and rearranging, we have

$$\begin{aligned}
\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \frac{\partial d_{21}}{\partial r}\left[\begin{array}{l}
\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{21}} \\
-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{21}} \\
-\iota_C X_C B_{t,T_1,r}\frac{\partial N\left(\iota_C\iota_U d_{21}\right)}{\partial d_{21}}
\end{array}\right] \\
&+\frac{\partial d_{22}}{\partial r}\left[\begin{array}{l}
\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)}{\partial d_{22}} \\
-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}\frac{\partial N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)}{\partial d_{22}}
\end{array}\right] \\
&-\iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\frac{\partial}{\partial r}B_{t,T_2,r}-\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\frac{\partial}{\partial r}B_{t,T_1,r}
\end{aligned}, \qquad (8.5.117)$$

Again, based on Lemma 2 and 3 and partials with respect to B, we have the compound option rho as

$$\begin{aligned}
\frac{\partial}{\partial r}CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) &= \iota_C\iota_U X_U B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)\left(T_2-t\right)B_{t,T_2,r} \\
&+\iota_C X_C N\left(\iota_C\iota_U d_{21}\right)\left(T_1-t\right)B_{t,T_1,r}
\end{aligned}, \qquad (8.5.118)$$

Validation of partial differential equation

Recall the compound option partial differential equation can be expressed as

$$r\left(t\right)CO_D=\frac{\partial CO_D}{\partial t}+\left[r\left(t\right)-\hat{q}\left(t\right)\right]\frac{\partial CO_D}{\partial S}S+\frac{1}{2}\sigma^2\left(t\right)S\frac{\partial^2 CO_D}{\partial S^2}. \qquad (8.5.119)$$

We validate this equation by solving for theta or

$$\frac{\partial CO_D}{\partial t} = r(t)CO_D - \left[r(t) - \hat{q}(t)\right]\frac{\partial CO_D}{\partial S}S - \frac{1}{2}\sigma^2(t)S\frac{\partial^2 CO_D}{\partial S^2}. \tag{8.5.120}$$

Recall

$$\begin{aligned} CO(S,t,T_1,T_2,\iota_C,\iota_U) &= \iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right) \\ &-\iota_C\iota_U X_U B_{t,T_2,r} N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C\rho\right) - \iota_C X_C B_{t,T_1,r} N\left(\iota_C d_{21}\right) \end{aligned}, \tag{8.5.121}$$

$$\Delta_{CO_D} \equiv \frac{\partial CO_D(S,t,T_1,T_2)}{\partial S} = \iota_C\iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right), \text{ and} \tag{8.5.122}$$

$$\Gamma_{CO_D} = \frac{\partial^2 CO_D(S,t,T_1,T_2)}{\partial S^2} = \frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C\iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right]. \tag{8.5.123}$$

Thus, substituting these results into the rearranged partial differential equation, we have

$$\begin{aligned} \frac{\partial CO_D}{\partial t} &= r(t)\begin{bmatrix} \iota_C\iota_U S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right) \\ -\iota_C\iota_U X_U B_{t,T_2,r} N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C\rho\right) - \iota_C X_C B_{t,T_1,r} N\left(\iota_C d_{21}\right) \end{bmatrix} \\ &-\left[r(t) - \hat{q}(t)\right]S\left[\iota_C\iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right)\right] \\ &-\frac{1}{2}\sigma^2(t)S\left[\frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{S_t}\left[N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C\iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right]\right] \end{aligned}. \tag{8.5.124}$$

Cancelling terms,

$$\begin{aligned} \frac{\partial CO_D}{\partial t} &= -\iota_C\iota_U r(t) X_U B_{t,T_2,r} N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C\rho\right) - \iota_C r(t) X_C B_{t,T_1,r} N\left(\iota_C d_{21}\right) \\ &+\hat{q}(t)S\left[\iota_C\iota_U B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right)\right] \\ &-\frac{B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}}{2}\sigma^2(t)\left[N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{11})}{\sigma_{t,T_1}} + \iota_C N_1\left(\iota_C\iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\frac{n_1(d_{12})}{\sigma_{t,T_2}}\right] \end{aligned}. \tag{8.5.125}$$

