Module 8.3: SRM GBM-Based Option Models

Learning objectives

- Explore measurement error between the binomial and closed-form model
- Explain how to compute Greeks based on the geometric Brownian motion option models
- Contrast no dividends and dividends with respect to Greek sensitivities
- Provide detailed graphical analysis of European-style and American-style options

Executive summary

Based on the notation presented in Module 5.4, we illustrate computing option Greeks within the geometric Brownian motion option valuation model (GBMOVM) for European-style options. We further explore differences between the Greeks based on the binomial option valuation model.

Central finance concepts

Based on materials covered in Modules 8.1 and 8.2, we assume a familiarity with the option Greeks. In this section, we will review various graphical insights related to GBMOVM and related Greeks. In the quantitative finance materials section below, we will present the mathematical details as well as several extensions.

GBMOVM Greeks

We now present graphical results based on the R code provided for the Greeks as calculated by the GBMOVM. Specifically, we explore delta, gamma, vega, theta, and rho. Recall delta measures an option value's sensitivity to changes in the underlying instrument's price. Gamma measures the delta's sensitivity to changes in the underlying instrument's price. Vega measures an option price's sensitivity to changes in the underlying asset's volatility. Theta measures an option price's sensitivity to changes in the time to maturity. Rho measures an option price's sensitivity to changes in the interest rate.

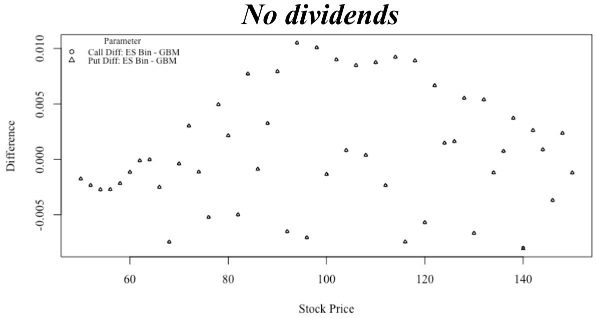
Measurement error with GBM binomial option valuation model

Our focus here is comparing the American-style option Greeks from the binomial option valuation model using the enhanced method and the GBMOVM. Before working our way through the Greeks, we first illustrate measurement error when comparing the binomial approach to European-style option value and the GBMOVM.

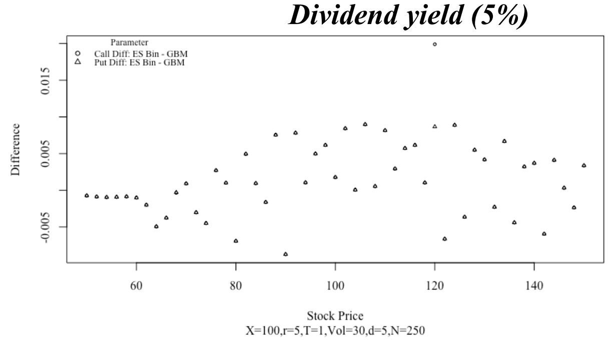
Figure 8.3.1 illustrates this error from three perspectives. Note throughout all graphs, ES denotes European-style and AS denotes American-style. In almost all cases, the call and put errors are indistinguishable. Panel A highlights the difference with respect to moneyness. The error in this case is almost always less than \$0.01. Panel B highlights the difference with respect to volatility with same boundaries but with a clear cyclical pattern. Panel C highlights the difference with respect to time to maturity. With longer time to maturity, the error declines.

Figure 8.3.1. Measurement error between binomial and GBMOVM
Panel A. Error with respect to stock price

No dividends

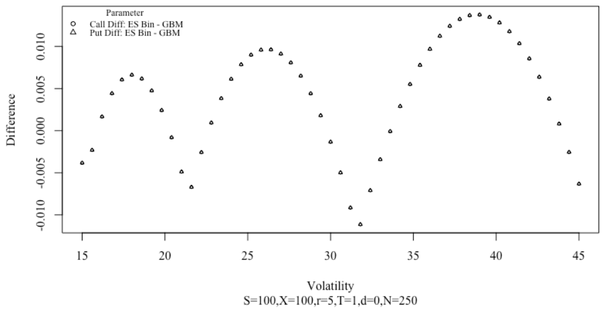


Dividend yield (5%)

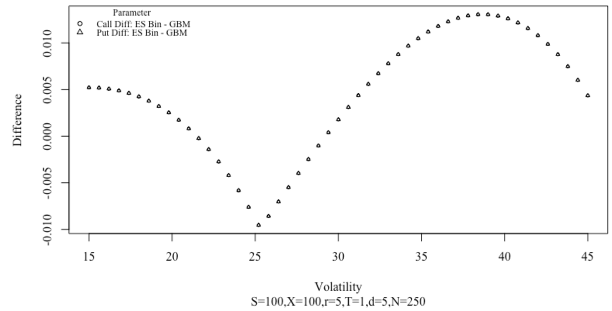


Panel B. Error with respect to volatility

No dividends

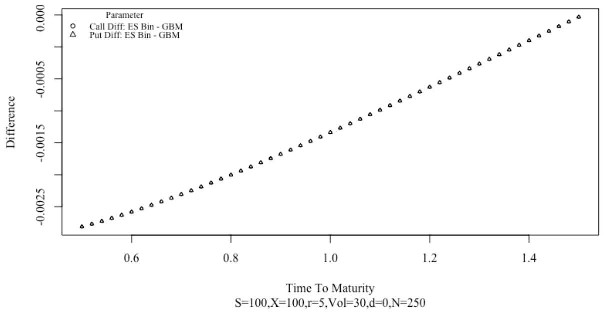


Dividend yield (5%)

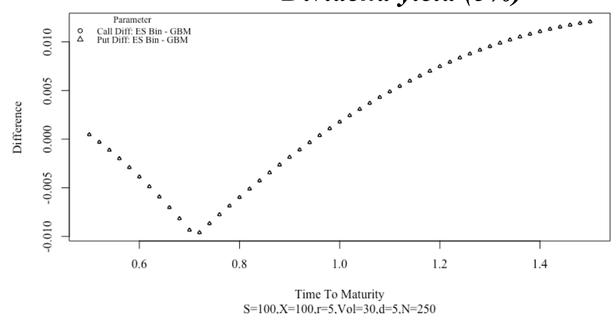


Panel C. Error with respect to time to maturity

No dividends



Dividend yield (5%)



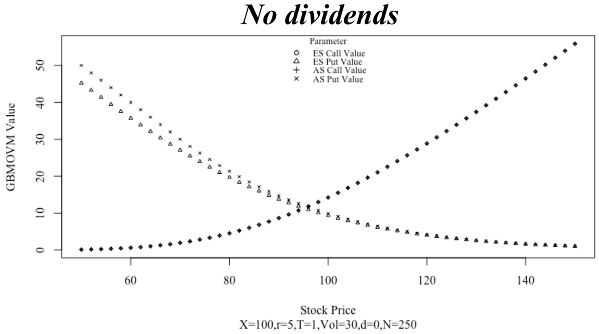
Delta

Figure 8.3.2 illustrates delta from several perspectives both without and with dividends (5%). Panel A just illustrates the relationship between the option values and the stock prices. Notice that when the interest rate equals the dividend yield the call and put equal each other at-the-money. The slopes of the lines in Panel A is delta as illustrated in Panel B. Panel B illustrates the influence of the early exercise feature on the delta for the American-style options. Panel C and D show the influence of volatility and time to maturity.

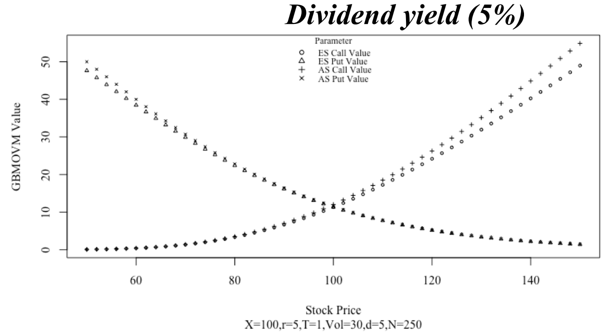
Figure 8.3.2. Call and put deltas based on GBMOVM with and without dividends

Panel A. Option value with respect to stock price

No dividends

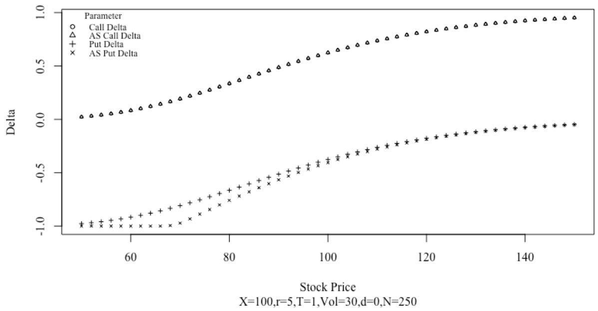


Dividend yield (5%)

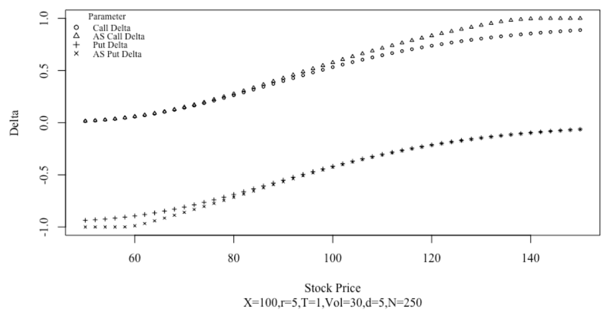


Panel B. Delta with respect to stock price

No dividends

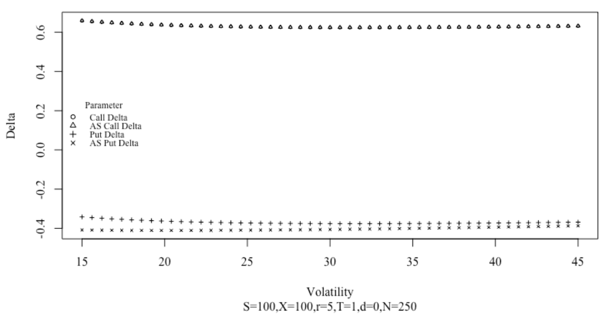


Dividend yield (5%)

