Module 8.1

Static Risk Measures

GBM-Based Binomial Models

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Overview

- Review GBM-Based binomial OVMs
- Addresses both European-style and American-style
- Explore five option Greeks
- Variety of plots generated in R

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Central Finance Concepts

- SRMs of GBM BOVM is primarily quantitative
- Review standard GBM-based BOVM objectives
- Address standard Greeks

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GBM-Based BOVM Objectives

- Multiplicative (constant relative volatility)
- Recombining (no exploding lattice)
- Incorporate dividends (escrow method)
- Address early exercise, if required

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Standard OVM Greeks

- Delta sensitivity to changes in underlying
- Gamma delta's sensitivity to changes in underlying
- Vega sensitivity to changes in volatility
- Theta sensitivity to changes in time
- Rho sensitivity to changes in the interest rate.

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Quantitative Finance Materials

- Review coherence conditions
- Option Greeks within GBM BOVM
  - Delta
  - Gamma
  - Theta
  - Vega
  - Rho

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Coherence Conditions

- No arbitrage boundary condition

$$0 < d < e^{(r-\delta)\Delta t} < u$$

- Probability condition

$$0 << \pi << 1$$

- No arbitrage condition

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u - d}$$

- Variance condition

$$Var_{\pi}\left[\ln\left(\frac{S_{\Delta t}}{S_0}\right)\right] = \left[\ln\left(\frac{u}{d}\right)\right]^2 \pi(1-\pi) = \sigma^2 \Delta t$$

u and d Conditions

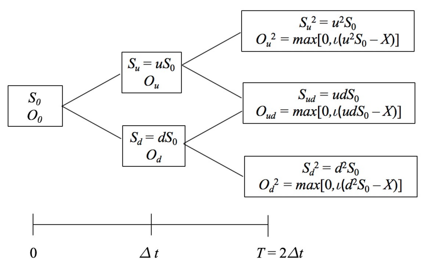
- Condition for u:

$$u = \frac{e^{(r-\delta)\Delta t + \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}$$

- Condition for d:

$$d = \frac{e^{(r-\delta)\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}$$

Recombining Binomial Model



Multiperiod BOVM

- Generic version:

$$O_0 = PV\left[E_{\pi}(O_T)\right] = \iota_U S_0 e^{-\delta T} Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U}$$

- Binomial summations:

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j (1-\pi_1)^{n-j},$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j (1-\pi_2)^{n-j},$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j (1-\pi_1)^{n-j},$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j (1-\pi_2)^{n-j},$$

Early Exercise

- At each node, evaluate

$$O_{i,j} = \max\left[O_{i,j}^B, O_{i,j}^X, O_{i,j}^L\right]$$

- Binomial model value

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + (1-\pi) O_{i+1,j}\right]$$

- Lower boundary condition

$$O_{i,j}^L = \max\left\{0, \iota_U\left[PV_{\delta,i,n-i}(S_{i,j}) - PV_{r,i,n-i}(X)\right]\right\}$$

- Early exercise value

$$O_{i,j}^X = \max\left[0, \iota_U (S_{i,j} - X)\right]$$

Option Greeks

- Static risk measure
- Mathematical derivatives
- Coherent
  - Delta (first derivative, underlying price)
  - Gamma (second derivative, underlying price)
  - Theta (first derivative, calendar time)
- Incoherent
  - Vega (first derivative, volatility)
  - Rho (first derivative, interest rate)

Delta

- Formal definition: $\Delta_O \equiv \frac{\partial O}{\partial S}$
- Standard binomial method

$$\Delta_{O,i,j} = \frac{O_{i+1,j+1} - O_{i+1,j}}{S_{i+1,j+1} - S_{i+1,j}}$$

- Enhanced binomial method

$$\Delta_{O,i,j} = \frac{O_{i,j+1} - O_{i,j-1}}{S_{i,j+1} - S_{i,j-1}}$$

- Numerical method

$$\Delta_{O,i,j} = \frac{O(S+h) - O(S-h)}{2h}$$

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Figure 9.1.1. Illustration of standard delta within GBM-based binomial model

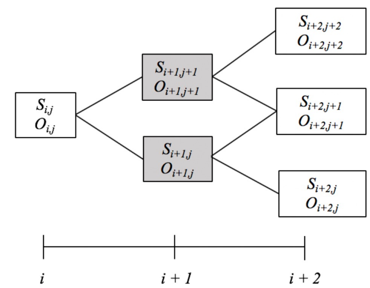

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Figure 9.1.2. Illustration of enhanced delta within GBM-based binomial model