Recall

$$\begin{aligned} &\Theta_{CO_D} \equiv \frac{\partial}{\partial t} CO_D(S,t,T_1,T_2,\iota_C,\iota_U) \\ &= -\frac{\sigma^2(t)}{2} S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta}\left\{N_1\left(\iota_U \frac{d_{12} - \rho d_{11}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1(d_{11})}{\sigma_{t,T_1}}\right] - \iota_C N_1\left(\iota_C\iota_U \frac{d_{11} - \rho d_{12}}{\sqrt{1-\rho^2}}\right)\left[\frac{n_1(d_{12})}{\sigma_{t,T_1}}\right]\right\} \\ &+\iota_C\iota_U \hat{q}(t) S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} N_2\left(\iota_C\iota_U d_{11}, \iota_U d_{12}; \iota_C\rho\right) - \iota_C\iota_U r(t) B_{t,T_2,r} X_U N_2\left(\iota_C\iota_U d_{21}, \iota_U d_{22}; \iota_C\rho\right) - \iota_C r(t) B_{t,T_1,r} X_C N_1\left(\iota_C d_{21}\right) \end{aligned}. \tag{8.5.126}$$

Thus, the PDE obtained from hedging with the underlying instrument is satisfied.

Lemma 1: Leibniz integral rule for double integral applied to bivariate normal

Assuming $N_2\left(a(x), b(x); \rho\right)$ is a standard bivariate normal cumulative distribution function, then

$$\frac{dN_2\left(a(x), b(x); \rho\right)}{dx} = \frac{dN_2\left(a(x), b(x); \rho\right)}{db(x)}\frac{db(x)}{dx} + \frac{dN_2\left(a(x), b(x); \rho\right)}{da(x)}\frac{da(x)}{dx}. \tag{8.5.127}$$

Lemma 1 Proof: Assuming $f(x, z_1, z_2)$ is a continuous function where the needed derivatives exist, then

$$y\left(x,z_1,z_2\right)\equiv\int_{l_2(x)}^{u_2(x)}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,z_2\right)dz_1\,dz_2=\int_{l_2(x)}^{u_2(x)}g\left(x,z_2\right)dz_2, \tag{8.5.128}$$

where

$$g\left(x,z_2\right)\equiv\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,z_2\right)dz_1, \tag{8.5.129}$$

and note

$$\frac{\partial y\left(x,z_1,z_2\right)}{\partial u_1\left(x\right)}=\int_{l_2(x)}^{u_2(x)}f\left(x,u_1\left(x\right),z_2\right)dz_2, \tag{8.5.130}$$

$$\frac{\partial y\left(x,z_1,z_2\right)}{\partial u_2\left(x\right)}=\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,u_2\left(x\right)\right)dz_1, \tag{8.5.131}$$

$$\frac{\partial y\left(x,z_1,z_2\right)}{\partial l_1\left(x\right)}=-\int_{l_2(x)}^{u_2(x)}f\left(x,l_1\left(x\right),z_2\right)dz_2,\text{ and} \tag{8.5.132}$$

$$\frac{\partial y\left(x,z_1,z_2\right)}{\partial l_2\left(x\right)}=-\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,l_2\left(x\right)\right)dz_1. \tag{8.5.133}$$

Applying the Leibniz integral rule to the outer integral

$$\begin{aligned}\frac{dy\left(x,z_1,z_2\right)}{dx}&=\int_{l_2(x)}^{u_2(x)}\frac{d}{dx}\left[g\left(x,z_2\right)\right]dz_2+\frac{du_2\left(x\right)}{dx}g\left(x,u_2\left(x\right)\right)-\frac{dl_2\left(x\right)}{dx}g\left(x,l_2\left(x\right)\right)\\&=\int_{l_2(x)}^{u_2(x)}\frac{d}{dx}\left[\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,z_2\right)dz_1\right]dz_2+\frac{du_2\left(x\right)}{dx}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,u_2\left(x\right)\right)dz_1-\frac{dl_2\left(x\right)}{dx}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,l_2\left(x\right)\right)dz_1\end{aligned}, \tag{8.5.134}$$

and also applying the Leibniz integral rule to the inner integral (first term in the equation above)

$$\frac{d}{dx}\left[\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,z_2\right)dz_1\right]=\int_{l_1(x)}^{u_1(x)}\frac{df\left(x,z_1,z_2\right)}{dx}dz_1+\frac{du_1\left(x\right)}{dx}f\left(x,u_1\left(x\right),z_2\right)-\frac{dl_1\left(x\right)}{dx}f\left(x,l_1\left(x\right),z_2\right). \tag{8.5.135}$$