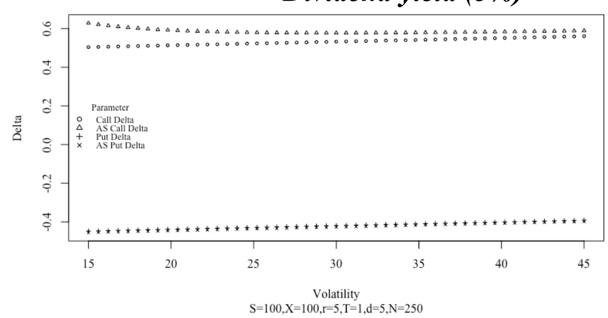


Panel C. Delta with respect to volatility

No dividends

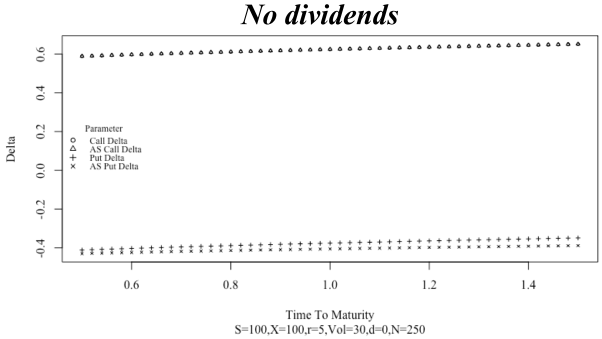


Dividend yield (5%)

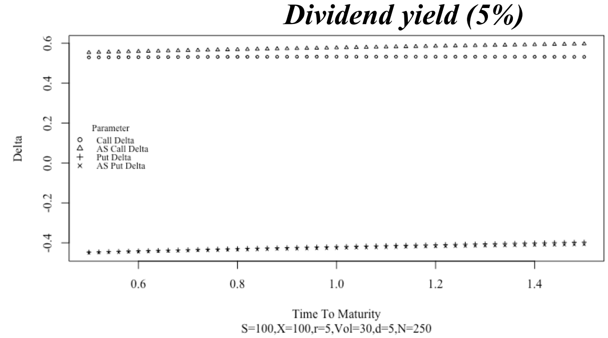


Panel D. Delta with respect to time to maturity

No dividends



Dividend yield (5%)



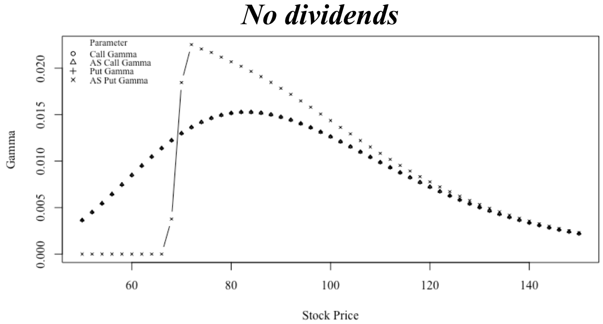
Gamma

Figure 8.3.3 highlights the role of gamma. The lognormal distribution assumption is clear in Panel A except when the early exercise feature impacts option gammas. Panels B and C show that the gamma increases with declining volatility and with declining time to maturity.

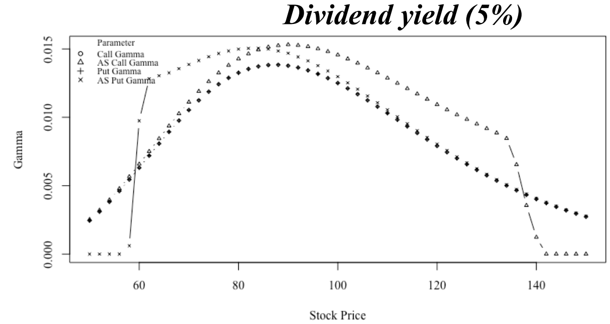
Figure 8.3.3. Call and put gamma based on GBMOVM with and without dividends

Panel A. Gamma with respect to stock price

No dividends

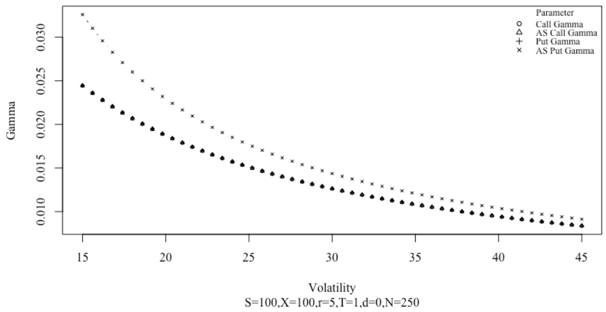


Dividend yield (5%)

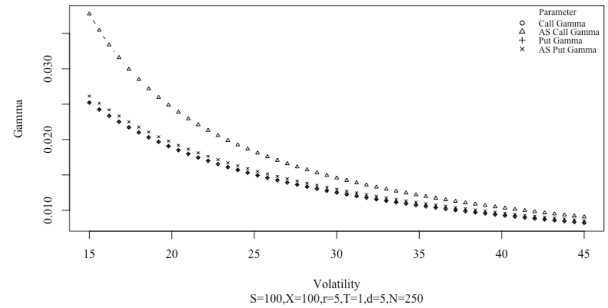


Panel B. Gamma with respect to volatility

No dividends

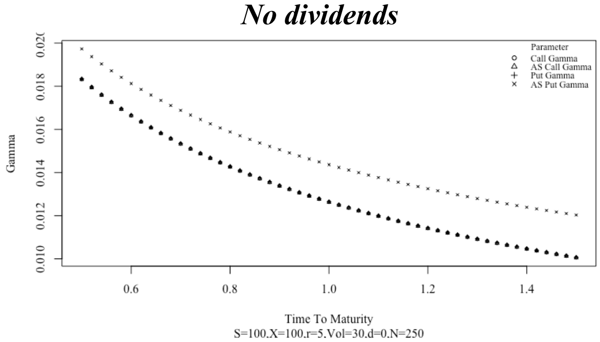


Dividend yield (5%)

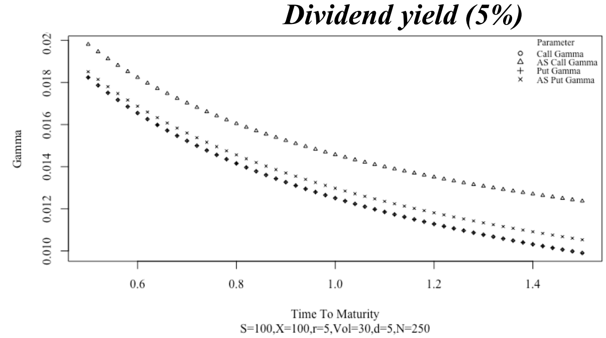


Panel C. Gamma with respect to time to maturity

No dividends



Dividend yield (5%)



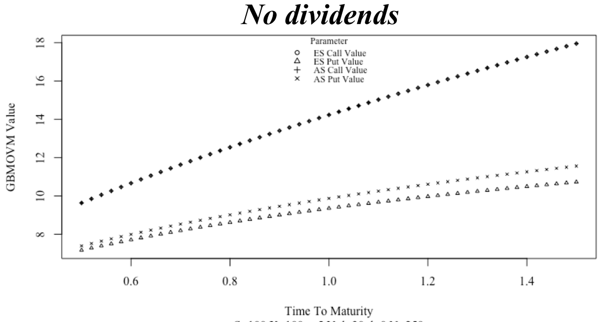
Theta

Figure 8.3.4 highlights the role of thetas. Panel A documents the well-known time value decay—option values decline with the mere passage of time. Panel B shows that the theta has a non-monotonic relationship with the stock price. At-the-money options have the highest time value; hence, they have the most negative time value decay or theta. Panel C shows that theta generally declines with higher volatility. Panel D shows that theta generally increases with time to maturity.

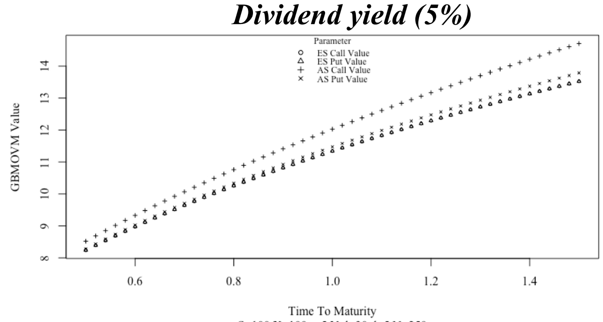
Figure 8.3.4. Call and put thetas based on GBMOVM with and without dividends

Panel A. Option value with respect to time to maturity

No dividends

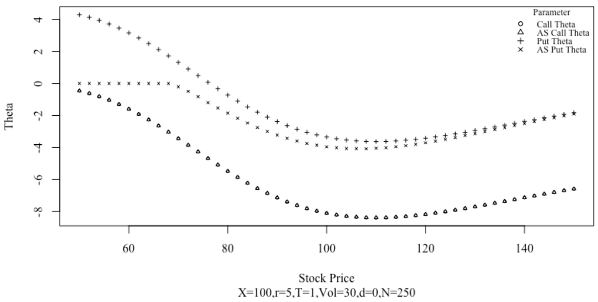


Dividend yield (5%)

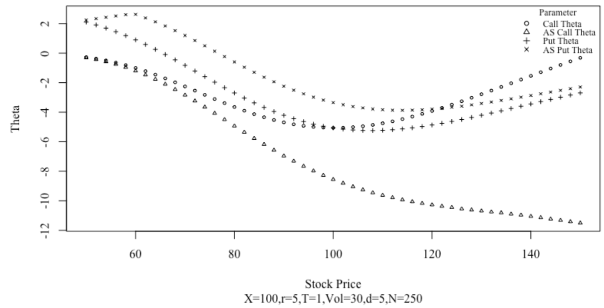


Panel B. Theta with respect to stock price

No dividends

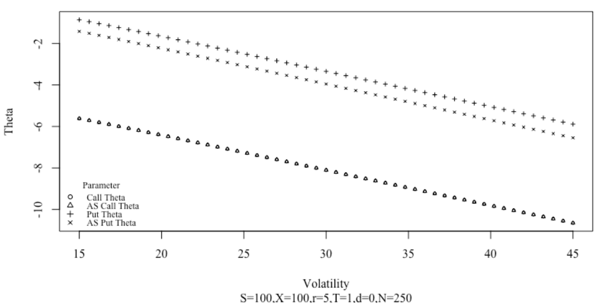


Dividend yield (5%)

