Module 7.2: Static Risk Management U. S. Treasuries

Learning objectives

- Review the basics of bond static risk management
- Quick survey of the history of bond risk management
- Distinguish between discrete and continuously compounded yield to maturity and its influence on duration calculations
- Review the basic properties of bond duration and convexity
- Introduce advanced static risk measures

Executive summary

In this module, we first review various aspects of traditional bond static risk measures. The traditional bond static risk measures include various forms of duration (Macaulay, modified, and effective) and a couple of forms of convexity (standard and effective). We explore the important role of compounding for both taking the present value of maturity varying cash flows as well as compounding methods for calculating holding period returns. After a brief tour of bond history, we review several important aspects of these measures.

With this foundation, we then move to advanced bond static risk measures based on an application of the LSC model. Within a detailed bond holding period return decomposition, we identify and explain numerous new measures of bond static risk. With these advanced measures, we explore bond expected holding period returns as well as its related variance. The module concludes with a brief explanation of selected R code.

Central finance concepts

Financial valuation at its core is simply estimating the present value of expected future cash flows. Though simple to express, the complexities lie in the details. Our focus here is related to the sensitivities of bond valuation to various underlying parameters. Though most of the module is rather technical, we discuss a few finance concepts here.

Origins of bond risk management

Macaulay (1938) is usually credited with providing the first mathematically detailed definition of an adjusted bond term to maturity, a definition Macaulay gave as duration.[1] Macaulay (1938) states "(f)or a study of the relations between long and short time interest rates, it would seem highly desirable to have some adequate measure of 'longness'. Let us use the word 'duration' to signify the essence of the time element in a loan. If one loan is essentially a longer term loan than another, we shall speak of it as having greater 'duration'." (p. 44) He further elaborates, "Now, if present value weighting be used, the 'duration' of a bond is an average of the durations of the separate single payment loans into which the bond may be broken up. To calculate this average the duration of each individual single payment loan must be weighted in proportion to the size of the individual loan; in other words, by the ratio of the present value of the individual future payment to the sum of all the present values, which is, of course, the price paid for the bond." (p. 48)

Traditional bond static risk measure definitions

Recall the coupon-bearing bond's value today can be modeled by taking the present value at some appropriate discount rate of the remaining coupons and principal payments. There are two traditional static bond risk measures, modified duration, and standard convexity. Modified duration measures the percentage change in the bond price (or portfolio) for a given change in the yield to maturity. Standard convexity measures the curvature of the price-yield relationship (again assuming discretely compounded yield to maturity) With Macaulay and modified duration, the change in the estimated price does not incorporate changes in cash flow. Effective duration is a measure of cash flow adjusted volatility. Effective convexity

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[1]Hawawini (1982) notes, however, that the duration concept may have been identified much earlier by Lidstone (1895). See Hawawini (1982), page 4.

similarly adjusts for cash flow effects on standard convexity by using a valuation model to compute bond prices that incorporate the consequences of interest rate changes on cash flows.

Immunization

Redington (1952) introduced the concept of immunization, a particularly useful idea in debt portfolio management. Redington states, "... I use the word 'immunization' to signify the investment of the assets in such a way that the existing business is immune to a general change in the rate of interest. The definition is not exact, but it should not mislead. Because of this definition immunization is to be regarded as a particular form of matching." (p. 289) Redington used a Taylor series to illustrate immunization and the role of duration. Interestingly, he asserts that in "... practice the first derivative is the most important for small changes in the rate of interest..." (p. 290) He further computed convexity, though he referred to it as spread, and provided some insights. Specifically, he states, "... in broad terms, the spread of the value of the asset-proceeds about the mean term should be greater than the spread of the value of the liability-outgo." (p. 291)

The intuition behind immunization can be applied widely in finance and is extremely useful, particularly considering the difficulties related to future forecasts. Forecasting the future entails various forms of sample error. By seeking to immunize portfolios, sample errors related to asset parameters are likely to be offset by sample errors related to liability parameters because often they are the same input parameters (e.g., discount rates). We now turn to selected figures related to this analysis.

Selected figures

Figure 7.2.1 presents the various bond values for different yield to maturities. Clearly, the higher coupon bonds are more valuable when everything else is the same.

Figure 7.2.1. Bond Value with Respect to Yield to Maturity with Different Coupon Rates

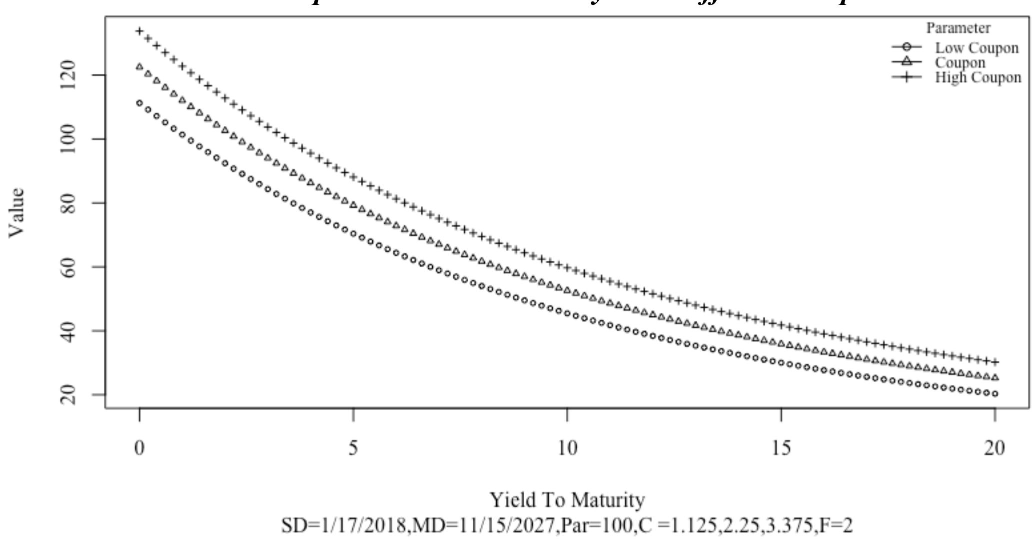


Figure 7.2.2 presents Macaulay duration for these three bonds. For duration, the higher the stated coupon, the lower is Macaulay duration. In all three cases, the higher the yield to maturity, the lower is Macaulay duration. Thus, in extremely low interest rate environments, bonds have higher sensitivity to changes in yield to maturity.

Figure 7.2.2. Bond Macaulay Duration with Respect to Yield to Maturity with Different Coupon Rates

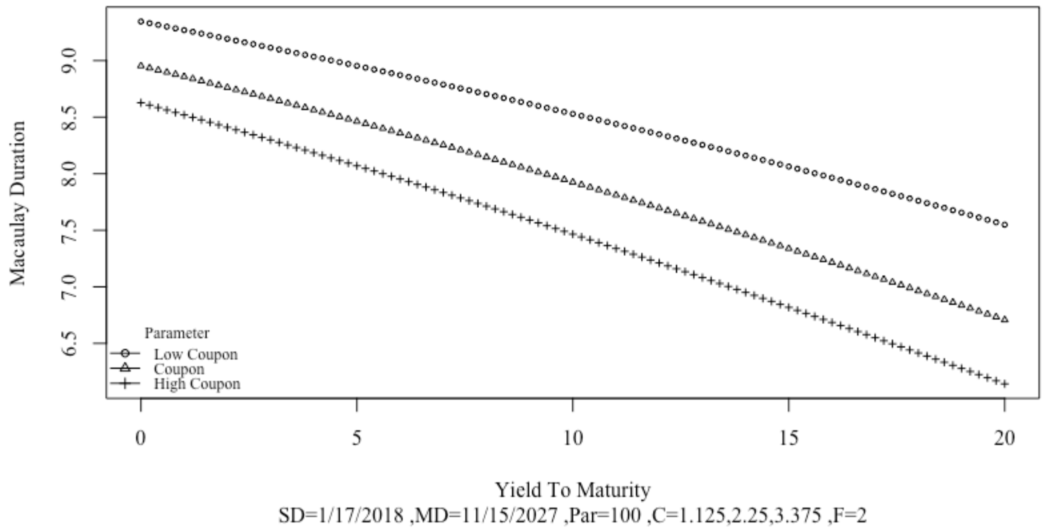
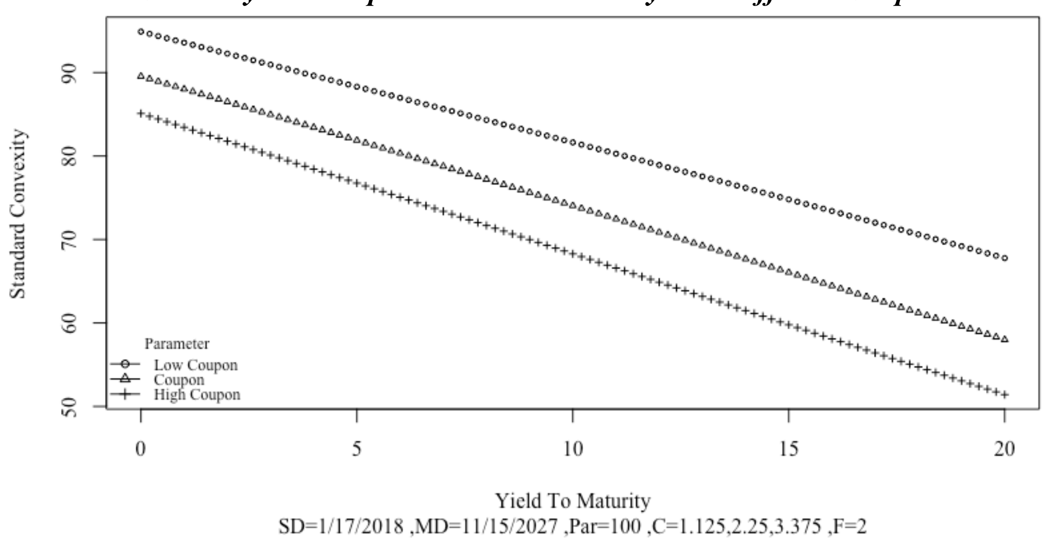


