Module 8.3

Static Risk Measures
GBM-Based Option Valuation Models

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Overview

- Review static risk measures related to the GBMOVM
- Contrast European-style with American-style (binomial) results
- Understand role of dividend yield
- Identify measurement error with binomial compared with GBMOVM (European-style)

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Central Finance Concepts

- GBMOVM option Greeks
- Measurement error
- Graphical analysis of option Greeks
  - Delta
  - Gamma
  - Theta
  - Vega
  - Rho

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Option Greeks

- Static risk measure
- Mathematical derivatives
- Coherent
  - Delta (first derivative, underlying price)
  - Gamma (second derivative, underlying price)
  - Theta (first derivative, calendar time)
- Incoherent
  - Vega (first derivative, volatility)
  - Rho (first derivative, interest rate)

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Measurement Error

- Contrast binomial OVM and GBMOVM
- European-style and w/ and w/o dividends
- Error with respect to:
  - Stock price
  - Volatility
  - Time to maturity
- Graphs illustrate N = 250

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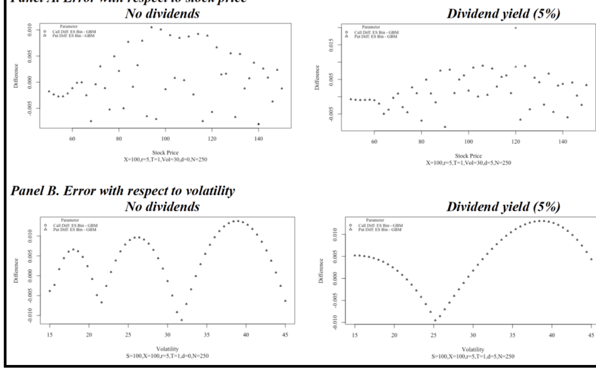
Binomial measurement error is relatively small when compared to GBMOVM when N = 250.

Figure 9.3.1. Measurement error between binomial and GBMOVM
Panel A. Error with respect to stock price



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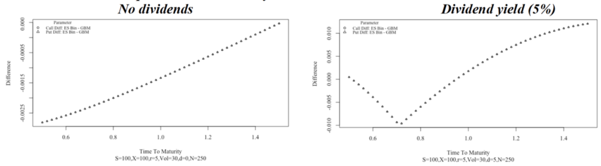
Binomial measurement error is relatively small when compared to GBMOVM when N = 250.

Panel C. Error with respect to time to maturity

No dividends *Dividend yield (5%)*



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Delta

- Panel A: The slopes of the lines is delta as illustrated in Panel B
- Panel B illustrates the influence of the early exercise feature on the delta for the American-style options
- Panel C and D show the influence of volatility and time to maturity

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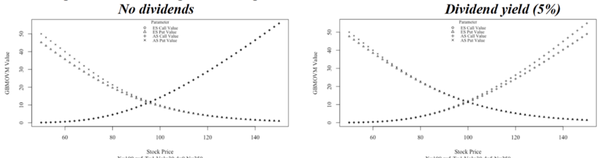
Figure 9.3.2. Call and put deltas based on GBMOVM with and without dividends

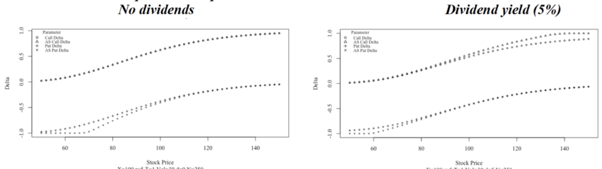
Panel A. Option value with respect to stock price