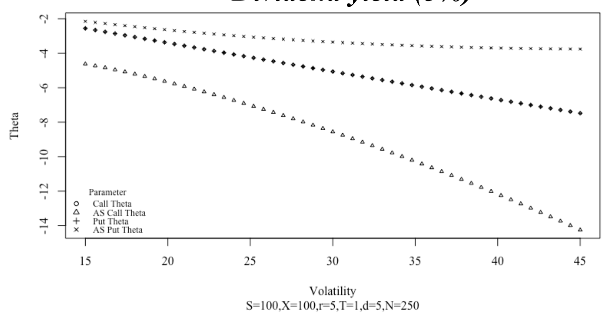


Panel C. Theta with respect to volatility

No dividends

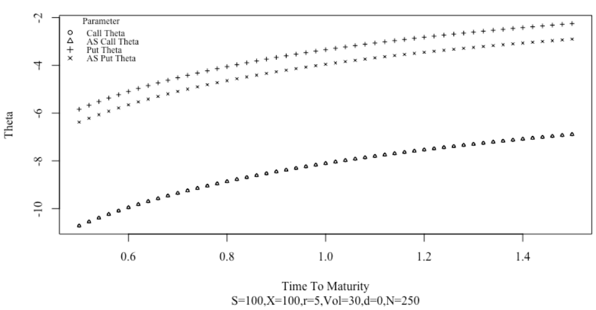


Dividend yield (5%)

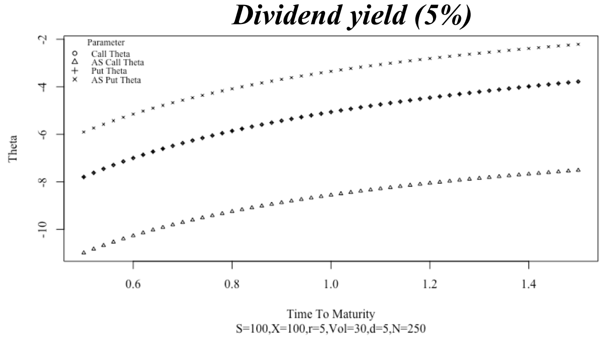


Panel D. Theta with time to maturity

No dividends



Dividend yield (5%)



Vega

Recall vega is the first derivative of the option value with respect to volatility. Neither the stock nor the risk-free interest rate is assumed to be influenced by changes in the stock's volatility. Volatility impacts both calls and puts the same.

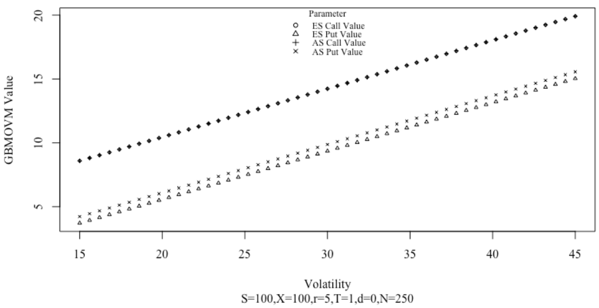
For call options that are deep out-of-the-money, the call price changes very little with a small change in volatility (it does not really change the probability of the stock reaching the strike price), hence the vega is close to zero. The same is true for deep out-of-the-money puts. For small changes in the volatility for deep in-the-money calls, the call price does not change much because it is already in-the-money, hence again the vega is close to zero.

Figure 8.3.5 highlights the role of vega. Panel A shows that the option values are increasing with respect to volatility. Panel B shows that the vega reflects the lognormal distribution. Panel C shows that vega has a complex relationship with volatility. Panel D shows that vega generally increases with time to maturity.

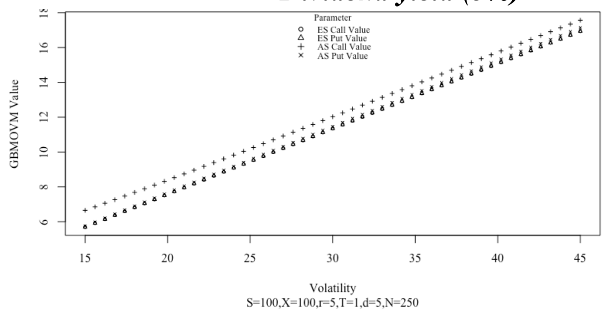
Figure 8.3.5. Call and Put Vegas Based on GBMOVM With and Without Dividends

Panel A. Option Value with Respect to Volatility

No dividends

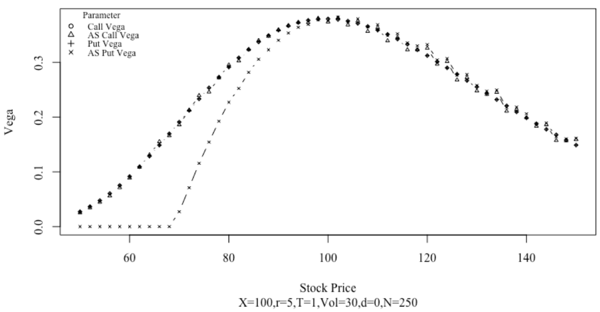


Dividend yield (5%)

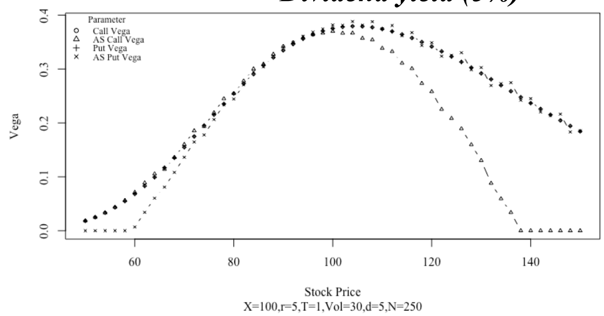


Panel B. Vega with Respect to Stock Price

No dividends

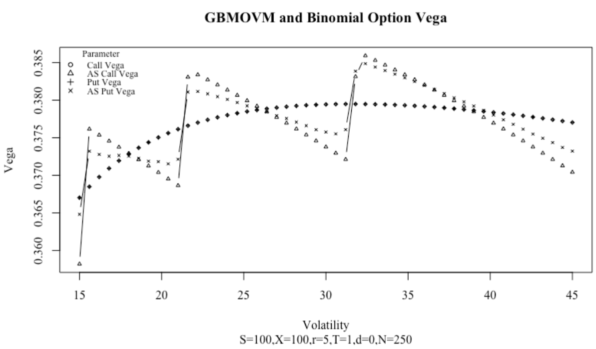


Dividend yield (5%)

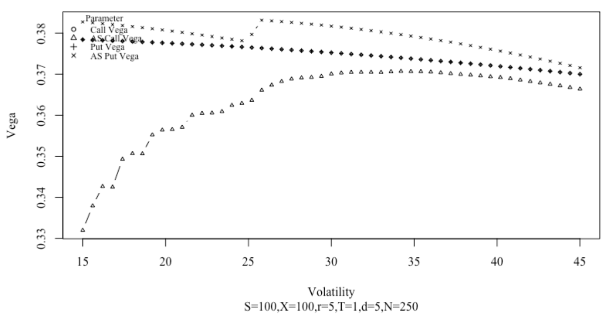


Panel C. Vega with Respect to Volatility

No Dividends

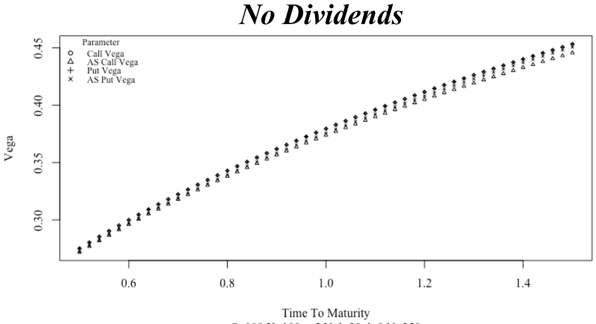


Dividend Yield (5%)

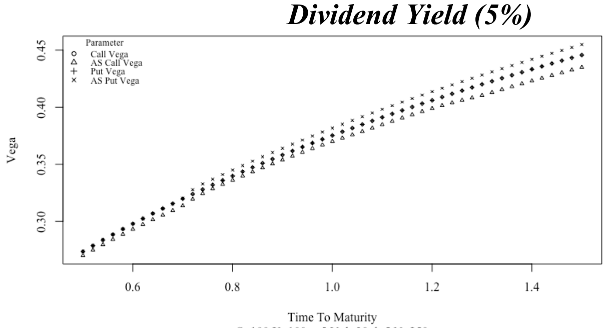


Panel D. Vega with Respect to Time to Maturity

No Dividends



Dividend Yield (5%)

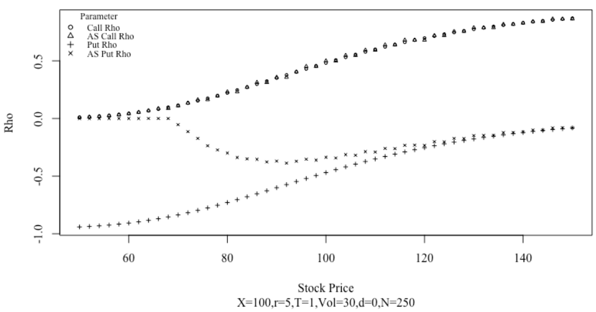


Rho

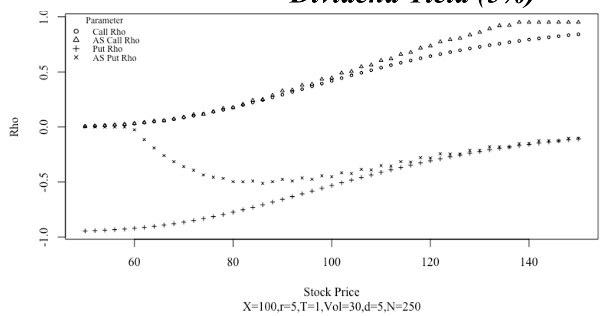
Figure 8.3.6 highlights the role of rho. Panel A shows that the rho generally increases with the stock price, except for American-style puts. Panel B shows that rho generally declines with volatility. Panel C shows that rho generally increases with time to maturity for calls and decreases for puts.