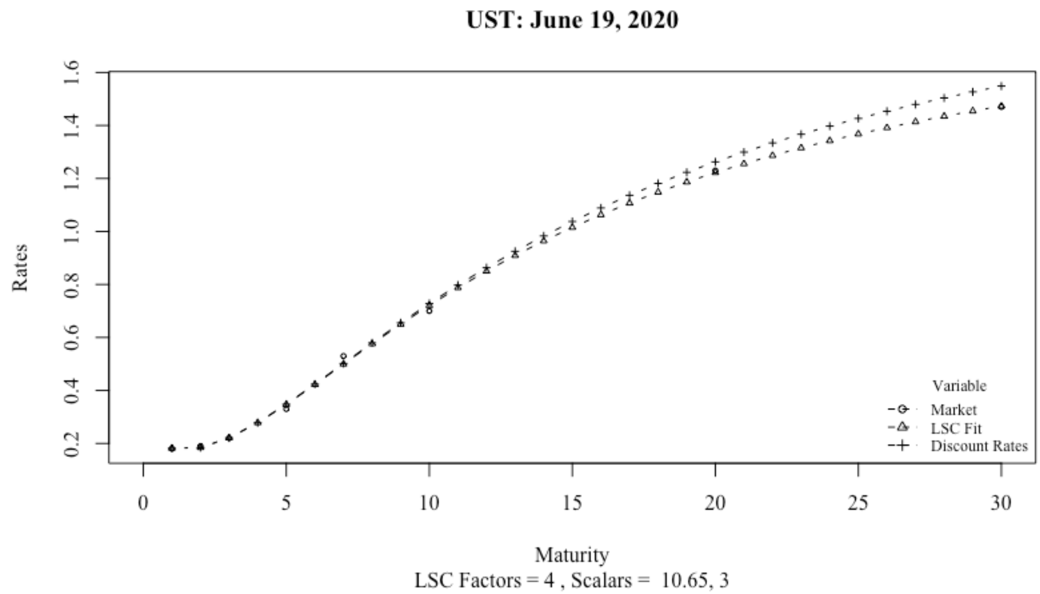
Figure 7.2.3 presents convexity for these three bonds. Bond convexity exhibits similar sensitivities to different coupon rates and yield to maturities.

Figure 7.2.3. Bond Convexity with Respect to Yield to Maturity with Different Coupon Rates



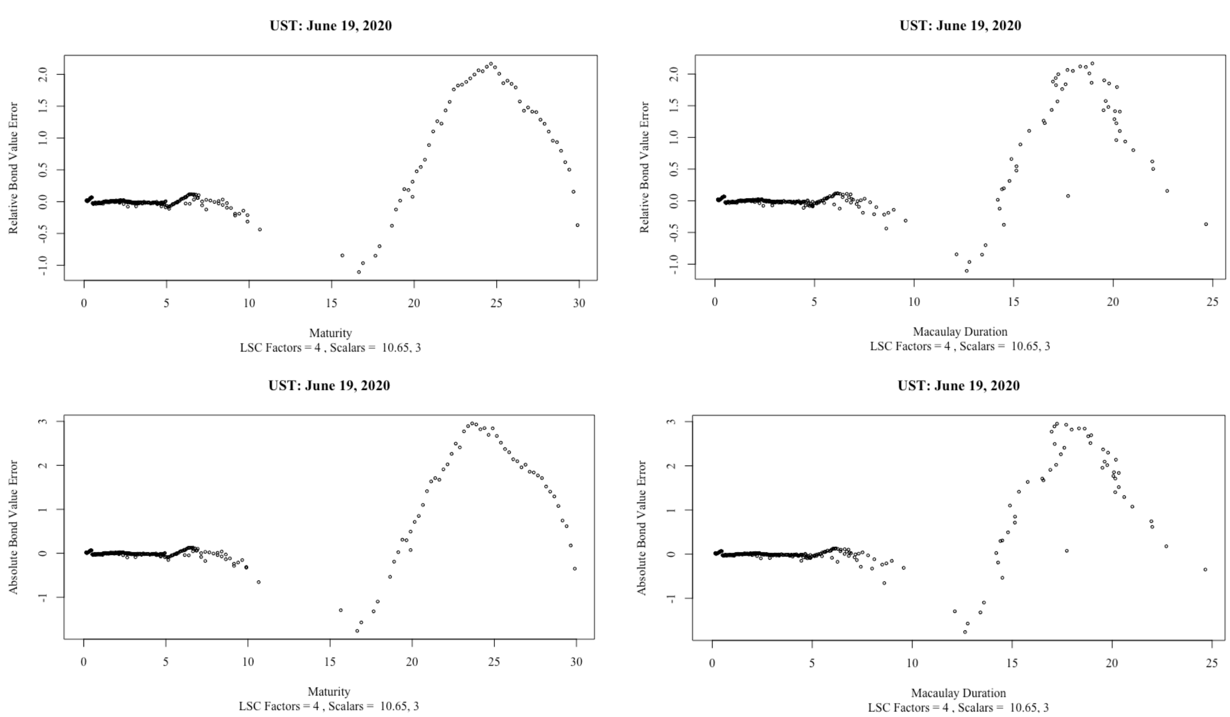
As illustrated in Module 4.1 with a different dataset, Figure 7.2.4 provides a plot of the three different rates, the original CMT rates, the fitted spot rates, and the fitted discount rates.

Figure 7.2.4. Four Factor LSC Model fit of UST CMT Rates Including Discount Rates



With the fitted LSC model, we can analyze the entire UST portfolio. The data set contains all UST notes and bonds except very short maturities. We produce several plots that examine estimation error introduced by the LSC model. Figure 7.2.5 provides a sampling illustrating relative and absolute bond value error based on maturity and duration as well as yield differential and yield to maturity. The scalars were set to be 10.65 and 3. Thus, the LSC model fit is best within the first 10 years or so.