Substituting this result into the previous equation, we have

$$\begin{aligned}\frac{dy\left(x,z_1,z_2\right)}{dx}&=\int_{l_2(x)}^{u_2(x)}\left[\int_{l_1(x)}^{u_1(x)}\frac{df\left(x,z_1,z_2\right)}{dx}dz_1+\frac{du_1\left(x\right)}{dx}f\left(x,u_1\left(x\right),z_2\right)-\frac{dl_1\left(x\right)}{dx}f\left(x,l_1\left(x\right),z_2\right)\right]dz_2\\&+\frac{du_2\left(x\right)}{dx}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,u_2\left(x\right)\right)dz_1-\frac{dl_2\left(x\right)}{dx}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,l_2\left(x\right)\right)dz_1\end{aligned}. \tag{8.5.136}$$

Thus

$$\begin{aligned}\frac{dy\left(x,z_1,z_2\right)}{dx}&=\int_{l_2(x)}^{u_2(x)}\int_{l_1(x)}^{u_1(x)}\frac{df\left(x,z_1,z_2\right)}{dx}dz_1\,dz_2\\&+\frac{du_1\left(x\right)}{dx}\int_{l_2(x)}^{u_2(x)}f\left(x,u_1\left(x\right),z_2\right)dz_2-\frac{dl_1\left(x\right)}{dx}\int_{l_2(x)}^{u_2(x)}f\left(x,l_1\left(x\right),z_2\right)dz_2,\\&+\frac{du_2\left(x\right)}{dx}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,u_2\left(x\right)\right)dz_1-\frac{dl_2\left(x\right)}{dx}\int_{l_1(x)}^{u_1(x)}f\left(x,z_1,l_2\left(x\right)\right)dz_1\end{aligned} \tag{8.5.137}$$

or

$$\frac{dy(x,z_1,z_2)}{dx}=\int_{l_2(x)}^{u_2(x)}\int_{l_1(x)}^{u_1(x)}\frac{df(x,z_1,z_2)}{dx}dz_1\,dz_2+\frac{\partial y(x,z_1,z_2)}{\partial u_1(x)}\frac{du_1(x)}{dx}+\frac{\partial y(x,z_1,z_2)}{\partial l_1(x)}\frac{dl_1(x)}{dx}$$
$$+\frac{\partial y(x,z_1,z_2)}{\partial u_2(x)}\frac{du_2(x)}{dx}+\frac{\partial y(x,z_1,z_2)}{\partial l_2(x)}\frac{dl_2(x)}{dx}. \quad (8.5.138)$$

QED

Example: Bivariate standard normal cumulative distribution function

Consider

$$N_2\left(a(x),b(x);\rho\right)\equiv\int_{-\infty}^{a(x)}\int_{-\infty}^{b(x)}\frac{\exp\left\{-\frac{z_1^2-2\rho z_1 z_2+z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_1\,dz_2=\int_{-\infty}^{a(x)}\int_{-\infty}^{b(x)}f(z_1,z_2)dz_1\,dz_2. \quad (8.5.139)$$

Based on Lemma 1, we have

$$\frac{dN_2\left(a(x),b(x);\rho\right)}{dx}=\int_{-\infty}^{a(x)}\int_{-\infty}^{b(x)}\frac{df(z_1,z_2)}{dx}dz_1\,dz_2+\frac{db(x)}{dx}\int_{-\infty}^{a(x)}f\left(b(x),z_2\right)dz_2-\frac{d(-\infty)}{dx}\int_{-\infty}^{a(x)}f(-\infty,z_2)dz_2$$
$$+\frac{da(x)}{dx}\int_{-\infty}^{b(x)}f\left(z_1,a(x)\right)dz_1-\frac{d(-\infty)}{dx}\int_{-\infty}^{b(x)}f(z_1,-\infty)dz_1 .\quad (8.5.140)$$