No dividends *Dividend yield (5%)*



Panel B. Delta with respect to stock price

No dividends *Dividend yield (5%)*



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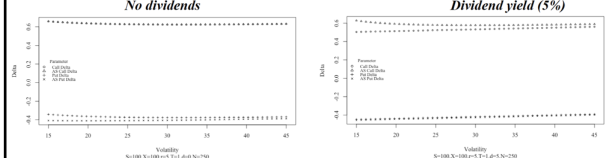
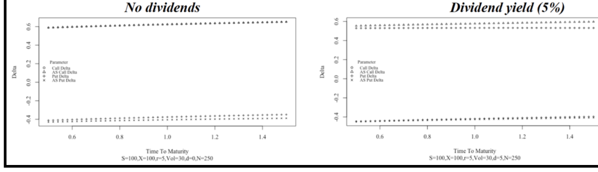
Panel C. Delta with respect to volatility

No dividends *Dividend yield (5%)*



Panel D. Delta with respect to time to maturity

No dividends *Dividend yield (5%)*



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Gamma

- The lognormal distribution assumption is clear in Panel A except when the early exercise feature impacts option gammas
- Panels B and C show that the gamma increases with declining volatility and with declining time to maturity

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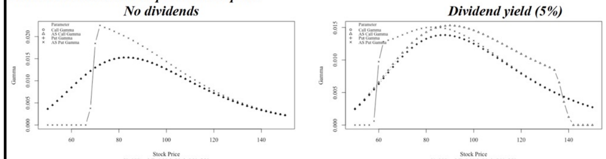
Figure 9.3.3. Call and put gamma based on GBMOVM with and without dividends

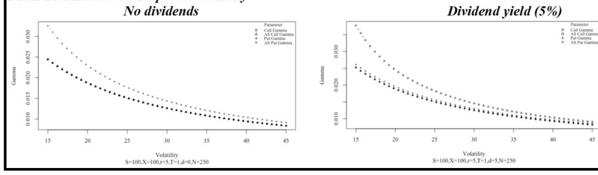
Panel A. Gamma with respect to stock price

No dividends *Dividend yield (5%)*



Panel B. Gamma with respect to volatility

No dividends *Dividend yield (5%)*



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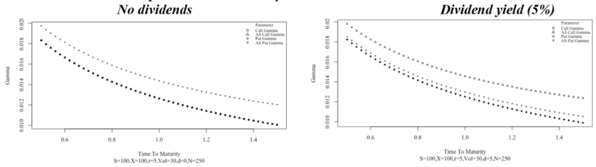
Panel C. Gamma with respect to time to maturity

No dividends | *Dividend yield (5%)*



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Theta

- Panel A: Well-known time value decay
- Panel B: Non-monotonic relationship with the stock price. At-the-money options have the highest time value
- Panel C shows that theta generally declines with higher volatility
- Panel D shows that theta generally increases with time to maturity

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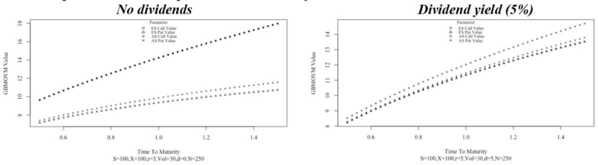
Figure 9.3.4. Call and put thetas based on GBMOVM with and without dividends

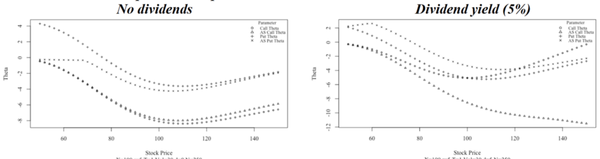
Panel A. Option value with respect to time to maturity

No dividends | *Dividend yield (5%)*



Panel B. Theta with respect to stock price

No dividends | *Dividend yield (5%)*



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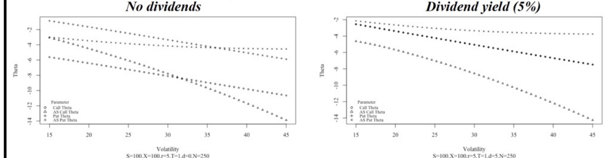
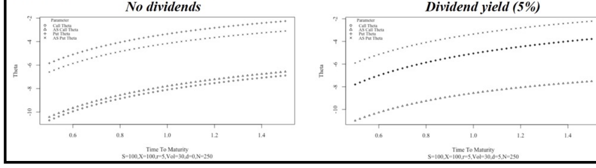
Panel C. Theta with respect to volatility