Figure 7.2.5. Selected Estimation Errors from LSC Model Fit



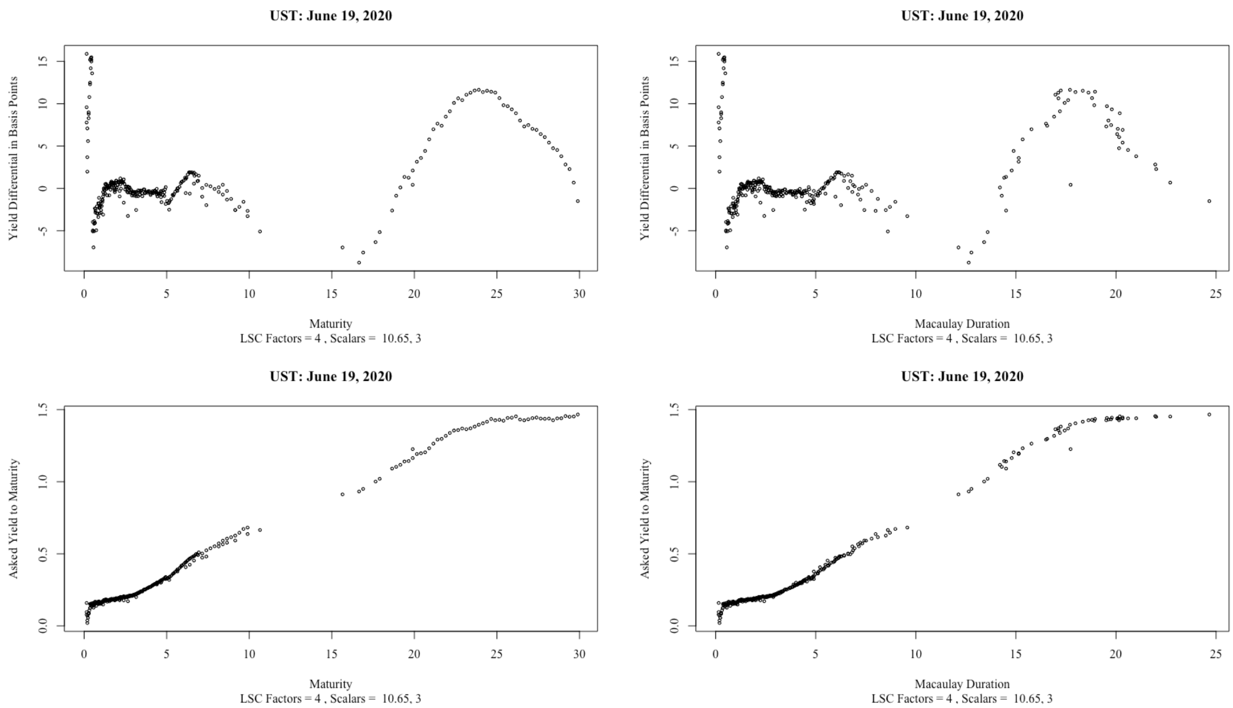


Figure 7.2.6 illustrates the UST bonds effective duration with respect to maturity for the selected date.

Figure 7.2.6. UST Bond Effective Duration with Respect to Maturity

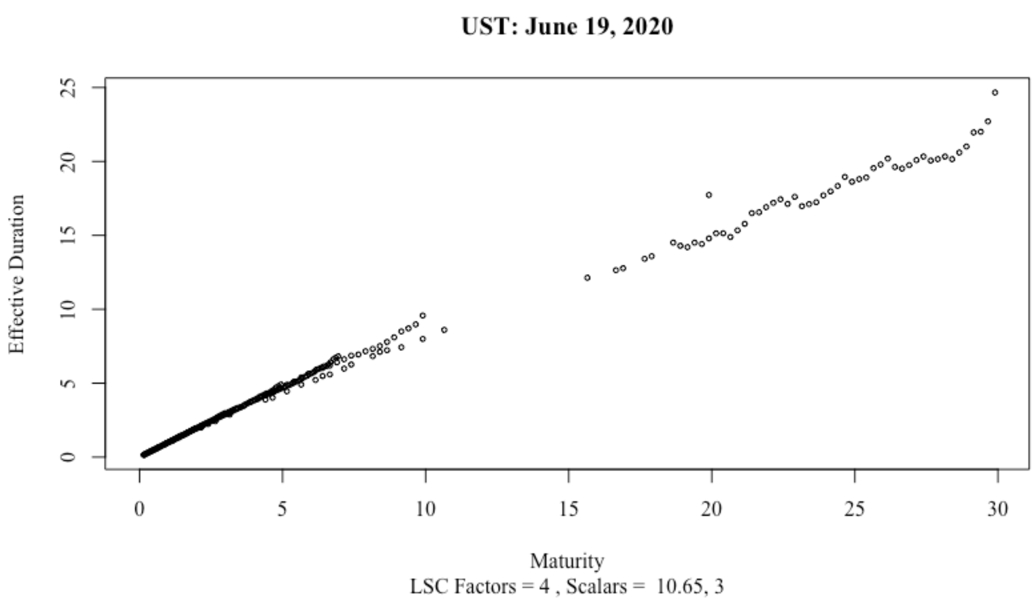


Figure 7.2.7 illustrates the UST bonds effective convexity with respect to maturity for the selected date.

Figure 7.2.7. UST Bond Effective Convexity with Respect to Maturity

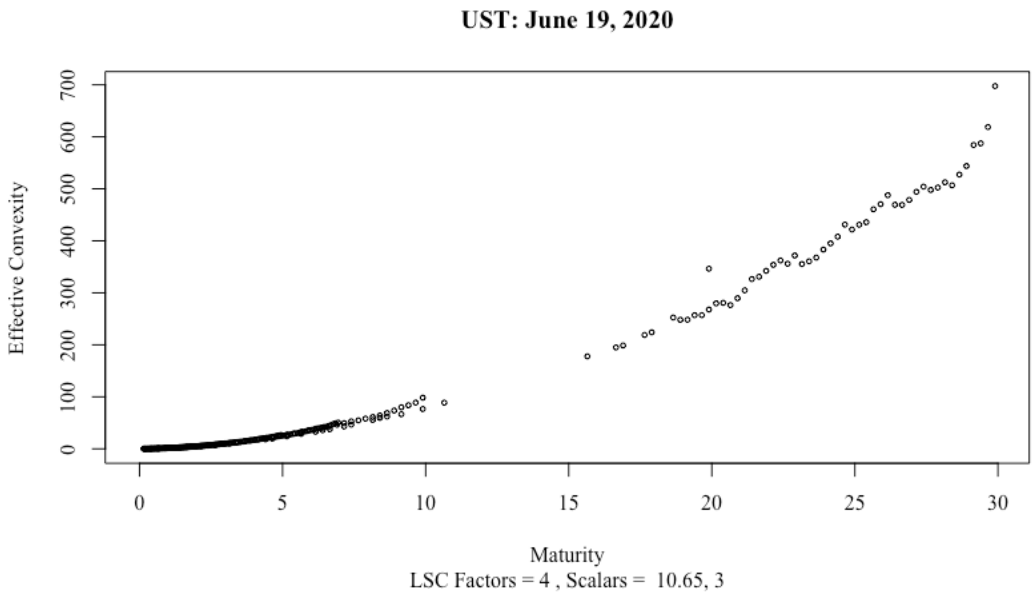


Figure 7.2.8 presents the effective convexity with respect to effective duration.

Figure 7.2.8. UST Bond Effective Convexity with Effective Duration

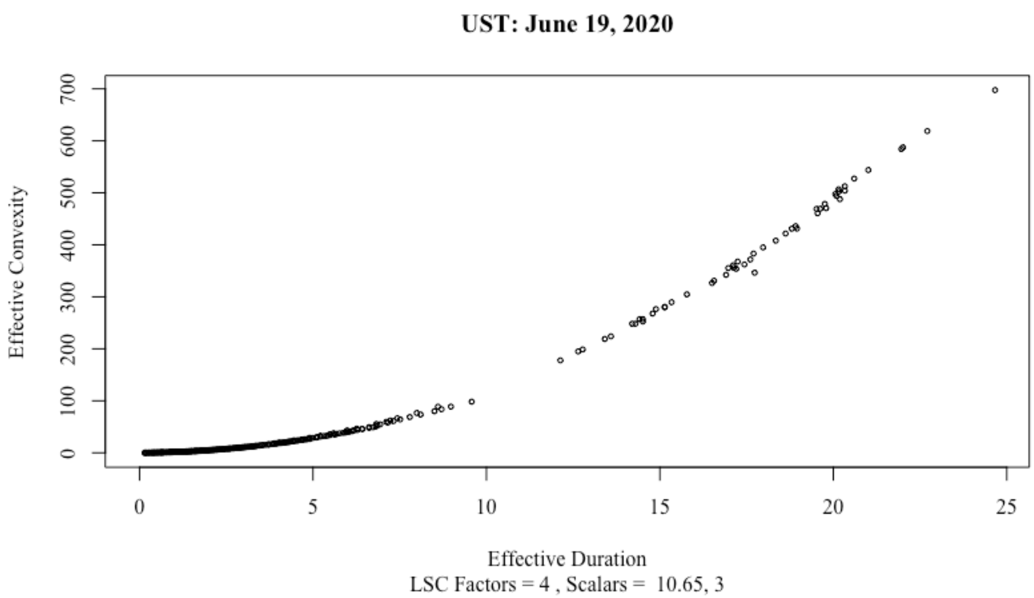


Figure 7.2.9 presents the three factor LSC model fit. Note that it is not as precise as the four factor model fitted earlier but it remains very close. Often, parsimony is highly valued and increased fit error is acceptable.