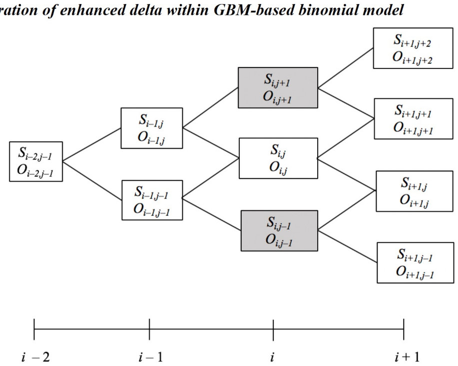

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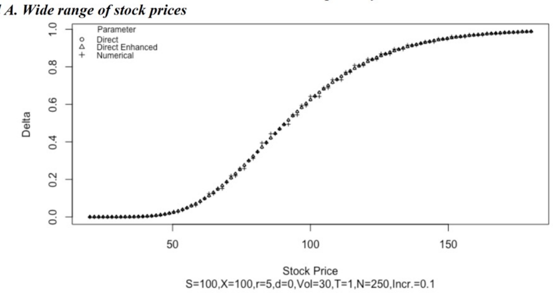
Figure 9.1.3. Three methods to estimate GBM-based European-style binomial call delta
Panel A. Wide range of stock prices



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Panel B. Narrow range of stock prices

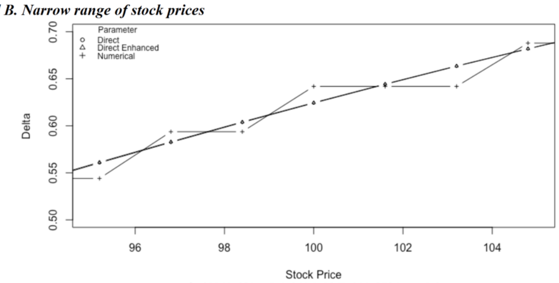

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Figure 9.1.4. Call and put values based on GBM-based European-style binomial model

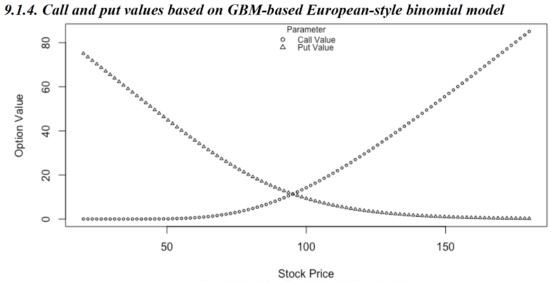

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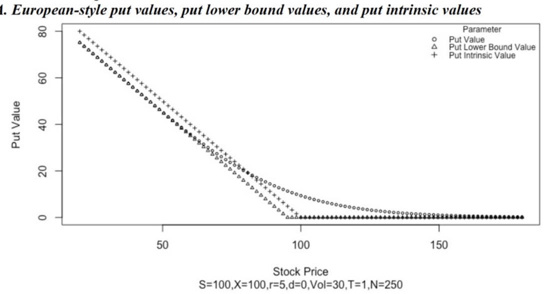
Figure 9.1.5. Call and put values based on GBM-based binomial model
Panel A. European-style put values, put lower bound values, and put intrinsic values



Panel B. European-style call values, call lower bound values, and call intrinsic values

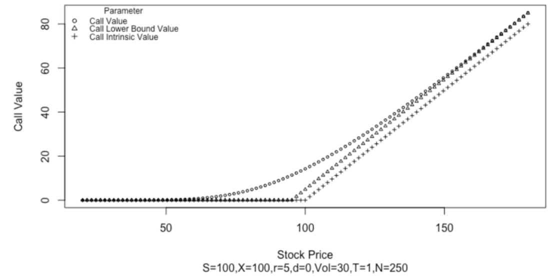


Figure 9.1.6. Call and put deltas based on GBM-based binomial model without dividends

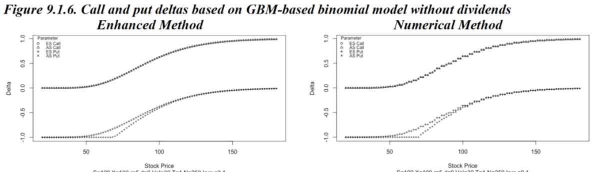
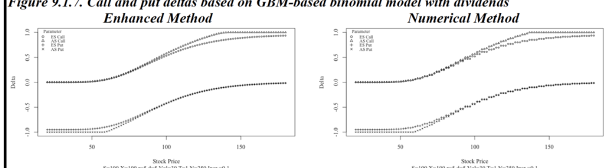