Figure 8.3.6. Call and Put Rhos based on GBMOVM With and Without Dividends
Panel A. Rho with Respect to Stock Price

No Dividends

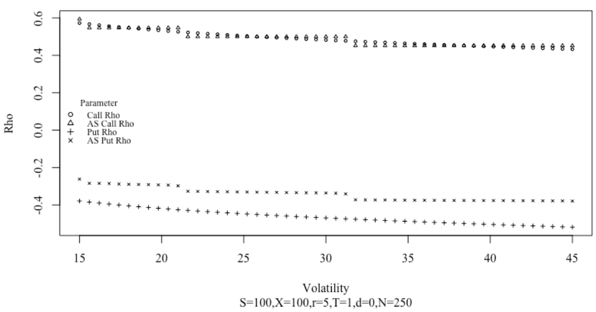


Dividend Yield (5%)

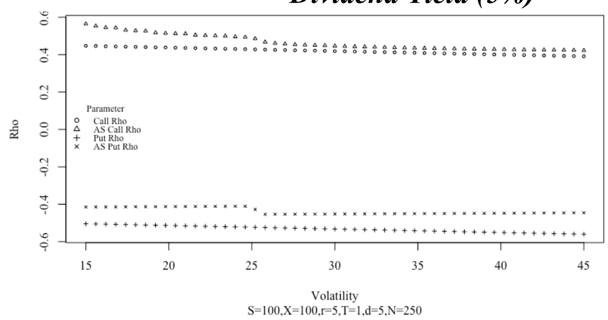


Panel B. Rho with Respect to Volatility

No Dividends

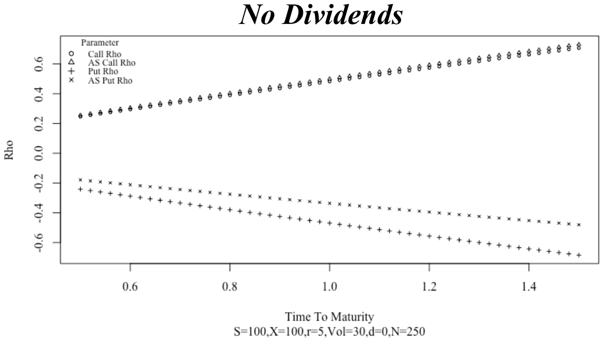


Dividend Yield (5%)

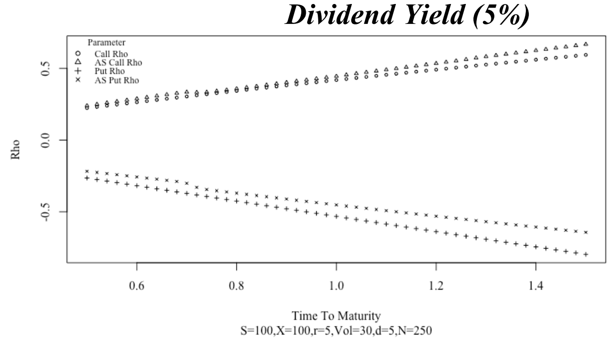


Panel C. Rho with Respect to Time to Maturity

No Dividends



Dividend Yield (5%)



Quantitative finance materials

We now examine the technical details of GBMOVM Greeks. But first, we review the GBMOVM.

GBM option valuation model (GBMOVM)

Recall based on a set of restrictive assumptions, the GBMOVM can be expressed as

$$O\left(S_t,t;\iota_U,X,T,r,\sigma\right)=\iota_U S_t B_\delta N\left(\iota_U d_1\right)-\iota_U X B_r N\left(\iota_U d_2\right), \tag{8.3.1}$$

where again the indicator functions is expressed as

$$\iota_U=\begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option}\end{cases}, \tag{8.3.2}$$

$$B_r=e^{-r(T-t)},\ \ B_\delta=e^{-\delta(T-t)} \tag{8.3.3}$$

$$N\left(d\right)=\int_{-\infty}^{d}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx\text{, (area under the standard cumulative normal distribution up to d)} \tag{8.3.4}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \text{ and} \tag{8.3.5}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{8.3.6}$$

If there is only a cash flow yield, then the call and put option equations can be expressed as

$$c_0 = S_0 e^{-\delta(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2) \text{ and} \tag{8.3.7}$$

$$p_0 = X e^{-r(T-t)} N(-d_2) - S_0 e^{-\delta(T-t)} N(-d_1). \tag{8.3.8}$$

Note that where convenient, we assume $t = 0$.

GBMOVM Greeks

We now cover the mathematical details related to Greeks as calculated by the GBMOVM.

Delta

Mathematically, delta is defined as

$$\Delta_O \equiv \frac{\partial O}{\partial S} = \iota_U B_\delta N(\iota_U d_1). \tag{8.3.9}$$

Sketch of proof: We cover a few preliminary concepts before providing a sketch of the derivation of delta. Based on put-call parity ($c = Se^{-\delta T} - Xe^{-rT} + p$), we know $\Delta_c = e^{-\delta T} + \Delta_p$. Also, note from the definitions of d_1 and d_2, we can express them as

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln(S)}{\sigma\sqrt{T}} + \frac{-\ln(X) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \text{ and} \tag{8.3.10}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S)}{\sigma\sqrt{T}} + \frac{-\ln(X) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}. \tag{8.3.11}$$

Therefore,

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T}}. \text{ (**Derivatives of d's relation**)} \tag{8.3.12}$$

From the definition of the standard normal cumulative distribution function ($N(d) = \int_{x=-\infty}^{x=d} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$), we have

$$\frac{\partial N(d)}{\partial d} \equiv n(d) = \frac{e^{-d^2/2}}{\sqrt{2\pi}}. \text{ (**Standard normal probability density function**)} \tag{8.3.13}$$

Lemma: $n(d_1) = \frac{Xe^{-rT} n(d_2)}{Se^{-\delta T}}$ or $n(d_2) = \frac{Se^{-\delta T} n(d_1)}{Xe^{-rT}}$.

Lemma proof: Rearranging, we have

$$\frac{Se^{-\delta T}}{Xe^{-rT}} = \frac{n(d_2)}{n(d_1)} = \frac{\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}}{\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}} = e^{\left(d_1^2 - d_2^2\right)/2}. \tag{8.3.14}$$

Focusing on the exponent,

$$\frac{d_1^2-d_2^2}{2}=\frac{1}{2}\left[d_1^2-\left(d_1-\sigma\sqrt{T}\right)^2\right]=\frac{1}{2}\left[d_1^2-d_1^2+2d_1\sigma\sqrt{T}-\sigma^2T\right]$$
$$=d_1\sigma\sqrt{T}-\frac{\sigma^2T}{2} \qquad (8.3.15)$$
$$=\left[\frac{\ln\left(\frac{S}{X}\right)+\left(r-\delta+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right]\sigma\sqrt{T}-\frac{\sigma^2T}{2}=\ln\left(\frac{S}{X}\right)+\left(r-\delta\right)T$$

Therefore,

$$e^{\left(d_1^2-d_2^2\right)/2}=e^{\ln\left(\frac{S}{X}\right)+\left(r-\delta\right)T}=\frac{Se^{-\delta T}}{Xe^{-rT}}. \qquad (8.3.16)$$

With this background information, we are now ready to sketch the call delta proof. Based on fundamental calculus rules and the Black, Scholes, Merton call formula, we have

$$\frac{\partial c}{\partial S}=e^{-\delta T}N\left(d_1\right)+Se^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial S}-Xe^{-rT}\frac{\partial N\left(d_2\right)}{\partial S}$$
$$=e^{-\delta T}N\left(d_1\right)+Se^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial d_1}\frac{\partial d_1}{\partial S}-Xe^{-rT}\frac{\partial N\left(d_2\right)}{\partial d_2}\frac{\partial d_2}{\partial S}. \qquad (8.3.17)$$

From the derivatives of d's relation and the standard normal probability density function, we have

$$\frac{\partial c}{\partial S}=e^{-\delta T}N\left(d_1\right)+Se^{-\delta T}n\left(d_1\right)\frac{1}{S\sigma\sqrt{T}}-Xe^{-rT}n\left(d_2\right)\frac{1}{S\sigma\sqrt{T}}. \qquad (8.3.18)$$

Finally, based on the lemma above, we have demonstrated that

$$\frac{\partial c}{\partial S}\equiv\Delta_c=e^{-\delta T}N\left(d_1\right). \qquad (8.3.19)$$

and based on put-call parity, we have (recall $N\left(-d_1\right)=1-N\left(d_1\right)$ due to the symmetry of the standard normal distribution)

$$\Delta_p=\Delta_c-e^{-\delta T}=e^{-\delta T}N\left(d_1\right)-e^{-\delta T}=e^{-\delta T}\left[N\left(d_1\right)-1\right]=-e^{-\delta T}N\left(-d_1\right). \qquad (8.3.20)$$

The value of the European call option at expiration is either \$0 if it is out-of-the-money ($S_T < X$) or the intrinsic value (the dollar amount it is in-the-money, $S_T - X$) when the stock ends up in-the-money ($S_T > X$). Prior to expiration, the option has time value as well as intrinsic value. The time value of an option depends on the relationship between the current stock price and the strike price.

Delta-neutral portfolio

A delta-neutral portfolio is a portfolio that has a portfolio delta of zero. A zero delta implies that the value of the portfolio does not change for infinitesimal changes in the stock price. Hence, the value of the portfolio is not affected by small changes in the stock price. Therefore, to hedge against small changes in the stock price, trades should be conducted such that the portfolio delta is zero.

We can create a synthetic call using bonds and the underlying asset. It can be demonstrated that a portfolio of $\Pi = -c + SN(d_1)$ is riskless because $N(d_1)$ is the delta. Hence the portfolio, Π, should grow at the risk-free interest rate. Rearranging this relationship, we have $c = SN(d_1) - \Pi$ which is the Black-Scholes formula with $\Pi = Xe^{-rT}N(d_2)$. Thus, a call option can be created synthetically using the underlying asset (buying $N(d_1)$ shares) and partially financing it with borrowing of $\Pi = Xe^{-rT}N(d_2)$.

The delta-neutral portfolio and portfolio insurance: Recall that portfolio insurance can be represented as long a stock and long a put or

$$S + p = S + Xe^{-rT}N(-d_2) + S(\Delta_p) = S(1 + \Delta_p) + Xe^{-rT}N(-d_2)$$

(by substituting the Black-Scholes-Merton formula and the definition of Δ). Hence, portfolio insurance is related to the concept of a delta neutral portfolio with additional exposure in stock. Recall that $\Delta_p < 0$ and $\Delta_c = 1 + \Delta_p$.