Figure 7.2.9. Three Factor LSC Model fit of UST CMT Rates Including Discount Rates

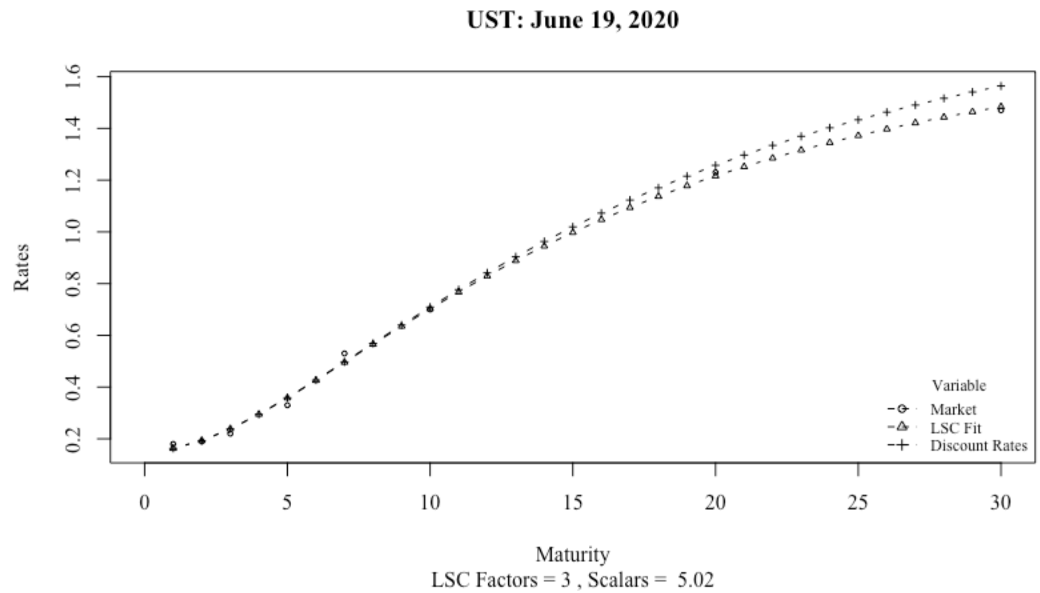


Figure 7.2.10 presents some of the input information, specifically the asked yield to maturity mapped to maturity.

Figure 7.2.10. Asked Yield to Maturity and Maturity

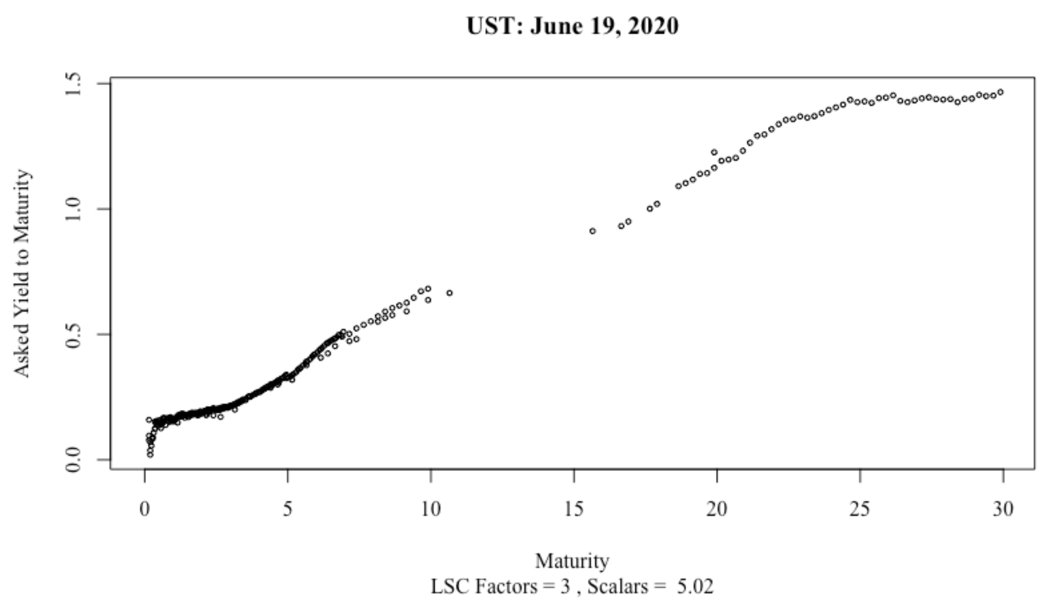


Figure 7.2.11 presents the level, slope, and curvature durations. The magnitude of these duration values is not the same nor are they proportionally the same across maturities.

Figure 7.2.11. Three Factor LSC Model Duration for UST Bonds

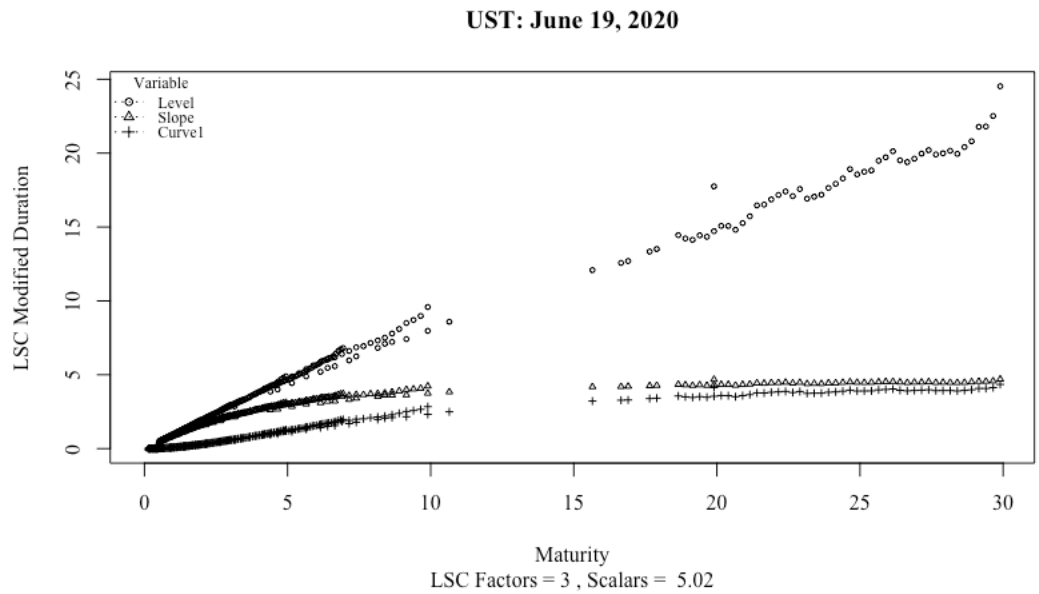


Figure 7.2.12 presents the level, slope, and curvature convexity. The magnitude of these convexity values is not the same nor are they proportionally the same across maturities.