Thus,

$$\frac{dN_2\left(a(x),b(x);\rho\right)}{dx}=\frac{db(x)}{dx}\int_{-\infty}^{a(x)}f\left(b(x),z_2\right)dz_2+\frac{da(x)}{dx}\int_{-\infty}^{b(x)}f\left(z_1,a(x)\right)dz_1$$
$$=\frac{db(x)}{dx}\int_{-\infty}^{a(x)}\frac{\exp\left\{-\frac{b(x)^2-2\rho b(x)z_2+z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_2+\frac{da(x)}{dx}\int_{-\infty}^{b(x)}\frac{\exp\left\{-\frac{z_1^2-2\rho z_1 a(x)+a(x)^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_1, \quad (8.5.141)$$

or

$$\frac{dN_2\left(a(x),b(x);\rho\right)}{dx}=\frac{dN_2\left(a(x),b(x);\rho\right)}{db(x)}\frac{db(x)}{dx}+\frac{dN_2\left(a(x),b(x);\rho\right)}{da(x)}\frac{da(x)}{dx}. \quad (8.5.142)$$

Lemma 2: Compound option partial with respect to d_{21}

We assert

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=0, \quad (8.5.143)$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}. \quad (8.5.144)$$

Lemma 2 Proof: Recall

$$d_{11}=d_{21}+\sigma_{t,T_1}, \quad (8.5.145)$$

and therefore

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{11}(S)}=\frac{\partial N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U d_{12}(S);\iota_C\rho\right\}}{\partial\left[d_{21}(S)+\sigma_{t,T_1}\right]}\frac{\partial\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\partial d_{21}(S)}$$
$$=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}. \quad (8.5.146)$$

Taking the derivative, we have

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}-\iota_C X_C B_{t,T_1,r}\frac{\partial N_1\left[\iota_C d_{21}(S)\right]}{\partial d_{21}}. \quad (8.5.147)$$

We now focus on $\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}$. Note

$$N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\int_{-\infty}^{\iota_U d_{22}(S)}n_2\left(z_1,z_2\right)dz_2\,dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\left[\int_{-\infty}^{\iota_U d_{22}(S)}n_1\left(z_2|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1=\int_{-\infty}^{\iota_U d_{22}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}n_1\left(z_1|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2. \quad (8.5.148)$$

Thus

$$N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\left[\int_{-\infty}^{\iota_U d_{22}(S)}n_1\left(z_2|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}N_1\left[\frac{\iota_U d_{22}(S)-\iota_C\rho z_1}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_1\right)dz_1, \quad (8.5.149)$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{21}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial\left[\iota_C\iota_U d_{21}(S)\right]}\frac{\partial\left[\iota_C\iota_U d_{21}(S)\right]}{\partial d_{21}(S)}$$
$$=\iota_C\iota_U\frac{\partial}{\partial\left[\iota_C\iota_U d_{21}(S)\right]}\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}N_1\left[\frac{\iota_U d_{22}(S)-\iota_C\rho z_1}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_1\right)dz_1$$
$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\iota_C\rho\iota_C\iota_U d_{21}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left[\iota_C\iota_U d_{21}(S)\right]$$
$$=\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\rho\iota_U d_{21}(S)}{\sqrt{1-\rho^2}}\right]n_1\left[\iota_C\iota_U d_{21}(S)\right]. \quad (8.5.150)$$

We now focus on $\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}$. Note

$$N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\int_{-\infty}^{\iota_U d_{12}(S)} n_2\left(z_1,z_2\right)dz_2\,dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\left[\int_{-\infty}^{\iota_U d_{12}(S)} n_1\left(z_2|z_1\right)n_1\left(z_1\right)dz_2\right]dz_1=\int_{-\infty}^{\iota_U d_{12}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{11}(S)} n_1\left(z_1|z_2\right)n_1\left(z_2\right)dz_1\right]dz_2 . \quad (8.5.151)$$

And

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{21}(S)}=\frac{\partial N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}}{\partial d_{21}(S)}, \quad (8.5.152)$$

$$N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_2\left(z_1,z_2\right)dz_2\,dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\left[\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_1\left(z_2|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1 \text{, and} \quad (8.5.153)$$
$$=\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]}\left[\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]} n_1\left(z_1|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2$$

$$\frac{\partial}{\partial d_{21}}N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}$$
$$=\frac{\partial}{\partial\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}}N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}\frac{\partial\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}}{\partial d_{21}}$$
$$=\iota_C\iota_U\frac{\partial}{\partial d_{21}}\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\left[\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_1\left(z_2|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1 . \quad (8.5.154)$$
$$=\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\iota_C\rho\left(\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}$$
$$=\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\rho\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\sqrt{1-\rho^2}}\right\}n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}$$

Finally

$$\frac{\partial}{\partial d_{21}}N_1\left[\iota_C d_{21}(S)\right]=\iota_C n_1\left[\iota_C d_{21}(S)\right]. \quad (8.5.155)$$