No dividends | *Dividend yield (5%)*



Panel D. Theta with time to maturity

No dividends | *Dividend yield (5%)*



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Vega

- Deep out-of-the-money: Vega is near zero
- Deep in-the-money: Vega is near zero
- Panel A: Option values increase with volatility
- Panel B: Vega mirrors the lognormal distribution
- Panel C: Vega has complex link with volatility
- Panel D: Vega generally increases with time to maturity.

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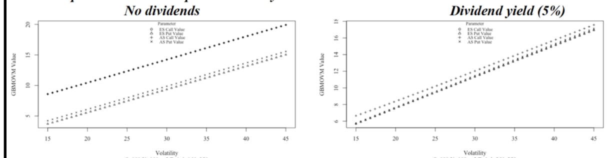
Figure 9.3.5. Call and put vegas based on GBMOVM with and without dividends

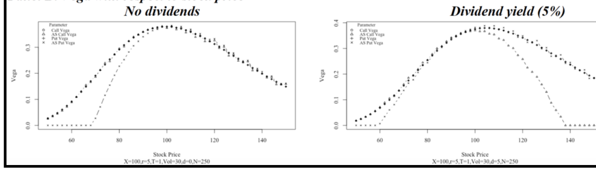
Panel A. Option value with respect to volatility

No dividends | *Dividend yield (5%)*



Panel B. Vega with respect to stock price

No dividends | *Dividend yield (5%)*



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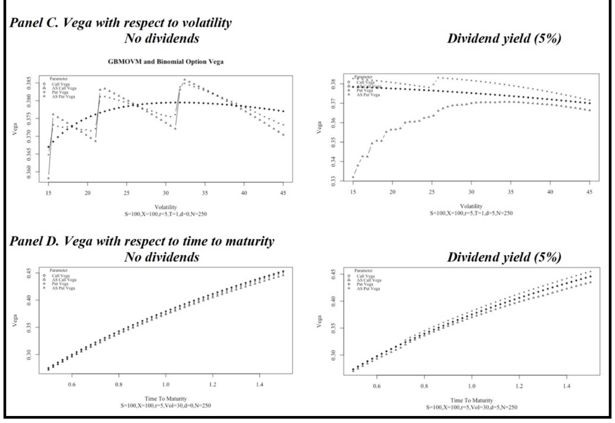

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Rho

- Panel A shows that the rho generally increases with the stock price, except for American-style puts
- Panel B shows that rho generally declines with volatility.
- Panel C shows that rho generally increases with time to maturity for calls and decreases for puts.

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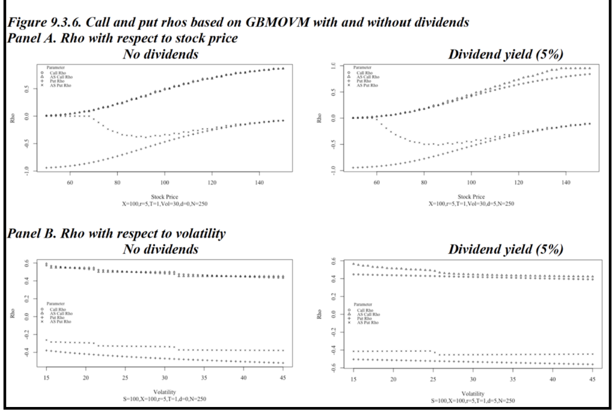

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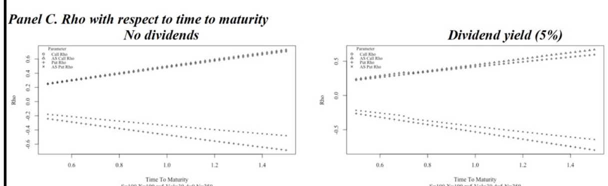