Figure 7.2.12. Three Factor LSC Model Convexity for UST Bonds

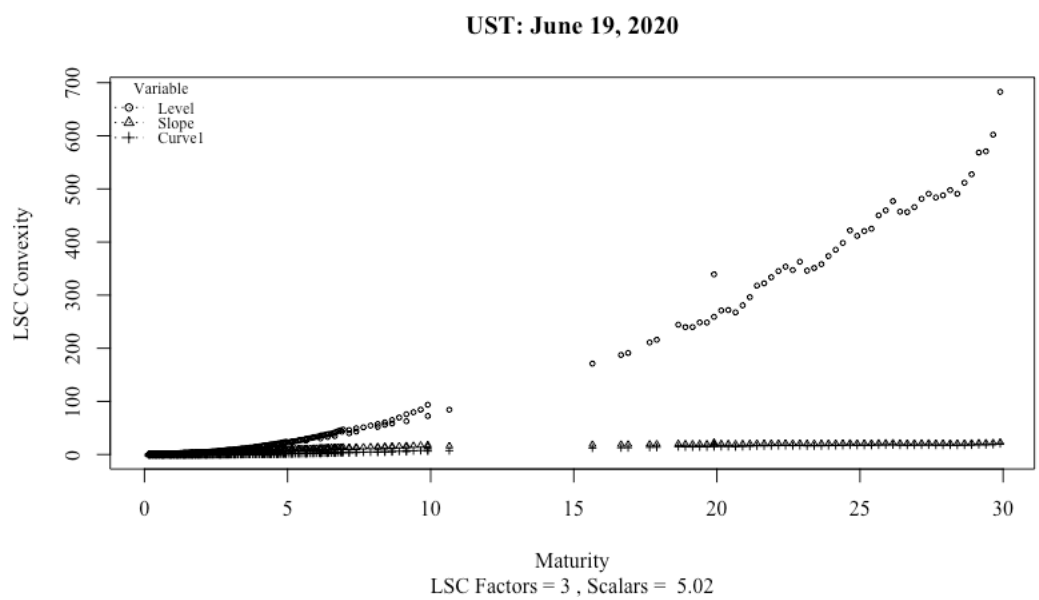
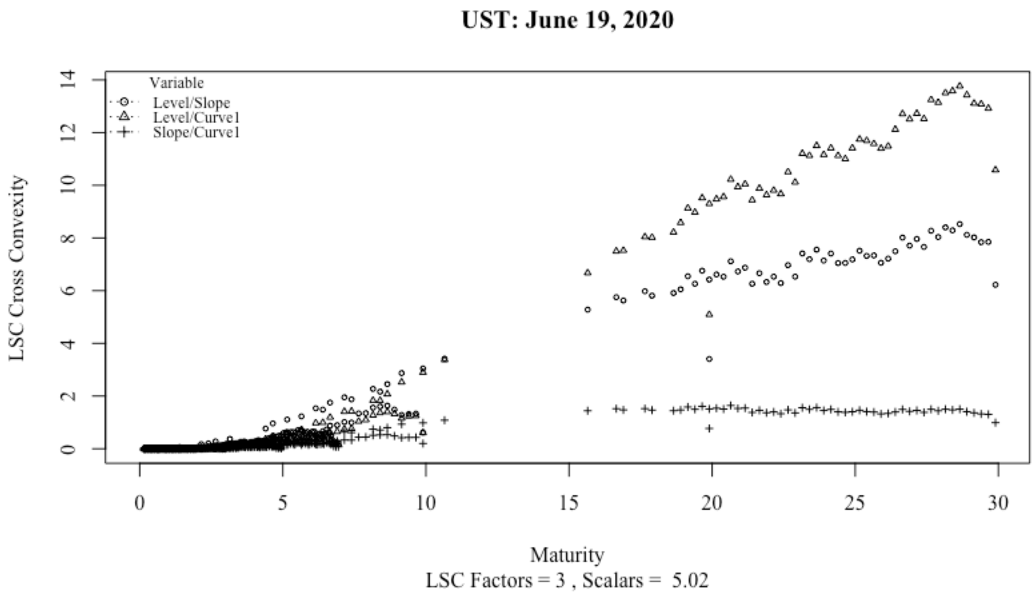


Figure 7.2.13 presents the cross convexity. The magnitude of these cross convexity values is relatively small.

Figure 7.2.13. Three Factor LSC Model Cross Convexity for UST Bonds



Quantitative finance materials

We now dive into the technicalities of bond static risk measurement.

Traditional bond static risk measure definitions

Recall the coupon-bearing bond's value today with discretely compounded yield, $V_{B,y}$, depends on the number of payments per year, m, the par value, Par, fraction of payment period elapsed already, f, the number of remaining coupon payments, N, the coupon rate expressed in annual decimal terms, CR, and the yield to maturity expressed in annual decimal terms, y,

$$\begin{aligned} V_{B,y} &= \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right) Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}} \\ &= \sum_{i=1}^{N} \frac{CF_i}{\left(1+\frac{y}{m}\right)^{i-f}} \end{aligned} . \tag{7.2.1}$$

There are two traditional static bond risk measures, modified duration and standard convexity. Modified duration measures the percentage change in the bond price (or portfolio) for a given change in the discretely compounded yield to maturity or

$$\begin{aligned} ModDur_{B,y} &\equiv -\frac{dV_{B,y}/V_B}{dy} \\ &= \frac{1}{m}\left[\sum_{i=1}^{N} \frac{(i-f)CF_i}{\left(1+\frac{y}{m}\right)^{i+1-f}}\right]\frac{1}{V_B} \end{aligned} . \tag{7.2.2}$$

Note that the first derivative depends on the bond valuation compounding convention, but the bond's market value does not depend on compounding convention. Standard convexity measures the curvature of the price-yield relationship (again assuming discretely compounded yield to maturity)

$$\begin{aligned} Convexity_{B,y} &\equiv \frac{1}{V_B}\frac{d^2V_{B,y}}{dy^2} \\ &= \frac{1}{m^2}\left[\sum_{i=1}^{N}\frac{(i-f)(i+1-f)CF_i}{\left(1+\frac{y}{m}\right)^{i+2-f}}\right]\frac{1}{V_B} \end{aligned}. \tag{7.2.3}$$

With Macaulay and modified duration, the change in the estimated price does not incorporate changes in cash flow. Effective duration is a measure of cash flow adjusted volatility. Specifically,

$$EffDur = \frac{V_{B-} - V_{B+}}{2V_B S}, \tag{7.2.4}$$

where V_B denotes the current price, V_{B-} denotes the price when rates fall by some shift in the term structure denoted by S, and V_{B+} denotes the price when rates rise by S. Note that if $\Delta V_B = (V_{B-}) - (V_{B+})$, then $\Delta y = 2S$. Assuming no changes in cash flow, then these two expressions for duration are identical. Effective duration is different when the consequences of the rate change result in a different valuation due to changes in cash flow, such as a call feature on a bond. We do not distinguish between yield compounding methods as we do not pursue cash flow adjusted static risk measures beyond an introduction.

Effective convexity corrects for this limitation of standard convexity by using a valuation model to compute bond prices that incorporate the consequences of interest rate changes on cash flows. At this point, you are not responsible for this valuation model (that is, the model that estimates V_{B-} and V_{B+}).

$$EffConv = \frac{V_{B-} + V_{B+} - 2V_B}{V_B S^2}. \tag{7.2.5}$$

The second derivative measures the change in the first derivative as we change the yield to maturity, hence $\Delta^2 V_B = \Delta V_{B-} - \Delta V_{B+} = (V_{B-} - V_B) - (V_B - V_{B+}) = V_{B-} + V_{B+} - 2V_B$. Also, $\Delta y^2 = S^2$. Hence, standard convexity and effective convexity are the same when cash flow considerations are ignored.

We first review the history of static risk management related to bonds.

Basic bond risk management

Recall with continuously compounded yield (r), we can express the bond value ($V_{B,r}$) as (where $t_i = i - f$)

$$\begin{aligned} V_{B,r} &= \sum_{i=1}^{N}\left(\frac{CR}{m}\right)Par\left(e^{-r\tau_i}\right) + Par\left(e^{-r\tau_N}\right) \\ &= \sum_{i=1}^{N} CF_i e^{-r\tau_i} \end{aligned}. \tag{7.2.6}$$

Based on a continuously compounded yield, Macaulay duration can be expressed simply as

$$\begin{aligned} MacDur_{B,r} &\equiv \sum_{i=1}^{N}\frac{CF_i e^{-r\tau_i}}{V_B}\tau_i \\ &= \sum_{i=1}^{N} w_i \tau_i \end{aligned}, \tag{7.2.7}$$

where r denotes the appropriate continuously compounded discount interest rate, V_B denotes the market value of the bond hence compounding method for the discount rate does not matter, CF_i denotes the appropriate cash flow at time i, and τ_i denotes the time until cash flow i. Note that

$$w_i = \frac{CF_i e^{-r\tau_i}}{V_B}, \tag{7.2.8}$$

is the proportion (or weight) that the present value of cash flow *i* contributes to the value of the bond. Macaulay assumed a constant yield for all cash flows.