Therefore,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)$$

$$=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\left(\iota_C\iota_U N_1\left\{\frac{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]-\rho\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}{\sqrt{1-\rho^2}}\right\}n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}\right). \quad (8.5.156)$$

$$-\iota_C\iota_U X_U B_{t,T_2,r}\left\{\iota_C\iota_U N_1\left[\frac{\iota_U d_{22}(S)-\rho\iota_U d_{21}(S)}{\sqrt{1-\rho^2}}\right]n_1\left[\iota_C\iota_U d_{21}(S)\right]\right\}$$

$$-\iota_C X_C B_{t,T_1,r}\left\{\iota_C n_1\left[\iota_C d_{21}(S)\right]\right\}$$

Note

$$n_1\left[\iota_C d_{21}(S)\right]=\frac{\exp\left\{-\frac{\left[\iota_C d_{21}(S)\right]^2}{2}\right\}}{\sqrt{2\pi}}=\frac{\exp\left\{-\frac{\left[\iota_C\iota_U d_{21}(S)\right]^2}{2}\right\}}{\sqrt{2\pi}}\text{, and} \quad (8.5.157)$$

$$=\frac{\exp\left[-\frac{d_{21}^2(S)}{2}\right]}{\sqrt{2\pi}}=n_1\left[\iota_C\iota_U d_{21}(S)\right]=n_1\left[d_{21}(S)\right]$$

$$n_1\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}=\frac{\exp\left(-\frac{\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]\right\}^2}{2}\right)}{\sqrt{2\pi}}=\frac{\exp\left\{-\frac{\left[d_{21}(S)+\sigma_{t,T_1}\right]^2}{2}\right\}}{\sqrt{2\pi}}$$

$$=n_1\left[d_{21}(S)\right]\exp\left[-\frac{2\sigma_{t,T_1}d_{21}(S)+\sigma_{t,T_1}^2}{2}\right]=n_1\left[d_{21}(S)\right]\exp\left[-\sigma_{t,T_1}d_{21}(S)-\frac{\sigma_{t,T_1}^2}{2}\right]$$

$$=n_1\left[d_{21}(S)\right]\exp\left(-\frac{\sigma_{t,T_1}^2}{2}\right)\exp\left[-\sigma_{t,T_1}d_{21}(S)\right]$$

$$=n_1\left[d_{21}(S)\right]\exp\left(-\frac{\sigma_{t,T_1}^2}{2}\right)\exp\left[-\sigma_{t,T_1}\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S_{T_1}^*}\right)-\frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}\right]$$

$$=\frac{S_{T_1}^* B_{t,T_1,r-\hat{q}}}{S_t}n_1\left[d_{21}(S)\right]. \quad (8.5.158)$$

Therefore,

$$\frac{\partial}{\partial d_{21}} CO_D\left(S,t,T_1,T_2\right)$$
$$= S_t B_{t,T_1,\hat{q}} B_{T_1,T_2,\delta} \iota_C \iota_U \left(\iota_C \iota_U N_1 \left\{ \frac{\iota_U \left[d_{22}(S) + \sigma_{t,T_2} \right] - \rho \iota_U \left[d_{21}(S) + \sigma_{t,T_1} \right]}{\sqrt{1-\rho^2}} \right\} \frac{S^*_{T_1} B_{t,T_1,r-\hat{q}}}{S_t} n_1 \left[d_{21}(S) \right] \right). \quad (8.5.159)$$
$$- X_U B_{t,T_2,r} \iota_C \iota_U \left\{ \iota_C \iota_U N_1 \left[\frac{\iota_U d_{22}(S) - \rho \iota_U d_{21}(S)}{\sqrt{1-\rho^2}} \right] n_1 \left[d_{21}(S) \right] \right\}$$
$$- X_C B_{t,T_1,r} \iota_C \left\{ \iota_C n_1 \left[d_{21}(S) \right] \right\}$$

Eliminating squared indicator functions and rearranging,

$$\frac{\partial}{\partial d_{21}} CO_D\left(S,t,T_1,T_2\right) = n_1\left[d_{21}(S) \right] B_{t,T_1,r} \left(\begin{array}{l} S^*_{T_1} B_{T_1,T_2,\delta} N_1 \left\{ \iota_U \frac{d_{22}(S) + \sigma_{t,T_2} - \rho \left[d_{21}(S) + \sigma_{t,T_1} \right]}{\sqrt{1-\rho^2}} \right\} \\ - X_U B_{T_1,T_2,r} N_1 \left[\iota_U \frac{d_{22}(S) - \rho d_{21}(S)}{\sqrt{1-\rho^2}} \right] - X_C \end{array} \right). \quad (8.5.160)$$