Figure 9.1.7. Call and put deltas based on GBM-based binomial model with dividends



Gamma

- Formal definition: $\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2}$
- Standard binomial method

$$\Gamma_{O,i,j} = \frac{\dfrac{O_{i+2,j+2} - O_{i+2,j+1}}{S_{i+2,j+2} - S_{i+2,j+1}} - \dfrac{O_{i+2,j+1} - O_{i+2,j}}{S_{i+2,j+1} - S_{i+2,j}}}{\dfrac{S_{i+2,j+2} - S_{i+2,j}}{2}}$$

- Enhanced binomial method

$$\Gamma_{O,i,j} = \frac{\dfrac{O_{i,j+1} - O_{i,j}}{S_{i,j+1} - S_{i,j}} - \dfrac{O_{i,j} - O_{i,j-1}}{S_{i,j} - S_{i,j-1}}}{\dfrac{S_{i,j+1} - S_{i,j-1}}{2}}$$

- Numerical method $\Gamma_{O,i,j} = \frac{\left[O(S+h) - O(S)\right] - \left[O(S) - O(S-h)\right]}{h^2}$

Figure 9.1.8. Illustration of standard gamma within GBM-based binomial model

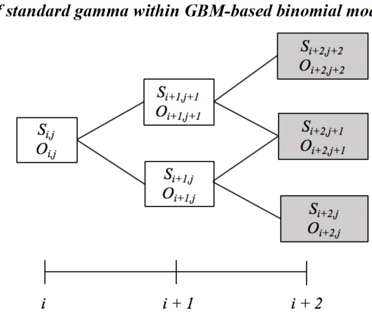


Figure 9.1.9. Illustration of enhanced gamma within GBM-based binomial model

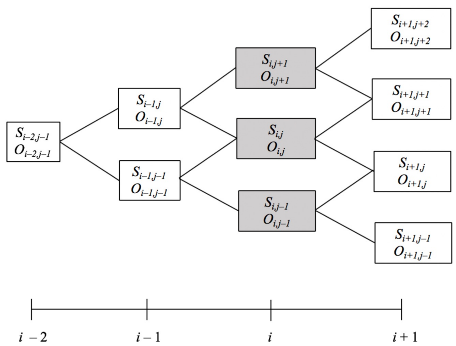


Figure 9.1.10. Three methods to estimate GBM-based European-style gamma without dividends

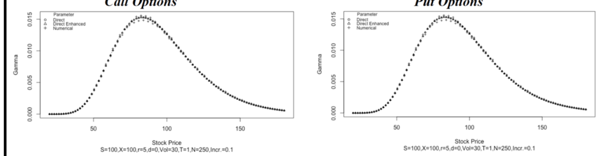


Figure 9.1.11. Three methods to estimate GBM-based gamma with dividends

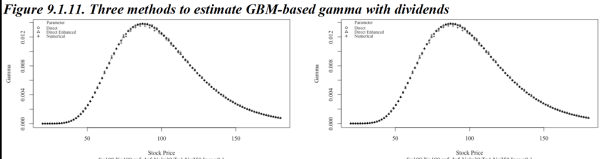


Figure 9.1.12. Call and put gammas based on GBM-based binomial model with no dividends

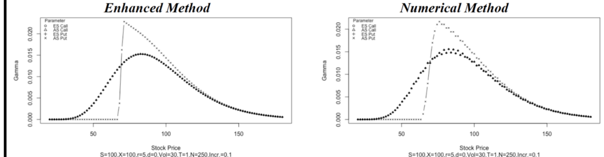
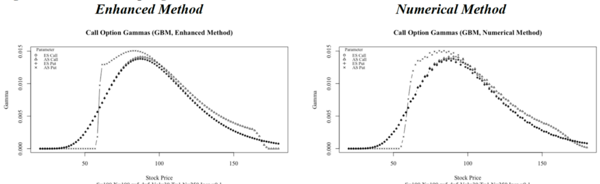


Figure 9.1.13. Call and put gammas based on GBM-based binomial model with dividends



Theta (*ud* = 1, required)

- Formal definition: $\theta_O \equiv \frac{\partial O}{\partial t}$
- Standard binomial method

$$\theta_{O,i,j} = \frac{O_{i+2,j+1} - O_{i,j}}{2\Delta t}$$

- Enhanced binomial method

$$\theta_{O,i,j} = \frac{O_{i+2,j+1} - O_{i-2,j-1}}{4\Delta t}$$

- Numerical method $\theta_{O,i,j} = \frac{O(t+h) - O(t-h)}{2h}$

Figure 9.1.14. Illustration of standard theta within GBM-based binomial model

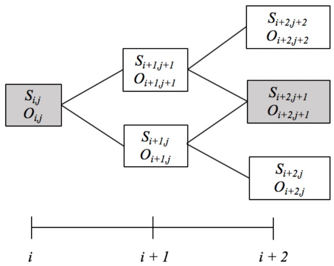


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Figure 9.1.15. Illustration of enhanced theta within GBM-based binomial model