Gamma

Mathematically, gamma is defined as

$$\Gamma_O \equiv \frac{\partial^2 O}{\partial S^2} = \frac{e^{-\delta T} n\left(d_1\right)}{S\sigma\sqrt{T}}. \tag{8.3.21}$$

Sketch of proof: Recall based on put-call parity $\Delta_p = \Delta_c - e^{-\delta T}$ and therefore we know

$$\frac{\partial \Delta_p}{\partial S} = \frac{\partial \Delta_c}{\partial S}. \tag{8.3.22}$$

From the definition of call delta,

$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta_c}{\partial S} = e^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial S} = e^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial d_1}\frac{\partial d_1}{\partial S} = \frac{e^{-qT} n\left(d_1\right)}{S\sigma\sqrt{T}}. \tag{8.3.23}$$

Theta

Mathematically, theta is defined as

$$\theta_O \equiv \frac{\partial O}{\partial t} = -\frac{\partial O}{\partial T} = -\frac{Se^{-\delta T} n\left(d_1\right)\sigma}{2\sqrt{T}} - \iota_U rXe^{-rT} N\left(\iota_U d_2\right) + \iota_U \delta Se^{-\delta T} N\left(\iota_U d_1\right). \tag{8.3.24}$$

Sketch of proof: Theta is defined as the change in the option value for a given change in calendar time, measured in years. As calendar time passes, the time to maturity declines. Therefore, there exist a negative relationship between calendar time and time to maturity. We derive results for time to maturity and then at the end switch the sign. Time to maturity appears in a variety of places in the option valuation formulas. Therefore, this proof is rather tedious. Consider the following preliminaries:

$$\frac{\partial e^{-rT}}{\partial T} = -re^{-rT}, \tag{8.3.25}$$

$$\frac{\partial e^{-\delta T}}{\partial T} = -\delta e^{-\delta T}, \text{ and} \tag{8.3.26}$$

$$\frac{\partial d_2}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{\sigma T^{-1/2}}{2}. \tag{8.3.27}$$

Therefore,

$$\begin{aligned} \frac{\partial c}{\partial T} &= S\frac{\partial e^{-\delta T}}{\partial T} N\left(d_1\right) + Se^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial T} - X\frac{\partial e^{-rT}}{\partial T} N\left(d_2\right) - Xe^{-rT}\frac{\partial N\left(d_2\right)}{\partial T} \\ &= -\delta Se^{-\delta T} N\left(d_1\right) + Se^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial d_1}\frac{\partial d_1}{\partial T} + rXe^{-rT} N\left(d_2\right) - Xe^{-rT}\frac{\partial N\left(d_2\right)}{\partial d_2}\frac{\partial d_2}{\partial T}. \\ &= -\delta Se^{-\delta T} N\left(d_1\right) + rXe^{-rT} N\left(d_2\right) + Se^{-\delta T} n\left(d_1\right)\frac{\partial d_1}{\partial T} - Xe^{-rT} n\left(d_2\right)\frac{\partial d_2}{\partial T} \end{aligned} \tag{8.3.28}$$

Based on $\frac{\partial d_2}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{\sigma T^{-1/2}}{2}$ and the lemma above ($n\left(d_2\right) = \frac{Se^{-\delta T} n\left(d_1\right)}{Xe^{-rT}}$), we have

$$\begin{aligned} \frac{\partial c}{\partial T} &= -\delta Se^{-\delta T} N\left(d_1\right) + rXe^{-rT} N\left(d_2\right) + Se^{-\delta T} n\left(d_1\right)\frac{\partial d_1}{\partial T} - Xe^{-rT}\left[\frac{Se^{-\delta T} n\left(d_1\right)}{Xe^{-rT}}\right]\left[\frac{dd_1}{dT} - \frac{\sigma T^{-1/2}}{2}\right] \\ &= -\delta Se^{-\delta T} N\left(d_1\right) + rXe^{-rT} N\left(d_2\right) + \frac{Se^{-\delta T} n\left(d_1\right)\sigma}{2\sqrt{T}} \end{aligned} . \tag{8.3.29}$$

Recall the sign change related to how time is measured, we have

$$\theta_c \equiv \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial T} = -\frac{Se^{-qT}n\left(d_1\right)\sigma}{2\sqrt{T}} - rXe^{-rT}N\left(d_2\right) + qSe^{-qT}N\left(d_1\right). \tag{8.3.30}$$

From put-call parity, we have

$$\frac{\partial p}{\partial T} = \frac{\partial c}{\partial T} + \delta Se^{-\delta T} - rXe^{-rT}. \tag{8.3.31}$$

Substituting for call results above,

$$\begin{aligned}\frac{\partial p}{\partial T} &= -\delta Se^{-\delta T}N\left(d_1\right) + rXe^{-rT}N\left(d_2\right) + \frac{Se^{-\delta T}n\left(d_1\right)\sigma}{2\sqrt{T}} + \delta Se^{-\delta T} - rXe^{-rT} \\ &= -\delta Se^{-\delta T}N\left(-d_1\right) + rXe^{-rT}N\left(-d_2\right) + \frac{Se^{-\delta T}n\left(d_1\right)\sigma}{2\sqrt{T}}\end{aligned}, \tag{8.3.32}$$

because of the normal distribution symmetry. Thus,

$$\theta_p \equiv \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial T} = -\frac{Se^{-\delta T}n\left(d_1\right)\sigma}{2\sqrt{T}} + rXe^{-rT}N\left(-d_2\right) - \delta Se^{-\delta T}N\left(-d_1\right). \tag{8.3.33}$$

Vega

Mathematically, vega is defined as

$$v_O \equiv \frac{\partial O}{\partial \sigma} = Se^{-\delta T}n\left(d_1\right)\sqrt{T} = Xe^{-rT}n\left(d_2\right)\sqrt{T}. \tag{8.3.34}$$

Sketch of proof: From put-call parity ($C = Se^{-\delta T} - Xe^{-rT} + P$), we know $\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma}$. Note

$$\begin{aligned}\frac{\partial c}{\partial \sigma} &= Se^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial \sigma} - Xe^{-rT}\frac{\partial N\left(d_2\right)}{\partial \sigma} \\ &= Se^{-\delta T}\frac{\partial N\left(d_1\right)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - Xe^{-rT}\frac{\partial N\left(d_2\right)}{\partial d_2}\frac{\partial d_2}{\partial \sigma}. \\ &= Se^{-\delta T}n\left(d_1\right)\frac{\partial d_1}{\partial \sigma} - Xe^{-rT}n\left(d_2\right)\frac{\partial d_2}{\partial \sigma}\end{aligned} \tag{8.3.35}$$

From the lemma above, we know

$$n\left(d_2\right) = \frac{Se^{-\delta T}n\left(d_1\right)}{Xe^{-rT}}. \tag{8.3.36}$$

Thus,

$$\begin{aligned}\frac{\partial c}{\partial \sigma} &= Se^{-\delta T}n\left(d_1\right)\frac{\partial d_1}{\partial \sigma} - Xe^{-rT}\left[\frac{Se^{-\delta T}n\left(d_1\right)}{Xe^{-rT}}\right]\frac{\partial d_2}{\partial \sigma} \\ &= Se^{-\delta T}n\left(d_1\right)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right] = Se^{-\delta T}n\left(d_1\right)\sqrt{T} = Xe^{-rT}n\left(d_2\right)\sqrt{T}\end{aligned}, \tag{8.3.37}$$

because $\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T}$ and by rearranging the lemma result as $n\left(d_1\right) = \frac{Xe^{-rT}n\left(d_2\right)}{Se^{-\delta T}}$.

Rho

Mathematically, rho is defined as

$$\rho_O \equiv \frac{\partial O}{\partial r} = \iota_U XTe^{-rT}N\left(\iota_U d_2\right). \tag{8.3.38}$$

Sketch of proof: From the definition of d_2, we know

$$\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r}, \tag{8.3.39}$$

and put call parity

$$\frac{\partial c}{\partial r} = rXe^{-rT} + \frac{\partial p}{\partial r}. \tag{8.3.40}$$

From the call option formula,

$$\begin{aligned}
\frac{\partial c}{\partial r} &= Se^{-\delta T}\frac{\partial N(d_1)}{\partial r} - X\frac{\partial e^{-rT}}{\partial r}N(d_2) - Xe^{-rT}\frac{\partial N(d_2)}{\partial r} \\
&= Se^{-\delta T}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r} + rXe^{-rT}N(d_2) - Xe^{-rT}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial r}. \\
&= rXe^{-rT}N(d_2) + Se^{-\delta T}n(d_1)\frac{\partial d_1}{\partial r} - Xe^{-rT}n(d_2)\frac{\partial d_2}{\partial r}
\end{aligned} \tag{8.3.41}$$

Based on $\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r}$ and the lemma above ($n(d_2) = \frac{Se^{-\delta T}n(d_1)}{Xe^{-rT}}$), we have

$$\frac{\partial c}{\partial T} = rXe^{-rT}N(d_2) + Se^{-\delta T}n(d_1)\frac{\partial d_1}{\partial r} - Xe^{-rT}\left[\frac{Se^{-\delta T}n(d_1)}{Xe^{-rT}}\right]\frac{\partial d_1}{\partial r} = rXe^{-rT}N(d_2). \tag{8.3.42}$$

Substituting this result into the put-call parity expression above,

$$\frac{\partial p}{\partial r} = \frac{\partial c}{\partial r} - rXe^{-rT} = rXe^{-rT}N(d_2) - rXe^{-rT} = -rXe^{-rT}N(-d_2). \tag{8.3.43}$$

We now turn to exploring other selected insights based on SRMs.