Note

$$\frac{d_{22} - \rho d_{21}}{\sqrt{1-\rho^2}} = \frac{d_{22} - \frac{\sigma_{t,T_1}}{\sigma_{t,T_2}} d_{21}}{\frac{\sigma_{T_1,T_2}}{\sigma_{t,T_2}}} = \frac{\sigma_{t,T_2} d_{22} - \sigma_{t,T_1} d_{21}}{\sigma_{T_1,T_2}}$$

$$= \frac{\sigma_{t,T_2} \left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})} B_{T_1,T_2,-(r-\delta)}}{X_U} \right) - \frac{\sigma^2_{t,T_2}}{2}}{\sigma_{t,T_2}} \right] - \sigma_{t,T_1} \left[\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}} \right) - \frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}} \right]}{\sigma_{T_1,T_2}}$$

$$= \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})} B_{T_1,T_2,-(r-\delta)}}{X_U} \right) - \frac{\sigma^2_{t,T_2}}{2} - \ln\left(\frac{S_t B_{t,T_1,-(r-\hat{q})}}{S^*_{T_1}} \right) + \frac{\sigma^2_{t,T_1}}{2}}{\sigma_{T_1,T_2}} = \frac{\ln\left(\frac{S_t B_{T_1,T_2,-(r-\delta)} S^*_{T_1}}{X_U S_t} \right) - \left(\frac{\sigma^2_{t,T_2}}{2} - \frac{\sigma^2_{t,T_1}}{2} \right)}{\sigma_{T_1,T_2}}$$

$$= \frac{\ln\left(\frac{S^*_{T_1} B_{T_1,T_2,-(r-\delta)}}{X_U} \right) - \frac{\sigma^2_{T_1,T_2}}{2}}{\sigma_{T_1,T_2}} = d^*_{2,T_1,T_2} \quad .(8.5.161)$$

And

$$\frac{d_{22}-\rho\left(d_{21}+\sigma_{t,T_1}\right)+\sigma_{t,T_2}}{\sqrt{1-\rho^2}}=\frac{d_{22}-\rho d_{21}}{\sqrt{1-\rho^2}}+\frac{\sigma_{t,T_2}-\rho\sigma_{t,T_1}}{\sqrt{1-\rho^2}}$$

$$=d^*_{2,T_1,T_2}+\frac{\sigma_{t,T_2}-\frac{\sigma_{t,T_1}}{\sigma_{t,T_2}}\sigma_{t,T_1}}{\frac{\sigma_{T_1,T_2}}{\sigma_{t,T_2}}}=d^*_{2,T_1,T_2}+\frac{\sigma^2_{t,T_2}-\sigma^2_{t,T_1}}{\sigma_{T_1,T_2}}=d^*_{2,T_1,T_2}+\frac{\sigma^2_{T_1,T_2}}{\sigma_{T_1,T_2}}=d^*_{1,T_1,T_2}. \tag{8.5.162}$$

Therefore,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=n_1\left[d_{21}(S)\right]B_{t,T_1,r}\left\{S^*_{T_1}B_{T_1,T_2,\delta}N_1\left(\iota_U d^*_{1,T_1,T_2}\right)-X_U B_{T_1,T_2,r}N_1\left(\iota_U d^*_{2,T_1,T_2}\right)-X_C\right\}, \tag{8.5.163}$$

and therefore

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=\iota_U n_1\left(d_{21}(S)\right)B_{t,T_1,r}\left\{\iota_U S^*_{T_1}B_{T_1,T_2,\delta}N_1\left(\iota_U d^*_{1,T_1,T_2}\right)-\iota_U X_U B_{T_1,T_2,r}N_1\left(\iota_U d^*_{2,T_1,T_2}\right)-\iota_U X_C\right\}. \tag{8.5.164}$$

Recall

$$\iota_U S^*_{T_1}B_{T_1,T_2,\delta}N_1\left(\iota_U d^*_{1,T_1,T_2}\right)-\iota_U X_U B_{T_1,T_2,r}N_1\left(\iota_U d^*_{2,T_1,T_2}\right)-\iota_U X_C=0. \tag{8.5.165}$$