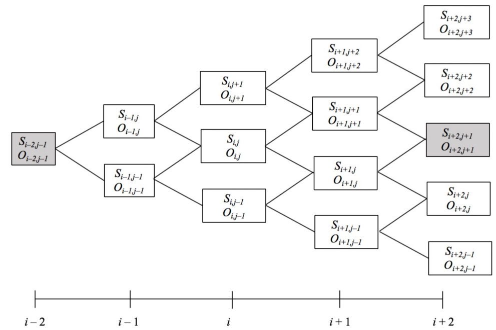

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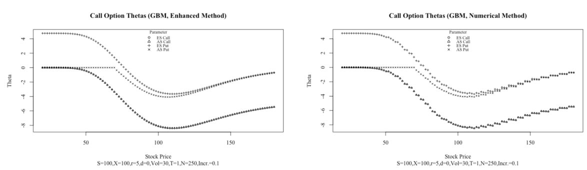
Figure 9.1.16. Illustration of theta within GBM-based binomial model

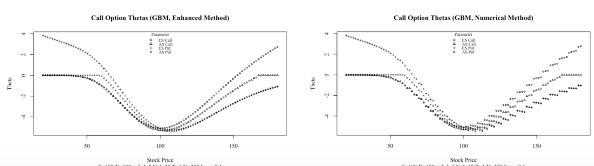
Panel A. Without dividends

Enhanced Method *Numerical Method*



Panel B. With dividends

Enhanced Method *Numerical Method*



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Vega

- Formal definition: $\nu_O \equiv \dfrac{\partial O}{\partial \sigma}$
- Numerical method

$$\nu_{O,i,j} = \frac{O_{\sigma+h,i,j} - O_{\sigma-h,i,j}}{2h}$$

- Other methods are not possible as vega is incoherent (but very important and useful)

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Figure 9.1.17. Illustration of vega within GBM-based binomial model without dividends

European-style *American-style*

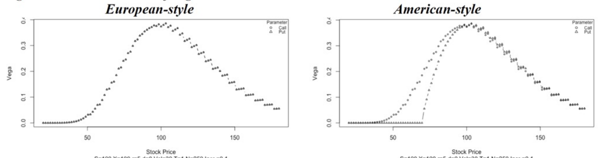

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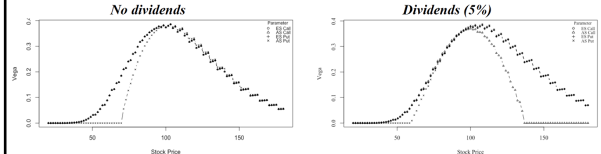
Figure 9.1.18. Vega with respect to stock price within GBM-based binomial model with and without dividends

No dividends *Dividends (5%)*



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Rho

- Formal definition: $\rho_O \equiv \frac{\partial O}{\partial r}$
- Numerical method

$$\rho_{O,i,j} = \frac{O_{r+h,i,j} - O_{r-h,i,j}}{2h}$$

- Other methods are not possible as rho is incoherent (and not that important in most cases)

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Figure 9.1.19. Illustration of rho within GBM-based binomial model without dividends

European-style *American-style*

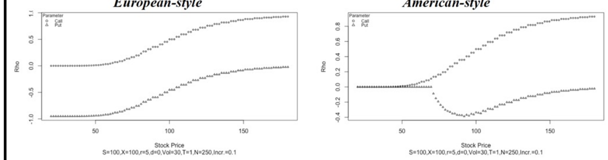

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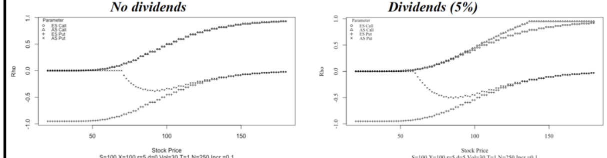
Figure 9.1.20. Rho with respect to stock price within GBM-based binomial model with and without dividends

No dividends *Dividends (5%)*



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Summary

- Reviewed GBM-Based binomial OVMs
- Addressed both European-style and American-style
- Explored five option Greeks
  - Coherent: Delta, Gamma, Theta
  - Incoherent: Vega, Rho
- Variety of plots generated in R reviewed

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