Hicks (1939) identified Macaulay duration as the elasticity of the bond value with respect to the discount factor. Hicks' elasticity measure "... is the *average length of time for which the various payments are deferred from the present, when the time of deferment are weighted by the discounted values of the payments*."[2] Let $\delta = e^{-r}$, a periodic discount factor. With continuously compounded yield, Hicks' elasticity measure can be expressed as

$$\begin{aligned} HE_{B,r} &\equiv \frac{dV_{B,r}/V_B}{d\delta/\delta} = \frac{dV_{B,r}}{d\delta}\frac{\delta}{V_B} \\ &= -\left(\sum_{i=1}^{N} \tau_i CF_i \delta^{\tau_i - 1}\right)\frac{\delta}{V_B} = -\left(\sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B}\tau_i\right), \\ &= -\sum_{i=1}^{N} w_i \tau_i = -ModDur_{B,r} \end{aligned} \tag{7.2.9}$$

and Hicks assigned the term "average period" to this measure. In our context of bonds, Hicks' insight is that if the average period of a bond is greater (less) than some benchmark, then a decline (rise) in interest rates will result in the change in the bond being greater (less) than the change in the benchmark. Thus, Hicks provides the tools for future asset-liability management.

Samuelson (1945) also observed similar insights as Hicks in the following theorem that provides "... conditions under which interest rates help or hurt a given person or institution: *Increased interest rates will help any organization whose (weighted) average time period of disbursements is greater than the average time period of its receipts*."[3] According to Poitras (2006), Samuelson's work was "... an extension of Hicks (1939) and an anticipation of Redington (1952)." (p. 14)

Immunization

Recall Redington (1952) introduced the concept of immunization, a particularly useful idea in debt portfolio management. Thus, using our notation, Redington's Taylor series approach approximating the new bond price with continuously compounded discount rates ($\hat{V}_{B,r}$) for a given change in the continuously compounded yield (r) is[4]

$$\begin{aligned} \hat{V}_{B,r} &= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}\left(\hat{r} - r\right)^i = \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}\Delta r^i \\ &\cong V_{B,r} + \left(\frac{\partial V_{B,r}}{\partial r}\right)\Delta r + \frac{1}{2}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right)\Delta r^2 \end{aligned}, \tag{7.2.10}$$

where the last equation an approximation based on the first two derivatives. As a measure of volatility, modified duration can be expressed as

$$ModDur_{B,r} \equiv -\frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right) = -\frac{\partial V_{B,r}/V_B}{\partial r}. \tag{7.2.11}$$

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[2]See Hicks (1939), p. 186. Italics in original.
[3]See Samuelson (1945), p. 11. Note particularly the footnote where he explains the theorem mathematically.
[4]Redington's approach was focused on the value of assets less the value of liabilities. For consistency here, we present just a single bond as an asset. Also, this expression is assuming discretely compounded bond returns.

Thus, modified duration addresses the percentage change in the bond's value with respect to the change in the continuously compounded yield. Standard convexity can be expressed as

$$Convexity_B \equiv \frac{1}{V_B}\left(\frac{\partial^2 V_B}{\partial r^2}\right). \tag{7.2.12}$$

Thus, the discretely compounded holding period return on the bond can be approximated as

$$\begin{aligned} R_{B,dc} &\equiv \frac{\hat{V}_{B,r}-V_B}{V_B} = \frac{1}{V_B}\sum_{i=1}^{\infty}\frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}\left(\hat{r}-r\right)^i = \frac{1}{V_B}\sum_{i=1}^{\infty}\frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}\Delta r^i \\ &\cong \frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right)\Delta r + \frac{1}{2}\frac{1}{V_B}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right)\Delta r^2 \qquad , \\ &= -ModDur_{B,r}\Delta r + \frac{1}{2}Convexity_{B,r}\Delta r^2 \end{aligned} \tag{7.2.13}$$

where[5]

$$\begin{aligned} ModDur_{B,r} &\equiv -\frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right) \\ &= \sum_{i=1}^{N}\frac{CF_i e^{-r\tau_i}}{V_B}\tau_i \text{ and} \\ &= \sum_{i=1}^{N} w_i \tau_i \end{aligned} \tag{7.2.14}$$

$$\begin{aligned} Convexity_{B,r} &\equiv \frac{1}{V_B}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right) \\ &= \sum_{i=1}^{N}\frac{CF_i e^{-r\tau_i}}{V_B}\tau_i^2 . \\ &= \sum_{i=1}^{N} w_i \tau_i^2 \end{aligned} \tag{7.2.15}$$

Both bond valuation methodology (continuously or discretely compounded discounting) and bond holding period returns (continuous or discrete compounding) influence bond risk measurement. Assuming continuously compounded bond holding period returns, we have[6]

$$\begin{aligned} \ln\left(\hat{V}_{B,r}\right) &= \sum_{i=0}^{\infty}\frac{1}{i!}\frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\left(\hat{r}-r\right)^i = \sum_{i=0}^{\infty}\frac{1}{i!}\frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\Delta r^i \qquad , \\ &\cong \ln\left(V_{B,r}\right) + \frac{\partial \ln\left(V_{B,r}\right)}{\partial r}\Delta r + \frac{1}{2}\frac{\partial \ln^2\left(V_{B,r}\right)}{\partial r^2}\Delta r^2 \end{aligned} \tag{7.2.16}$$

and thus[7]

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[5]This perspective of duration became known as modified duration as opposed to Macaulay duration. If continuous compounding for computing bond value is used, then these two perspectives of duration are identical. This equality is not true with discrete compounding for computing bond values.
[6]The second derivative produces an additional term with continuous compounding. See Barber (1995).
[7]The second derivative is the result of applying the product rule and the chain rule in calculus.

$$
\begin{aligned}
R_{B,cc} &\equiv \ln\left(\frac{\hat{V}_{B,r}}{V_B}\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\left(\hat{r}-r\right)^i = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i} \Delta r^i \\
&\cong \frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right)\Delta r + \frac{1}{2}\left[\frac{1}{V_B}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right) + \left(\frac{-1}{V_B^2}\right)\left(\frac{\partial V_{B,r}}{\partial r}\right)^2\right]\Delta r^2 \\
&= -ModDur_{B,r}\Delta r + \frac{1}{2}\left(Convexity_{B,r} - ModDur_{B,r}^2\right)\Delta r^2
\end{aligned}
\quad . \tag{7.2.17}
$$

Thus, duration can be viewed as a present value weighted average of time (Macaulay's perspective) or a measure of the sensitivity to interest rate changes (Redington's perspective). Redington illustrates the important role of convexity in asset-liability management. Though somewhat suggestive from the Taylor series, Redington makes no effort to examine higher order effects beyond convexity.

Bond risk management empirical evidence

Barber (1995) demonstrates that approximating continuously compounded rates of return using duration only or duration and convexity is much more accurate than the more traditional discretely compounded rates of return. Durand (1957) extended duration to growth stocks to identify particularly long duration assets. Fisher (1966) applies duration to facilitate computation of realized rates of return. Fisher also illustrates duration with continuously compounded interest rates, noting that the analysis is much simpler, which we also exploit.

Fisher and Weil (1971) provide a thorough empirical analysis of bond immunization strategies based on data from 1925 through 1968. They conclude that the reductions in a bond portfolio risk measure "... are so dramatic that we conclude that a properly chosen portfolio of long-term bonds is essentially riskless." (p. 423) Granito (1984) provides a thorough examination of immunization strategies, including contingent immunization and dedicated bond portfolios.

Starting in the early 1980s several extensions to duration and convexity began to appear suggesting duration and convexity were ineffective bond risk measures.[8] Recognizing that the term structure of interest rates does not shift parallel, Fong and Vasicek (1983, 1984) as well as Fong and Fabozzi (1985) proposed the M-square measure. M-square captures the risk of a bond portfolio to non-parallel term structure shifts and is expressed as

$$
M_B^2 \equiv \sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B}\left(\tau_i - H\right)^2 = \sum_{i=1}^{N} w_i\left(\tau_i - H\right)^2, \tag{7.2.18}
$$

where H denotes the target horizon.

Nawalkha and Chambers (1996) offer the M-absolute measure that has the advantage of not also requiring duration. They find that the M-absolute measure results in significantly lower risk than simple duration, suggesting that slope, curvature, and other issues may play an important role.

$$
M_B^A \equiv \sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B}\left|\tau_i - H\right|^2 = \sum_{i=1}^{N} w_i\left|\tau_i - H\right|^2. \tag{7.2.19}
$$

Recognizing that real-world shifts in term structure may be combinations of level changes, slope changes, and curvature changes, several authors introduced more complex models. See, for example, Chambers, Carleton, and McEnally (1988), Prisman and Shores (1988), and Bierwag, Kaufman, and Latta (1988).