Thus,

$$\frac{\partial}{\partial d_{21}}CO_D\left(S,t,T_1,T_2\right)=0. \tag{8.5.166}$$

Lemma 3: Compound option partial with respect to d_{22}
We assert

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)=0, \tag{8.5.167}$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{12}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}. \tag{8.5.168}$$

Lemma 3 Proof: Recall

$$d_{12}=d_{22}+\sigma_{t,T_2}, \tag{8.5.169}$$

and follow same logic as the first part of Lemma 2. Taking the derivative,

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}. \tag{8.5.170}$$

We now focus on $\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$. Note

$$N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\int_{-\infty}^{\iota_U d_{22}(S)}n_2\left(z_1,z_2\right)dz_2\,dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}\left[\int_{-\infty}^{\iota_U d_{22}(S)}n_1\left(z_2|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1=\int_{-\infty}^{\iota_U d_{22}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{21}(S)}n_1\left(z_1|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2. \tag{8.5.171}$$

Thus

$$N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_U d_{22}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{21}(S)} n_1\left(z_1\middle|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2$$
$$=\int_{-\infty}^{\iota_U d_{22}(S)} N\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho z_2}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_2\right)dz_2 , \quad (8.5.172)$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}=\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial\left[\iota_U d_{22}(S)\right]}\frac{\partial\left[\iota_U d_{22}(S)\right]}{\partial d_{22}(S)}$$
$$=\iota_U\frac{\partial}{\partial\left[\iota_U d_{22}\right]}\int_{-\infty}^{\iota_U d_{22}(S)} N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho z_2}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(z_2\right)dz_2 . \quad (8.5.173)$$

Therefore,

$$\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}=\iota_U N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho\iota_U d_{22}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left[\iota_U d_{22}(S)\right]. \quad (8.5.174)$$

We now focus on $\dfrac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$. Note

$$N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\int_{-\infty}^{\iota_U d_{12}(S)} n_2\left(z_1,z_2\right)dz_2\,dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U d_{11}(S)}\left[\int_{-\infty}^{\iota_U d_{12}(S)} n_1\left(z_2\middle|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1=\int_{-\infty}^{\iota_U d_{12}(S)}\left[\int_{-\infty}^{\iota_C\iota_U d_{11}(S)} n_1\left(z_1\middle|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2 . \quad (8.5.175)$$

And

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}=\frac{\partial N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}}{\partial d_{22}(S)}, \quad (8.5.176)$$

$$N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_2\left(z_1,z_2\right)dz_2\,dz_1$$
$$=\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}\left[\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]} n_1\left(z_2\middle|z_1\right)dz_2\right]n_1\left(z_1\right)dz_1 , \quad (8.5.177)$$
$$=\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]}\left[\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]} n_1\left(z_1\middle|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2$$

$$\frac{\partial}{\partial d_{22}}N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}$$

$$=\frac{\partial}{\partial\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}}N_2\left\{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right],\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right];\iota_C\rho\right\}\frac{\partial\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}}{\partial d_{22}}$$

$$=\iota_U\frac{\partial}{\partial d_{22}}\int_{-\infty}^{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]}\left[\int_{-\infty}^{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]}n_1\left(z_1|z_2\right)dz_1\right]n_1\left(z_2\right)dz_2 \quad , \qquad (8.5.178)$$

$$=\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}$$

and

$$\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$

$$=\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}. \qquad (8.5.179)$$

Substituting these two partial derivatives,

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\frac{\partial N_2\left[\iota_C\iota_U d_{11}(S),\iota_U d_{12}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$

$$-\iota_C\iota_U X_U B_{t,T_2,r}\frac{\partial N_2\left[\iota_C\iota_U d_{21}(S),\iota_U d_{22}(S);\iota_C\rho\right]}{\partial d_{22}(S)}$$

$$=\iota_C\iota_U S_t B_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}. \qquad (8.5.180)$$

$$-\iota_C\iota_U X_U B_{t,T_2,r}\iota_U N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho\iota_U d_{22}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left[\iota_U d_{22}(S)\right]$$