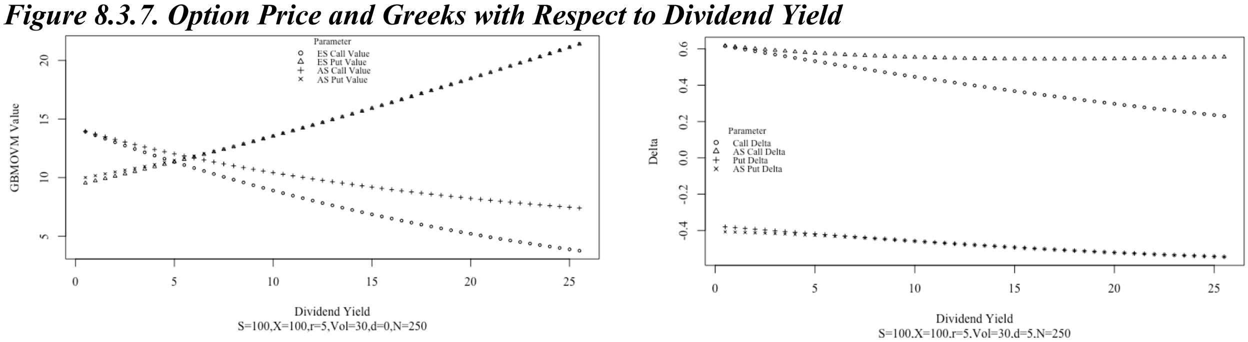
Selected other insights based on Greeks

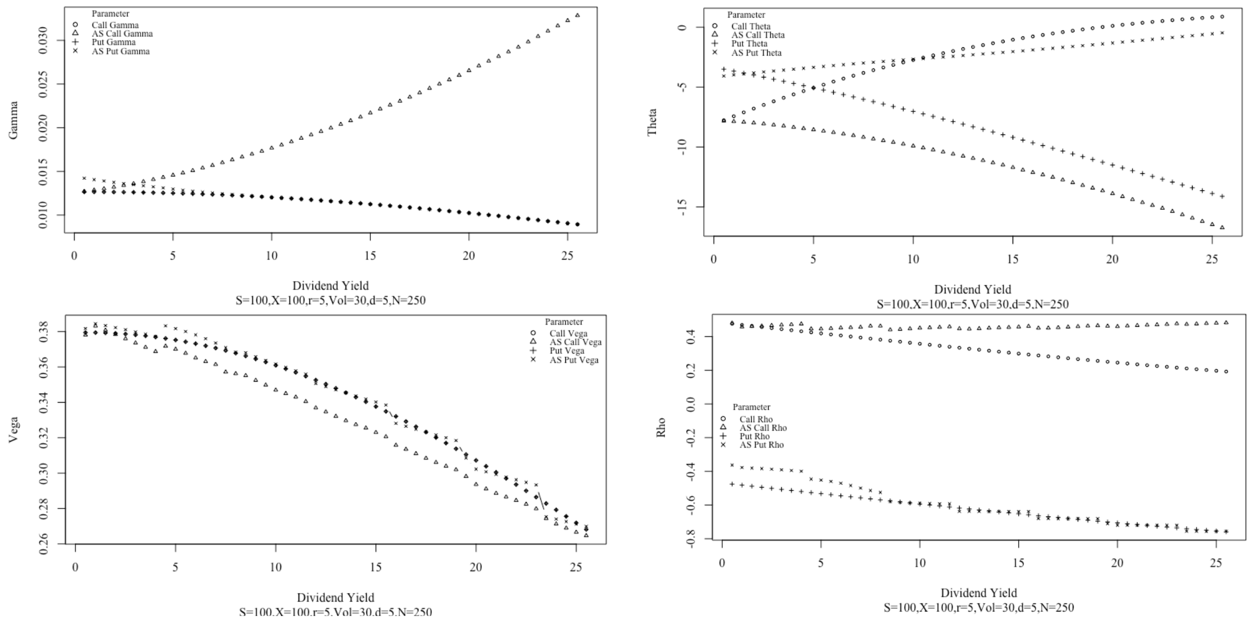
We now examine a few other aspects related to the GBMOVM. Specifically, we explore the sensitivity of the Greeks to dividend yield as well as a few extended Greeks. Finally, we explore the use of selected Greeks to estimate option price changes.

Sensitivity to dividend yield

Figure 8.3.7 provides several graphs related to dividend yield. The dividend yield on the horizontal axis and the option prices or Greeks are on the vertical axis. For option values, the negative sloped line is the call and the positive sloped line is the put. Note that when the interest rate equals the dividend yield, the value of the call equals the value of the put. Each Greek is sensitive to dividend yield.

Figure 8.3.7. Option Price and Greeks with Respect to Dividend Yield





Extended Greeks
One way to categorize static risk measures, such as delta and gamma, is based on the derivative order. The option valuation model can be represented as a function of underlying parameters of underlying instrument value (S), time to maturity (T), volatility (σ), risk-free rate (r), strike price (X), and dividend yield (δ).

$$O = f\left(S, T; \sigma, r, X, \delta\right)$$

First order risk measures include:
- Delta (S)
- Theta (t)
- Vega (σ)
- Rho (r)
- (X)
- (δ)

Second order risk measures include:
- Gamma (S,S), (S,t), (S,σ), (S,r), (S,X), (S,δ)
- (t,S), (t,t), (t,σ), (t,r), (t,X), (t,δ)
- (σ,S), (σ,t), Vanna (σ,σ), (σ,r), (σ,X), (σ,δ)
- (r,S), (r,t), (r,σ), (r,r), (r,X), (r,δ)
- (X,S), (X,t), (X,σ), (X,r), (X,X), (X,δ)
- (δ,S), (δ,t), (δ,σ), (δ,r), (δ,X), (δ,δ)

Higher order risk measures are also feasible.
Estimating option price changes
Based on delta and gamma along with the Taylor series approximation, we can estimate the dollar price change in the option for a given change in the underlying stock. We first briefly review the univariate Taylor series approximation.
Theorem: Univariate Taylor Series
Assume a continuous function $f\left(x\right)$, where $-\infty < x < \infty$ and $-\infty < f\left(x\right) < \infty$. Also assume at $f\left(x_0\right)$ has derivatives of all orders. Then the Taylor series of f about the number x_0 can be expressed as

$$f(x)=\sum_{i=0}^{\infty}\frac{f^{(i)}(x_0)}{i!}(x-x_0)^i. \tag{8.3.44}$$

The n[th] Taylor polynomial p_n of f about x_0 is

$$p_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \tag{8.3.45}$$

The n[th] Taylor remainder r_n of f about x_0 is

$$r_n(x)=f(x)-p_n(x). \tag{8.3.46}$$

It can be shown than

$$r_n(x)=\frac{f^{(n+1)}(x_z)}{(n+1)!}(x-x_0)^{n+1}, \tag{8.3.47}$$

for some $x_0 < x_z < x$.[1]

We apply this theorem to improved estimates of changes in option prices for given changes in the underlying instrument's value.

Example 1: Long call option estimated with delta only

The 1[st] order Taylor polynomial $O_1(S)$ of S about S_0 is

$$O_1(S)=O(S_0)+O'(S_0)(S-S_0). \tag{8.3.48}$$

From the definition of delta

$$\Delta_o=O'(S_0)=\left.\frac{\partial O(S)}{\partial S}\right|_{S=S_0}=\iota_O e^{-\delta T}N(\iota_O d_1). \tag{8.3.49}$$

We have with some rearranging and substitutions,

$$dO_1=O_1(S_0+\Delta S)-O(S_0)=\Delta_o(\Delta S). \tag{8.3.50}$$

Note in the limit and based on the BSMOVM GBM assumption (no dividends),

$$dS=\mu Sdt+\sigma Sdz. \tag{8.3.51}$$

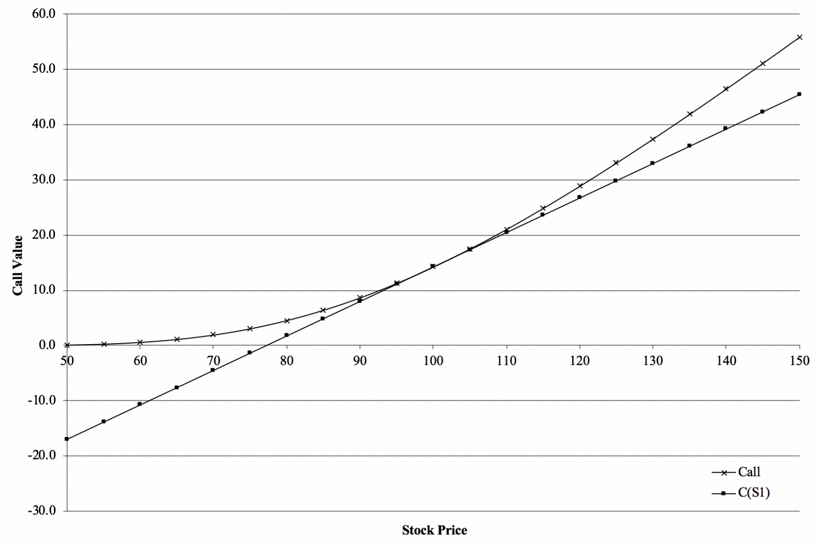
Thus, the change in the option can be roughly estimated as

$$dO_1=\Delta_o\mu Sdt+\Delta_o\sigma Sdz$$

Figure 8.3.8 illustrates the delta approximation of the change in the option price for a given change in the underlying instrument. We assume the base case is a non-dividend paying stock with an initial stock price of $100, strike price of $100, risk-free interest rate of 5%, volatility of 30%, and time to expiration of 1 year. The call price, according to the GBMOVM is $14.23.

[1] The univariate Taylor series can be found in just about any calculus book. See, for example, Ellis and Gulick [1982], 477.

Figure 8.3.8. Delta Approximation of the GBMOVM



Clearly, based on Figure 8.3.8, we need a more precise estimate. Note that for a sufficiently large change in the underlying instrument value, we would estimate a change in the option price when compared to its original value, a very counterintuitive result. We now examine applying second order Taylor series.

Example 2: Long call option with delta and gamma

The 2nd Taylor polynomial $O_2(S)$ of S about S_0 is

$$O_2(S) = O(S_0) + O'(S_0)(S - S_0) + \frac{O''(S_0)}{2!}(S - S_0)^2. \tag{8.3.52}$$

From the definition of delta as before and gamma

$$\Gamma_o = O''(S_0) = \left.\frac{\partial O^2(S)}{\partial S^2}\right|_{S=S_0} = \frac{e^{-\delta T} n(d_1)}{S\sigma\sqrt{T}}, \tag{8.3.53}$$

where

$$n(d) = \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}}. \tag{8.3.54}$$

We have with some rearranging and substitutions,

$$dO_2 = O_2(S_0 + \Delta S) - O(S_0) = \Delta_o(\Delta S) + \frac{\Gamma_o}{2}(\Delta S)^2. \tag{8.3.55}$$

Again, based on the BSMOVM and GBM framework, the change in the option can be roughly estimated as

$$\begin{aligned} dO_2 &= \Delta_o(\mu S dt + \sigma S dz) + \frac{\Gamma_o}{2}(\mu S dt + \sigma S dz)^2 \\ &= \Delta_o \mu S dt + \Delta_o \sigma S dz + \frac{\Gamma_o}{2}\left(\mu^2 S^2 dt^2 + 2\mu\sigma S^2 dt dz + \sigma^2 S^2 dz^2\right) \end{aligned}. \tag{8.3.56}$$