In the quest to further improve bond risk measures, duration vector measures have been introduced to immunize against a variety of non-parallel changes in the term structure of interest rates. The duration vector measure is expressed as

$$
D_B\left(m\right) \equiv \sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B} g\left(\tau_i\right)^m = \sum_{i=1}^{N} w_i g\left(\tau_i\right)^m, \tag{7.2.20}
$$

where the duration measure is linear in $g\left(\tau_i\right)^m$. Each duration vector measure is set equal to the horizon raised to the mth power. This generalized approach permits more weight on specific locations on the term

[8]See Nawalkha, Soto, and Beliaeva (2005) for a thorough review.

structure based on known empirical term structure properties. (See, for example, Chambers, Carleton, and McEnally (1988).)

Finally, Ho (1992) suggests key rate duration measures where key spot rates are selected, and bond risk measures are linear interpolations of changes in these key spot rates. Nawalkha, Soto and Beliaeva (2005) argue that a weakness of this approach is requires selecting an arbitrary number of key rates and hence larger hedged portfolios with greater rebalancing expenses.

Compounding plays an important role in bond static risk measurement. We now carefully highlight this role.

Tradition duration and convexity review

We now review several selected concepts related to the traditional bond static risk measures.

Duration as measure of risk

Suppose we had two 20-year bonds, 9%- and 11%-coupon bond (semi-annual coupon payments). If we plug different yields to maturity into the bond price equation, we construct the Table 7.2.1.

Table 7.2.1. Bond Prices and Percentage Changes for Two Bonds

Yield to Maturity	9% Coupon Bond Price	11% Coupon Bond Price	9% Coupon % Change in Price	11% Coupon %Change in Price
7.0%	$121.36	$142.71	***10.43%****	***10.04%***
8.0%	$109.90	$129.69	***9.90%***	***9.53%***
9.0%	$100.00	$118.40	9.38%	9.05%
10.0%	$91.42	$108.58	***8.89%***	8.58%
11.0%	$83.95	$100.00	***8.42%***	8.14%
12.0%	$77.43	$92.48		

* 10.43% = (121.36 - 109.90)/109.90

Notice that the percentage change in the 11% coupon bond price when rates move from 7% to 8% compared with 8% to 9% is 51 basis points (10.04% - 9.53%) whereas the percentage change in the 9% coupon bond price for the same rate moves is 53 basis points. Therefore, the 9% coupon bond is viewed as being riskier. Comparing the percentage change in the 9% coupon bond for different yield to maturities (see 10% to 11% compared with 11% to 12% having a 47 basis point difference (8.89% - 8.42%), we see that lower yield to maturities are riskier. From this table, we see that lower coupon bonds are riskier than higher coupon bonds with the same maturity (because with higher coupon, you get your money back sooner; its as if the maturity is shorter). We would like to have a single number indicative of bond risk that takes both maturity and coupon into account. Duration, or more specifically, modified duration, is that number.

We also see from Table 7.2.1 that the lower the yield to maturity, the higher the percentage change in the bond price (for a 1% change in yield to maturity). The lower the yield to maturity, the higher the present values of more distant coupon payments, and therefore the duration is longer. The more volatile are interest rates, the more bond prices move. The higher the duration, the price volatility will have a greater effect.

Duration can be used to estimate a bond portfolio rate of return

$$\frac{\Delta V_B}{V_B} \cong -D\frac{\Delta y}{\left(1+\dfrac{y}{m}\right)} = -ModDur_B \Delta y. \tag{7.2.21}$$

There is a one-to-one relationship between bond price changes and duration, for a given change in the required yield. Finally, recall that duration is a weighted-average of cash flows. Generally, the longer the maturity bond the greater is the duration. Hence, modified duration can be used as a risk measure.

Calculating duration and convexity

Table 7.2.2 illustrates the related calculations for a four-year, annual 10% coupon bond, yield to maturity of 10%, and par value of $100. Thus, m = 1, V_B = $100, and $y = C = 0.10$.

Table 7.2.2. Manual Calculation of Duration and Convexity

1	2	3	4	5	6
Maturity	**Cash Flow**	**PV(CF)**	**w(t)**	**t*w(t)**	**t*(t+1)*w(t)**
1	10	9.091	0.091	0.091	0.182
2	10	8.264	0.083	0.165	0.496
3	10	7.513	0.075	0.225	0.902
4	110	75.131	0.751	3.005	15.026
	Sum	100.000	1.000	3.487	16.606
				Convexity	***13.724***
				Duration	***3.487***
				Modified Duration	***3.170***

Note: Convexity = 16.606/(1^2(1+0.1)^2) = 13.724, Duration = sum(t*w(t)) = 3.487, and Modified Duration = 3.487/(1 + 0.1) = 3.170.

Now suppose we have a 12-year, 10% coupon, 10% yield to maturity, and $100 Par bond. Macaulay duration is 7.495 years, modified duration is 6.814, and convexity is 66.754. Assuming the yield to maturity rises 100 basis points (1%) we can compute the following:

Using just duration:

$$\text{ModDur}_B = \frac{D}{(1+y)} = \frac{7.495}{1.10} = 6.814 \text{ and thus} \tag{7.2.22}$$

$$\begin{aligned}\text{Percentage price change} &= -\text{ModDur}_B\left(\Delta y\right) \\ &= -6.814(0.01) = -6.814\%\end{aligned}. \tag{7.2.23}$$

Using the convexity:

$$\begin{aligned}\text{Percentage price change due to convexity} &= \frac{1}{2}\left(\text{Convexity}_B\right)(\Delta y)^2 \\ &= 0.5(66.754)(0.01)^2 = 0.334\%\end{aligned}. \tag{7.2.24}$$

Putting duration and convexity together: Using duration alone gives a first approximation to percentage price change, based on the straight line that is tangent to the actual, convex price-yield curve. Adding the percentage price change due to convexity produces a better approximation.

$$\begin{aligned}\text{\% Price Change due to duration and convexity} &= -\text{ModDur}_B\left(\Delta y\right) + \frac{1}{2}\text{Convexity}_B(\Delta y)^2 \\ &= -6.814\% + 0.334\% = -6.48\%\end{aligned}. \tag{7.2.25}$$

The actual percentage price change is –6.49%. Hence, modified duration over-estimates the percentage change in price (duration-based estimate is –6.814%); however the convexity adjustment over-corrects the duration-based error for a total error of only +1 basis point.

One other example is useful. Assuming the yield to maturity falls 100 basis points, we can compute the following (yield change is = –0.01):

$$\begin{aligned}\text{\%Price Change due to duration and convexity} &= -6.814(-0.01) + 0.5(66.754)(-0.01)^2 \\ &= 6.814\% + 0.334\% = 7.15\%\end{aligned}. \tag{7.2.26}$$

The actual percentage price change is 7.16%. Hence, duration under-estimates the percentage change in price (duration estimate is +6.814%), however the convexity adjustment under-corrects the duration-based error for a total error of only –1 basis point.

In both cases, the convexity correction is additive. A bond with greater convexity than the bond in this example would not experience as much price decrease when rates rise and would experience more price increase when rates fall. We might say ‘more convexity is better’ or the ‘benter the better.’

Modified duration and convexity analysis is concerned, for the most part, with bond price volatility. By price volatility, in this context, we don't mean a standard deviation, which is the type of volatility discussed in connection with options, but price changes (sometimes percentages, sometimes dollars). A bond that tends to experience larger percentage price changes, both positive and negative, would be considered a more volatile bond, and thus a riskier bond.

Modified duration and convexity are useful bond management tools. Modified duration is a first approximation of price changes and therefore a measure of volatility. Convexity can be used to adjust the duration-based price approximations as well as be used to enhance bond portfolio performance. Managers will be seeking higher convexities, assuming all other parameters are held constant.

Immunization illustrated

For longer holding periods, the disadvantage of price risk is offset by the advantage of reinvestment risk. That is, if interest rates go up, the bond loses its current value, but the future coupon payments can be invested at a higher interest rate. Managing the tradeoff between price risk and reinvestment risk is the motivation behind the process known as immunization.

Immunization refers to designing a bond portfolio so that its future performance is insensitive to future changes in interest rates. For example, if we had monies to invest for 3.5 years and wanted to "immunize" the investment, one method would be to simply invest in zero-coupon bonds that mature in 3.5 years. Assuming no taxes and no default risk, then we know exactly when and what our payoffs will be in the future. The holding period rate of return will be the yield to maturity on the zero coupon bonds.