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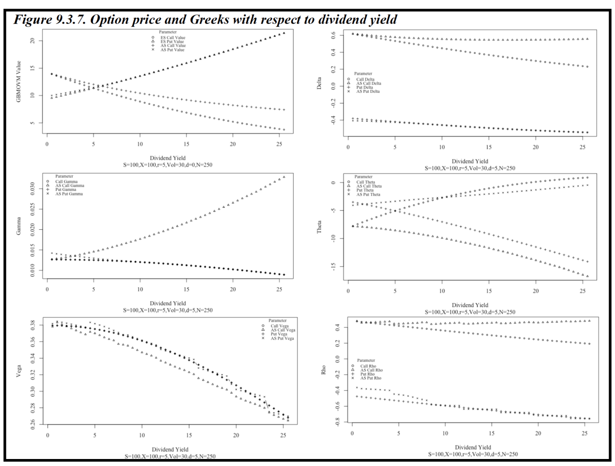

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Quantitative Finance Materials

- Review GBMOVM
- Define option Greeks with selected insights
- Review sensitivity to dividend yield
- Introduce extended Greeks
- Estimating option price changes with Taylor series

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Dividend Yield Only

- Call model

$$C_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2)$$

- Put model

$$P_0 = X e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$$

$$N(d) = \int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \quad d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad d_2 = d_1 - \sigma\sqrt{T}$$

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Delta

- Based on GBMOVM, generic delta

$$\Delta_O \equiv \frac{\partial O}{\partial S} = \iota_U e^{-\delta T} N(\iota_U d_1)$$

- Put-call parity: $c = Se^{-\delta T} - Xe^{-rT} + p$

$$\Delta_c = e^{-\delta T} + \Delta_p$$

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Delta-Neutral Portfolio

- Portfolio with zero delta
- Insensitive to infinitesimal changes in S
- Single stock
  - Riskless portfolio: $\Pi = -c + SN(d_1)$
  - Rearranged: $c = SN(d_1) - \Pi$
  - Thus, $\Pi = Xe^{-rT}N(d_2)$

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Gamma

- Based on GBMOVM, generic gamma

$$\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2} = \frac{e^{-\delta T} n(d_1)}{S\sigma\sqrt{T}}$$

- Put-call parity: $c = Se^{-\delta T} - Xe^{-rT} + p$

$$\Delta_c = e^{-\delta T} + \Delta_p \qquad \Gamma_c = \frac{\partial \Delta_p}{\partial S} = \frac{\partial \Delta_c}{\partial S} = \Gamma_p$$

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Theta

- Based on GBMOVM, generic theta

$$\theta_O \equiv \frac{\partial O}{\partial t} = -\frac{\partial O}{\partial T} = -\frac{Se^{-\delta T} n(d_1)\sigma}{2\sqrt{T}} - \iota_U rXe^{-rT} N(\iota_U d_2) + \iota_U \delta S e^{-\delta T} N(\iota_U d_1)$$

- Put-call parity: $c = Se^{-\delta T} - Xe^{-rT} + p$

$$\frac{\partial p}{\partial T} = \frac{\partial c}{\partial T} + \delta S e^{-\delta T} - rXe^{-rT}$$

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Vega

- Based on GBMOVM, generic vega

$$v_O \equiv \frac{\partial O}{\partial \sigma} = Se^{-\delta T} n(d_1)\sqrt{T} = Xe^{-rT} n(d_2)\sqrt{T}$$

- Put-call parity: $c = Se^{-\delta T} - Xe^{-rT} + p$

$$\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma}$$

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Rho

- Based on GBMOVM, generic rho

$$\rho_O \equiv \frac{\partial O}{\partial r} = \iota_U XTe^{-rT} N\left(\iota_U d_2\right)$$