Note

$$n_1\left(\iota_U d_{22}\right)=\frac{\exp\left[-\frac{\left(\iota_U d_{22}\right)^2}{2}\right]}{\sqrt{2\pi}}=\frac{\exp\left(-\frac{d_{22}^2}{2}\right)}{\sqrt{2\pi}}=n_1\left(d_{22}\right), \qquad (8.5.181)$$

and

$$n_1\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}=\frac{\exp\left(-\frac{\left\{\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right\}^2}{2}\right)}{\sqrt{2\pi}}=n_1\left[d_{22}(S)+\sigma_{t,T_2}\right]$$

$$=\frac{\exp\left[-\frac{\left(d_{22}+\sigma_{t,T_2}\right)^2}{2}\right]}{\sqrt{2\pi}}=n_1\left(d_{21}\right)\exp\left(-\frac{2\sigma_{t,T_2}d_{22}+\sigma_{t,T_2}^2}{2}\right)=n_1\left(d_{21}\right)\exp\left(-\sigma_{t,T_2}d_{22}-\frac{\sigma_{t,T_2}^2}{2}\right)$$

$$=n_1\left(d_{22}\right)\exp\left(-\frac{\sigma_{t,T_2}^2}{2}\right)\exp\left(-\sigma_{t,T_2}d_{22}\right) \qquad . \qquad (8.5.182)$$

$$=n_1\left(d_{22}\right)\exp\left(-\frac{\sigma_{t,T_2}^2}{2}\right)\exp\left\{-\sigma_{t,T_2}\frac{\ln\left[S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}\big/X_U\right]-\frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}\right\}$$

$$=\frac{X_U}{S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}n_1\left(d_{22}\right)$$

Substituting

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)$$

$$=\iota_C\iota_U S_tB_{t,T_1,\hat{q}}B_{T_1,T_2,\delta}\iota_U N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}\frac{X_U}{S_tB_{t,T_1,-(r-\hat{q})}B_{T_1,T_2,-(r-\delta)}}n_1\left(d_{22}\right). \qquad (8.5.183)$$

$$-\iota_C\iota_U X_UB_{t,T_2,r}\iota_U N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho\iota_U d_{22}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]n_1\left(d_{22}\right)$$

Rearranging and cancelling terms,

$$\frac{\partial}{\partial d_{22}}CO_D\left(S,t,T_1,T_2\right)$$

$$=\iota_C X_UB_{t,T_2,r}n_1\left(d_{22}\right)\left(\begin{array}{l}N_1\left\{\frac{\iota_C\iota_U\left[d_{21}(S)+\sigma_{t,T_1}\right]-\iota_C\rho\left(\iota_U\left[d_{22}(S)+\sigma_{t,T_2}\right]\right)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right\}\\-N_1\left[\frac{\iota_C\iota_U d_{21}(S)-\iota_C\rho\iota_U d_{22}(S)}{\sqrt{1-\left(\iota_C\rho\right)^2}}\right]\end{array}\right). \qquad (8.5.184)$$

Thus,

$$\frac{\partial}{\partial d_{22}} CO_D\left(S,t,T_1,T_2\right)$$

$$= \iota_C X_U B_{t,T_2,r} n_1\left(d_{22}\right)\begin{pmatrix} N_1\left\{\iota_C \iota_U \frac{\left[d_{21}(S)+\sigma_{t,T_1}\right]-\rho\left[d_{22}(S)+\sigma_{t,T_2}\right]}{\sqrt{1-\rho^2}}\right\} \\ -N_1\left[\iota_C \iota_U \frac{d_{21}(S)-\rho d_{22}(S)}{\sqrt{1-\rho^2}}\right] \end{pmatrix}. \quad (8.5.185)$$

Note

$$\frac{d_{21}+\sigma_{t,T_1}-\rho\left(d_{22}+\sigma_{t,T_2}\right)}{\sqrt{1-\rho^2}} = \frac{d_{21}-\rho d_{22}}{\sqrt{1-\rho^2}} + \frac{\sigma_{t,T_1}-\rho\sigma_{t,T_2}}{\sqrt{1-\rho^2}}, \quad (8.5.186)$$

and recall $\rho = \frac{\sigma_{t,T_1}}{\sigma_{t,T_2}}$.

QED

Summary

In this module, we introduced various static risk measures related to geometric Brownian motion-based compound option valuation models. We examined the sensitivity of the compound option values and Greeks to changes in selected underlying parameters. After reviewing the valuation model, we derived several Greeks. Finally, we concluded this module with a brief contrast between numerical and analytic Greek computations illustrated with R code.

References

See Module 5.7.