Note that in the limit, based on well-known properties of Brownian motion,

$$dt^2 = 0, \tag{8.3.57}$$

$$dtdz = 0, \text{ and} \tag{8.3.58}$$

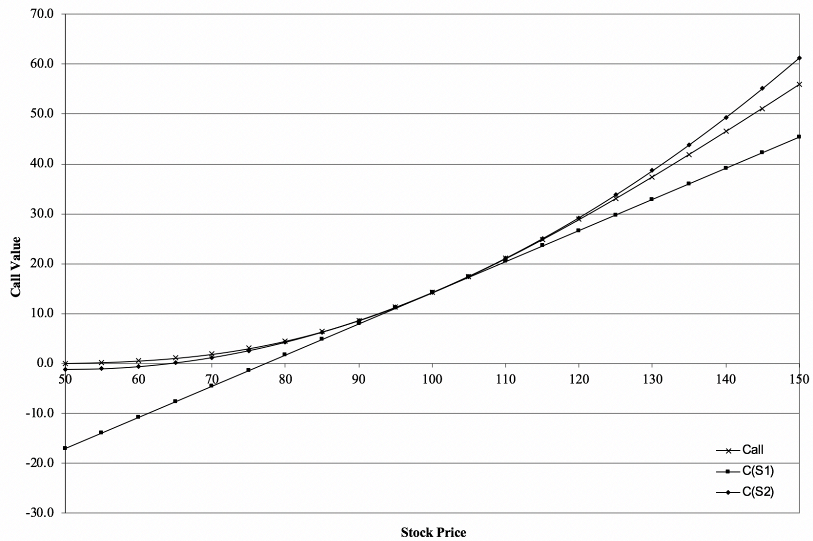
$$dz^2 = dt. \tag{8.3.59}$$

Thus, in the limit, we know

$$dO = \left(\Delta_o \mu S + \frac{\Gamma_o}{2}\sigma^2 S^2 \right) dt + \Delta_o \sigma S dz . \tag{8.3.60}$$

As illustrated in Figure 8.3.9 below, the delta-gamma estimate is an improvement, but we could use an even more precise estimate for larger changes. Again, for a sufficiently large change in the underlying instrument value, we would estimate a change in the option price more than its original value, a very counterintuitive result. To improve our Taylor series estimation process as well as better understand the GBMOVM, we consider a multivariate Taylor series expansion.

Figure 8.3.9. Delta and Gamma Approximations of GBMOVM



For the next approximation, we need the multivariate version of Taylor series.

Theorem: Multivariate Taylor Series[2]

Assume a continuous function $f(X)$, where $X(x_1, x_2, \ldots, x_n)$ is a vector with n elements, and $-\infty < x_i < \infty, i = 1, \ldots, n$. Also assume at $f(X^0)$ has derivatives of all orders. Let $D_i = \partial/\partial x_i$ be the operators of partial differentiation where

$$D_i f = \partial f / \partial x_i , \tag{8.3.61}$$

and

$$D_i^m f = \partial^m f / \partial x_i^m , \tag{8.3.62}$$

and in the multidimensional case

$$D_{i_1} D_{i_2} \ldots D_{i_k} f = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_K}},$$

where the required partial derivatives are assumed to exist. Then the Taylor series of f about the point X^0 is

$$f(X) = \sum_{i=0}^{\infty} \frac{\left[\sum_{j=1}^{n} \left(x_j - x_j^0 \right) D_j \right]^i}{i!} f(X^0). \tag{8.3.63}$$

The k[th] degree Taylor polynomial of n variables $p_{n,k}$ of f about X^0 is

---

[2] See, for example, Aramanovich, et. al. [1965].

$$p_{n,k}\left(X\right)=\sum_{i=0}^{k}\frac{\left[\sum_{j=1}^{n}\left(x_j-x_j^0\right)D_j\right]^i}{i!}f\left(X^0\right). \tag{8.3.64}$$

The k[th] Taylor remainder $r_{n,k}$ of f about X^0 is

$$r_{n,k}\left(X\right)=f\left(X\right)-p_{n,k}\left(X\right).$$

It can be shown than

$$r_{n,k}\left(X\right)=\frac{\left[\sum_{j=1}^{n}\left(x_j-x_j^0\right)D_j\right]^{k+1}}{\left(k+1\right)!}f\left(X^z\right),$$

for some $X^0<X^z<X$.

We now consider the case where the size of the vector X is $n=2$. Assume a continuous function $f\left(x_1,x_2\right)$, where $-\infty<x_1,x_2<\infty$ and $-\infty<f\left(x_1,x_2\right)<\infty$. Also assume at $f\left(x_1^0,x_2^0\right)$ has derivatives of all orders. Then the Taylor series of f about the numbers x_1^0,x_2^0 is

$$f\left(x_1,x_2\right)=\sum_{i=0}^{\infty}\frac{1}{i!}\left[\left(x_1-x_1^0\right)\frac{\partial}{\partial x}+\left(x_2-x_2^0\right)\frac{\partial}{\partial y}\right]^i f\left(x_1^0,x_2^0\right). \tag{8.3.65}$$

Therefore, the 2[nd] Taylor polynomial $p_2\left(x_1,x_2\right)$ of f about x_1^0 and x_1^0 is

$$\begin{aligned}p_2\left(x_1,x_2\right)&=f\left(x_1^0,x_2^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_1}\left(x_1-x_1^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_2}\left(x_2-x_2^0\right)\\&+\frac{1}{2}\frac{\partial f^2\left(x_1^0,x_2^0\right)}{\partial x_1^2}\left(x_1-x_1^0\right)^2+\frac{1}{2}\frac{\partial f^2\left(x_1^0,x_2^0\right)}{\partial x_2^2}\left(x_2-x_2^0\right)^2+\frac{\partial f^2\left(x_1^0,x_2^0\right)}{\partial x_1\,\partial x_2}\left(x_1-x_1^0\right)\left(x_2-x_2^0\right)\end{aligned}. \tag{8.3.66}$$

Example 3: Long call option with delta, gamma, and theta

The 2[nd] degree Taylor polynomial of both variables $O_{2,2}\left(S,t\right)$ of O about $\left(S,t\right)^0$ is

$$\begin{aligned}O_{2,2}\left(S,t\right)&=O\left(S_0,t_0\right)+\frac{\partial O\left(S_0,t_0\right)}{\partial S}\left(S-S_0\right)+\frac{\partial O\left(S_0,t_0\right)}{\partial t}\left(t-t_0\right)\\&+\frac{1}{2!}\frac{\partial^2 O\left(S_0,t_0\right)}{\partial S^2}\left(S-S_0\right)^2+\frac{1}{2!}\frac{\partial^2 O\left(S_0,t_0\right)}{\partial t^2}\left(t-t_0\right)^2+\frac{1}{2!}\frac{\partial^2 O\left(S_0,t_0\right)}{\partial S\,\partial t}\left(S-S_0\right)\left(t-t_0\right)\end{aligned}. \tag{8.3.67}$$

From the definition of delta and gamma as before, as well as

$$\theta_o=\left.\frac{\partial O}{\partial t}\right|_{S=S_0,t=t_0}, \tag{8.3.68}$$

$$\hat{\theta}_o=\left.\frac{\partial^2 O}{\partial t^2}\right|_{S=S_0,t=t_0}, \text{ and} \tag{8.3.69}$$

$$Charm_o=\left.\frac{\partial^2 O}{\partial S\,\partial t}\right|_{S=S_0,t=t_0}. \tag{8.3.70}$$

Again, based on the GBMOVM, the change in the option can be roughly estimated as

$$dO_{2,2} = \Delta_o\left(\mu Sdt + \sigma Sdz\right) + \theta_o dt + \frac{\Gamma_o}{2}\left(\mu Sdt + \sigma Sdz\right)^2$$
$$+\frac{\hat{\theta}_o}{2}dt^2 + \frac{Charm_o}{2}\left(\mu Sdt + \sigma Sdz\right)dt$$
$$= \Delta_o \mu Sdt + \Delta_o \sigma Sdz + \theta_o dt + \frac{\Gamma_o}{2}\left(\mu^2 S^2 dt^2 + 2\mu\sigma S^2 dtdz + \sigma^2 S^2 dz^2\right) \quad (8.3.71)$$
$$+\frac{\hat{\theta}_o}{2}dt^2 + \frac{Charm_o}{2}\left(\mu Sdt^2 + \sigma Sdtdz\right)$$

Again, in the limit, based on well-known properties of Brownian motion,

$$dt^2 = 0, \quad (8.3.72)$$
$$dtdz = 0, \text{ and} \quad (8.3.73)$$
$$dz^2 = dt. \quad (8.3.74)$$

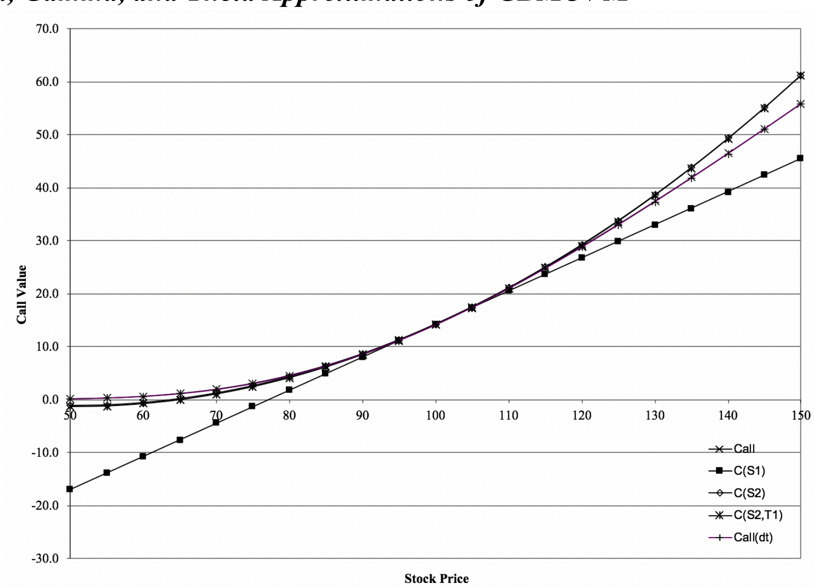
Thus,

$$dO_{2,2} = \left(\theta_o + \Delta_o \mu S + \frac{\Gamma_o}{2}\sigma^2 S^2\right)dt + \Delta_o \sigma Sdz. \quad (8.3.75)$$

Note that this is the Itô's Lemma result of the standard GBMOVM.

Figure 8.3.10 illustrate incorporating theta where one week is assumed to have passed. Unfortunately, by assuming the mere passage of time, the basic option valuation mapping changes. If we remap the option valuation results, then the delta-gamma result becomes the delta-gamma-theta result. Thus, to improve the estimate, we need higher order Greeks with respect to the underlying instrument.