- Put-call parity: $c = Se^{-\delta T} - Xe^{-rT} + p$

$$\frac{\partial p}{\partial r} = \frac{\partial c}{\partial r} - rXe^{-rT} = rXe^{-rT} N\left(d_2\right) - rXe^{-rT} = -rXe^{-rT} N\left(-d_2\right)$$

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Extended Greeks

- First order
  - Coherent: Delta, Gamma, Theta
  - Incoherent: Vega, Rho
- Second order
  - Gamma (S,S), (S,t), (S,σ), (S,r), (S,X), (S,δ)
  - (t,S), (t,t), (t,σ), (t,r), (t,X), (t,δ)
  - (σ,S), (σ,t), Vanna (σ,σ), (σ,r), (σ,X), (σ,δ)
  - (r,S), (r,t), (r,σ), (r,r), (r,X), (r,δ)
  - (X,S), (X,t), (X,σ), (X,r), (X,X), (X,δ)
  - (δ,S), (δ,t), (δ,σ), (δ,r), (δ,X), (δ,δ)

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Option Change (Delta)

- Long call option price (Taylor series)

$$O_1(S) = O(S_0) + O'(S_0)(S - S_0)$$

- In continuous time

$$dO_1 = \Delta_o \mu S dt + \Delta_o \sigma S dz$$

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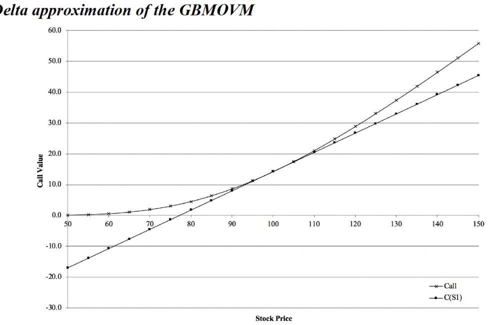
Figure 9.3.8. Delta approximation of the GBMOVM



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Option Change (Delta/Gamma)

- Long call option price (Taylor series)

$$O_2(S) = O(S_0) + O'(S_0)(S - S_0) + \frac{O''(S_0)}{2!}(S - S_0)^2$$

- In continuous time

$$dO_2 = \Delta_o\left(\mu S dt + \sigma S dz\right) + \frac{\Gamma_o}{2}\left(\mu S dt + \sigma S dz\right)^2$$

$$= \Delta_o \mu S dt + \Delta_o \sigma S dz + \frac{\Gamma_o}{2}\left(\mu^2 S^2 dt^2 + 2\mu\sigma S^2 dt dz + \sigma^2 S^2 dz^2\right)$$

$dt^2 = 0$, $dtdz = 0$, and $dz^2 = dt$.

$$dO = \left(\Delta_o \mu S + \frac{\Gamma_o}{2}\sigma^2 S^2\right)dt + \Delta_o \sigma S dz$$

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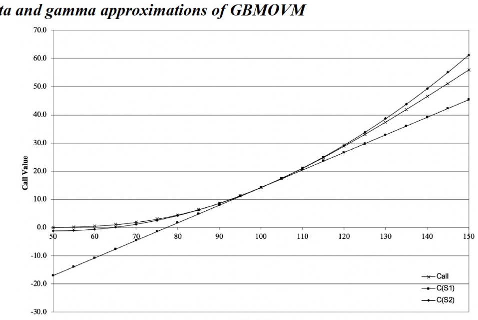
Figure 9.3.9. Delta and gamma approximations of GBMOVM



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Option Change (Delta/Gamma/Theta)

- Long call option price (Taylor series)

$$O_{2,2}(S,t) = O(S_0,t_0) + \frac{\partial O(S_0,t_0)}{\partial S}(S-S_0) + \frac{\partial O(S_0,t_0)}{\partial t}(t-t_0)$$
$$+\frac{1}{2!}\frac{\partial^2 O(S_0,t_0)}{\partial S^2}(S-S_0)^2 + \frac{1}{2!}\frac{\partial^2 O(S_0,t_0)}{\partial t^2}(t-t_0)^2 + \frac{1}{2!}\frac{\partial^2 O(S_0,t_0)}{\partial S\,\partial t}(S-S_0)(t-t_0)$$

- In continuous time

$$dO_{2,2} = \left(\theta_o + \Delta_o \mu S + \frac{\Gamma_o}{2}\sigma^2 S^2\right)dt + \Delta_o \sigma S dz$$

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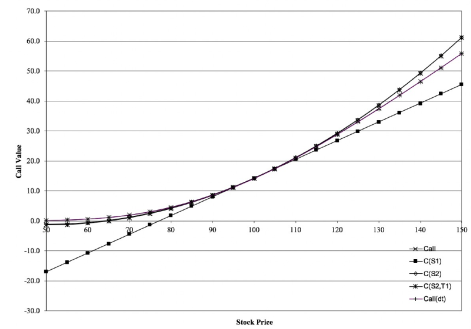
Figure 9.3.10. Delta, gamma, and theta approximations of GBMOVM



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Delta/Gamma/Speed Approx.

- Long call option price (Taylor series)

$$O_3(S) = O(S_0) + O'(S_0)(S-S_0) + \frac{O''(S_0)}{2!}(S-S_0)^2 + \frac{O'''(S_0)}{3!}(S-S_0)^3$$

$$Speed_o = O'''(S_0) = \left.\frac{\partial^3 O(S)}{\partial S^3}\right|_{S=S_0} = -\Gamma_o d_3 \qquad d_3 = \frac{d_1 + \sigma\sqrt{T}}{S\sigma\sqrt{T}}$$

- In continuous time

$$dO = \left(\Delta_o \mu S + \frac{\Gamma_o}{2}\sigma^2 S^2\right)dt + \Delta_o \sigma S dz$$

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Figure 9.3.11. Delta, gamma, and speed approximations of GBMOVM

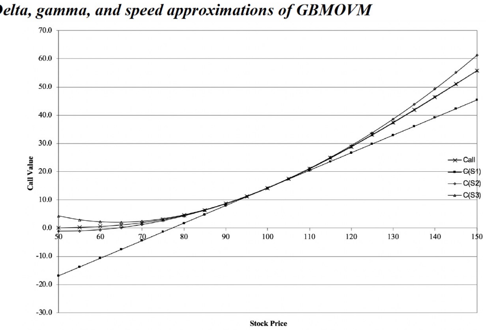

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Figure 9.3.12. Fourth derivative approximations of GBMOVM

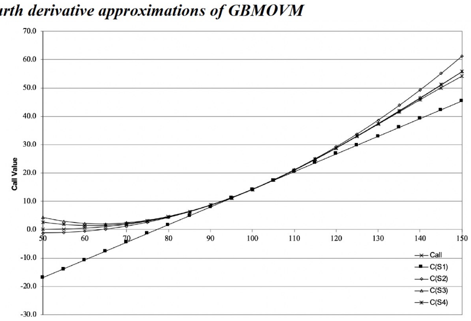

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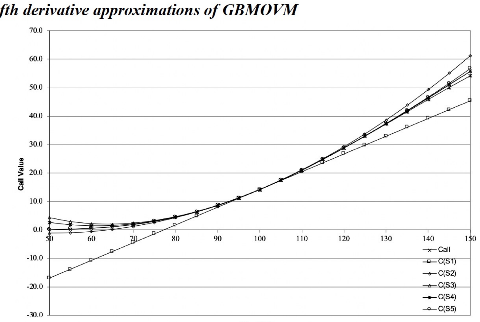
Figure 9.3.13. Fifth derivative approximations of GBMOVM



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Summary

- Reviewed static risk measures related to the GBMOVM
- Contrasted European-style with American-style (binomial) results
- Examined role of dividend yield
- Identified measurement error with binomial compared with GBMOVM (European-style)