Figure 8.3.10. Delta, Gamma, and Theta Approximations of GBMOVM



Example 4: Long call option with delta, gamma, and speed

The 3rd Taylor polynomial $O_3(S)$ of S about S_0 is

$$O_3(S) = O(S_0) + O'(S_0)(S - S_0) + \frac{O''(S_0)}{2!}(S - S_0)^2 + \frac{O'''(S_0)}{3!}(S - S_0)^3.$$

From the definition of delta and gamma as before and speed

$$Speed_o = O'''(S_0) = \left.\frac{\partial^3 O(S)}{\partial S^3}\right|_{S=S_0} = -\Gamma_o d_3, \text{ (speed)} \quad (8.3.76)$$

where

$$d_3 = \frac{d_1 + \sigma\sqrt{T}}{S\sigma\sqrt{T}}. \quad (8.3.77)$$

We have with some rearranging and substitutions,

$$dO_3 = O_3(S_0 + \Delta S) - O(S_0) = \Delta_o(\Delta S) + \frac{\Gamma_o}{2}(\Delta S)^2 + \frac{Speed_o}{6}(\Delta S)^3. \quad (8.3.78)$$

Again, based on the GBMOVM, the change in the option can be roughly estimated as

$$dO_3 = \Delta_o(\mu S dt + \sigma S dz) + \frac{\Gamma_o}{2}(\mu S dt + \sigma S dz)^2 + \frac{Speed_o}{6}(\mu S dt + \sigma S dz)^3. \quad (8.3.79)$$

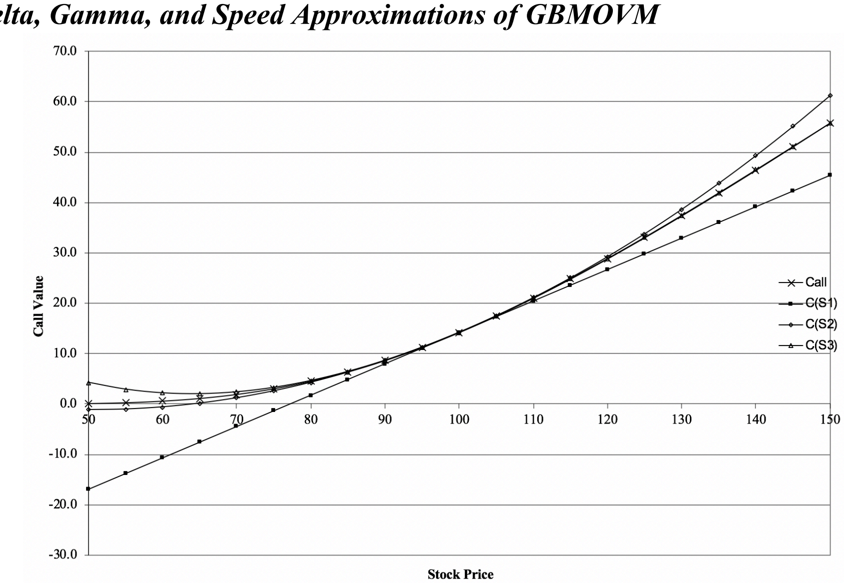
Note that in the limit, based on well-known properties of Brownian motion, we continue to have

$$dO = \left(\Delta_o \mu S + \frac{\Gamma_o}{2}\sigma^2 S^2\right)dt + \Delta_o \sigma S dz. \quad (8.3.80)$$

as all higher order terms are eliminated. Note that this is the same result when Speed is ignored. Higher order terms do not influence limiting results even though they do influence discrete changes.

As illustrated in Figure 8.3.11, the delta-gamma-speed estimate is an improvement, but we could use an even more precise estimate for larger changes.

Figure 8.3.11. Delta, Gamma, and Speed Approximations of GBMOVM



We now briefly sketch higher order derivatives. There is no evidence that knowledge of these higher order derivatives is helpful in financial risk management activities.

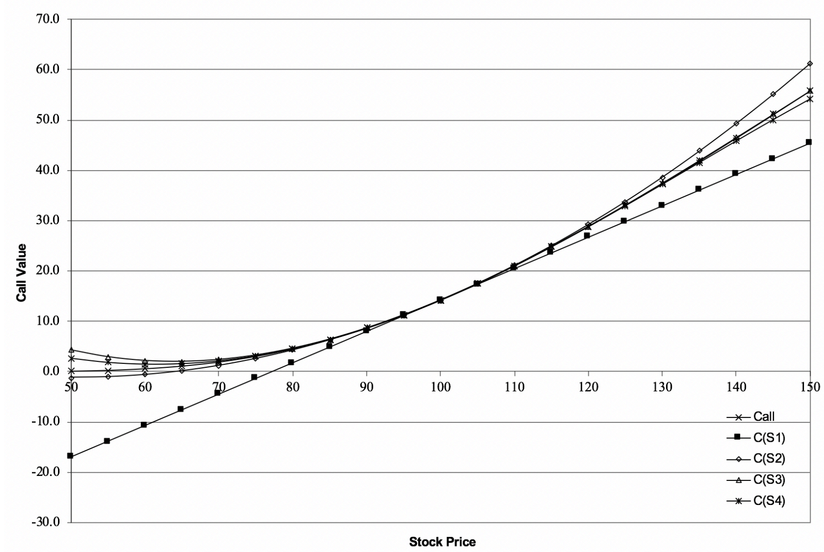
Fourth derivative:

$$O_{4S} = -\left[O_{3S} d_3 + O_{2S} d_{3,S}\right], \quad (8.3.81)$$

where

$$d_{3,S} = \frac{\partial d_3}{\partial S} = \frac{-d_1\sigma\sqrt{T} - \sigma^2 T + 1}{S^2\sigma^2 T}.$$

Figure 8.3.12. Fourth Derivative Approximations of GBMOVM



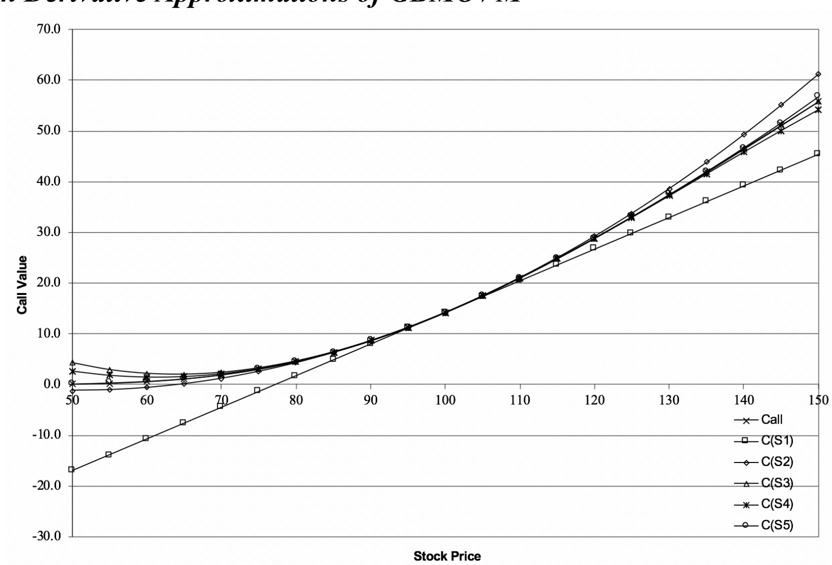
Fifth derivative:

$$O_{5S} = -\left[O_{4S} d_3 + 2O_{3S} d_{3,S} + O_{2S} d_{3,2S} \right], \tag{8.3.82}$$

where

$$d_{3,2S} = \frac{\partial^2 d_3}{\partial S^2} = \frac{2d_1 \sigma \sqrt{T} + 2\sigma^2 T - 3}{S^3 \sigma^2 T}. \tag{8.3.83}$$

Figure 8.3.13. Fifth Derivative Approximations of GBMOVM



Sixth derivative:

$$O_{6S} = -\left[O_{5S} d_3 + 3O_{4S} d_{3,S} + 3O_{3S} d_{3,2S} + O_{2S} d_{3,3S} \right], \tag{8.3.84}$$

where

$$d_{3,3S} = \frac{\partial^3 d_3}{\partial S^3} = \frac{-6d_1 \sigma \sqrt{T} - 6\sigma^2 T + 11}{S^4 \sigma^2 T}. \tag{8.3.85}$$

Seventh derivative:

$$O_{7S} = -\left[O_{6S} d_3 + 4O_{5S} d_{3,S} + 6O_{4S} d_{3,2S} + 4O_{3S} d_{3,3S} + O_{2S} d_{3,4S} \right], \tag{8.3.86}$$

where

$$d_{3,4S} = \frac{\partial^4 d_3}{\partial S^4} = \frac{24 d_1 \sigma \sqrt{T} + 24\sigma^2 T - 50}{S^5 \sigma^2 T}. \tag{8.3.87}$$

Eighth derivative:

$$O_{8S} = -\left[O_{7S} d_3 + 5O_{6S} d_{3,S} + 10O_{5S} d_{3,2S} + 10O_{4S} d_{3,3S} + 5O_{3S} d_{3,4S} + O_{2S} d_{3,5S} \right], \tag{8.3.88}$$

where

$$d_{3,5S} = \frac{\partial^5 d_3}{\partial S^5} = \frac{-120 d_1 \sigma \sqrt{T} - 120\sigma^2 T + 274}{S^6 \sigma^2 T}. \tag{8.3.89}$$

Ninth derivative:

$$O_{9S} = -\left[O_{8S} d_3 + 6O_{7S} d_{3,S} + 15O_{6S} d_{3,2S} + 20O_{5S} d_{3,3S} + 15O_{4S} d_{3,4S} + 6O_{3S} d_{3,5S} + O_{2S} d_{3,6S} \right], \tag{8.3.90}$$

where

$$d_{3,6S} = \frac{\partial^6 d_3}{\partial S^6} = \frac{720 d_1 \sigma \sqrt{T} + 720\sigma^2 T - 1{,}764}{S^7 \sigma^2 T}. \tag{8.3.91}$$

Tenth derivative:

$$O_{10S} = -\left[O_{9S} d_3 + 7O_{8S} d_{3,S} + 21O_{7S} d_{3,2S} + 35O_{6S} d_{3,3S} + 35O_{5S} d_{3,4S} + 21O_{4S} d_{3,5S} + 7O_{3S} d_{3,6S} + O_{2S} d_{3,7S} \right], \tag{8.3.92}$$

where

$$d_{3,7S} = \frac{\partial^7 d_3}{\partial S^7} = \frac{-5{,}040 d_1 \sigma \sqrt{T} - 5{,}040\sigma^2 T + 13{,}068}{S^7 \sigma^2 T}. \tag{8.3.93}$$

Summary

Based on the notation presented in Module 5.4, we illustrated computing option Greeks within the geometric Brownian motion option valuation model (GBMOVM) for European-style options. We further explore differences between the Greeks based on the binomial option valuation model. We also examined various extension available given a SRM approach to GBMOVM.

References

See Module 5.4.