If we purchase longer maturity zero-coupon bonds, then we should be concerned about a rise in interest rates causing a decline in the bond price when it is sold in 3.5 years. This is known as price risk and notice that risk is an *increase* in interest rates. If we purchase shorter maturity zero coupon bonds, then we should be concerned about a fall in interest rates causing a decline in the reinvestment rate earned when these zero coupon bonds mature and we need reinvest the proceeds at a lower interest rate. This is known as reinvestment risk and notice that risk is a *decrease* in interest rate. If we purchase 3.5 year bonds, we have neither price nor reinvestment rate risk.

Now consider a four-year, 10% annual coupon-paying bond with a yield to maturity of 8.5%. The current bond price is $104.913, and the duration is 3.5 years. Also suppose that there were two other candidate four-year bonds to purchase, 20% coupon bond priced at $137.669 with a duration of 3.24 years and a zero coupon bond priced at $72.157 with a duration of 4.0 years. The bond that immunizes the portfolio assuming the desired holding period is 3.5 years is the 10% coupon bond. The 20% coupon bond has net reinvestment rate risk and hence will suffer if rates fall, and the zero-coupon bond has net price risk and hence will suffer if rates rise.

Table 7.2.3 illustrates the annualized holding period returns from these three potential investments assuming a variety of future interest rates.

Table 7.2.3. Illustration of Holding Period Returns and Duration

Interest Rate	**Zero Coupon Bond**	**10% Coupon Bond**	**20% Coupon Bond**
3%	9.31%	8.54%	8.13%
5%	9.01%	8.52%	8.25%
7%	8.72%	8.50%	8.39%
8.5%	8.50%	8.50%	8.50%
10%	8.29%	8.50%	8.62%
12%	8.01%	8.51%	8.78%
14%	7.74%	8.54%	8.95%

Notice that the immunized 10% bonds in the above table exhibit the slight benefit of positive convexity for large moves in interest rates.

The key to understanding the relationship with convexity is to remember that convexity increases with increases in duration. Hence, convexity is higher for lower coupon bonds, lower yield to maturity, and longer maturity bonds.

Effective duration and convexity revisited
For callable bonds, as rates *fall* the bond will behave more like a bond maturing on the first call date (in Table 7.2.4, 0.5 years to first call date) because issuers will desire to exercise their call option and issue bonds at a lower coupon rate. We see in Table 7.2.4, Bond X's effective duration falls from 4.2 at +250 to 0.5 at –250. Hence this 5-year callable bond's effective duration dramatically falls with *falling* interest rates. Because modified duration does not adjust for cash flow effects, a callable bond's modified duration does not change that much with changes in interest rates (see Bond M in Table 7.2.5). We also see in Table 7.2.4, Bond X's effective convexity falls from 21.0 at +250 to 0.5 at –250. Hence this 5-year callable bond's effective convexity dramatically falls with *falling* interest rates. Because standard convexity does not adjust for cash flow effects, a callable bond's standard convexity does not change that much with changes in interest rates (see Bond M in Table 7.2.5).

For putable bonds, as rates *rise* the bond will behave more like a bond maturing on the first put date (in Table 7.2.4, 0.5 years to first put date) because investors will desire to exercise their put option and reinvest proceeds at a higher coupon rate. We see in Table 7.2.4, Bond Y's effective duration falls from 4.4 at –250 to 0.5 at +250. Hence this 5-year putable bond's effective duration dramatically falls with *rising* interest rates. Because modified duration does not adjust for cash flow effects, a putable bond's modified duration does not change that much with changes in interest rates (see Bond O in Table 7.2.5). We also see in Table 7.2.4, Bond Y's effective convexity falls from 22.5 at –250 to 0.5 at +250. Hence this 5-year putable bond's effective convexity dramatically falls with *rising* interest rates. Because standard convexity does not adjust for cash flow effects, a putable bond's standard convexity does not change that much with changes in interest rates (see Bond O in Table 7.2.5).

Table 7.2.4. Effective Duration and Effective Convexity

	Bond X Callable		**Bond Y Putable**		**Bond Z Straight**	
Shift	**EffDur***	**EffCon***	**EffDur**	**EffCon**	**EffDur**	**EffCon**
-250	0.5	0.5	4.4	22.5	4.4	22.5
+250	4.2	21.0	0.5	0.5	4.2	21.0

Note: *EffDur denotes effective duration and EffCon denotes effective convexity

Table 7.2.5. Modified Duration and Standard Convexity

	Bond M Callable		**Bond N Straight**		**Bond O Putable**	
Shift	**MD***	**SC***	**MD**	**SC**	**MD**	**SC**
-250	4.3	21.9	4.4	22.5	4.4	22.5
+250	4.2	21.0	4.2	21.0	4.3	21.7

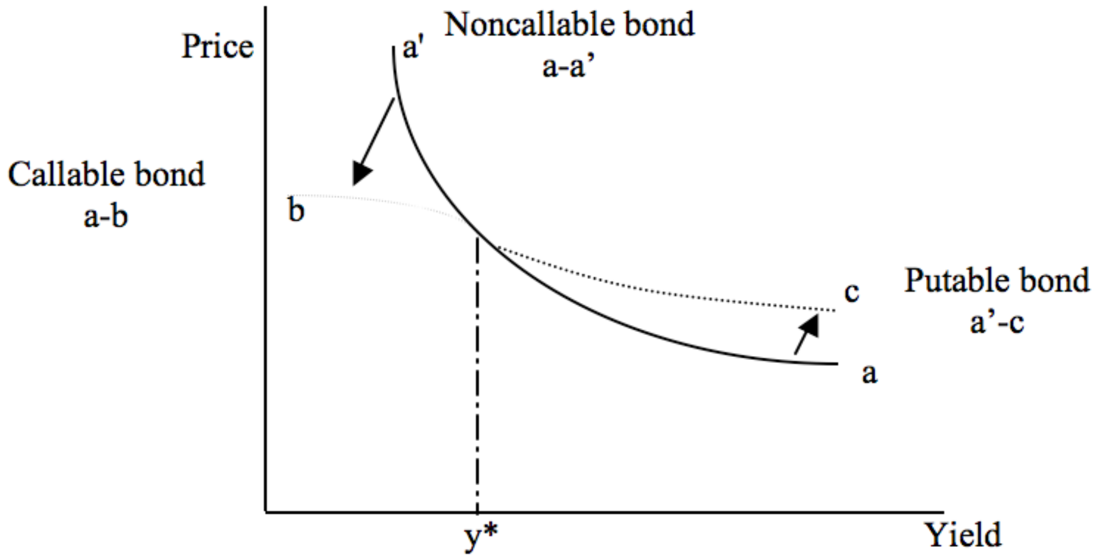
Note: *MD denotes modified duration and SC denotes standard convexity

The call feature allows the issuer to make the bonds mature on the call date rather than the maturity date. Hence the maturity of the bond is uncertain. Effective duration and effective convexity capture the valuation aspects of this uncertainty. Both effective duration and effective convexity fall dramatically when interest rates fall (the bonds are more likely to be called).

The put feature allows the investor to make the bonds mature on the put date rather than the maturity date. Hence the maturity of the bond is uncertain. Effective duration and effective convexity capture the valuation aspects of this uncertainty. Both effective duration and effective convexity fall dramatically when interest rates rise (the bonds are more likely to be put back to the issuer).

Figure 7.2.1 illustrates these effects. The call feature introduces negative convexity in the price-yield relationship when rates fall. The put feature lowers the convexity of the price-yield relationship when rates rise.

Figure 7.2.1. Price-Yield Relationship of Callable, Putable, and Straight Bonds



One way to keep these relationships straight is to remember that the call feature introduces essentially two maturities, short and long. The issuer will choose the long maturity if rates rise and short maturity if rates fall. Hence the effective duration and effective convexity are lowered when rates fall (bonds will be called). For putable bonds, the investor will choose the long maturity if rates fall and the short maturity if rates rise. Hence the effective duration and effective convexity are lowered when rates rise (bonds will be put back to the issuer).

- For *callable* bonds, both the effective duration and effective convexity fall as rates *fall*. Remember the bond *issuer* will decide to shorten the maturity of the bonds.
- For *putable* bonds, both the effective duration and effective convexity fall as rates *rise*. Remember the bond *investor* will decide to shorten the maturity of the bonds.

Asset liability management

Managing interest rate risk is difficult for a variety of reasons. First, interest rates are derived numbers based on market prices. The interest rate reported depends upon the calculation methodology deployed. Second, interest rates are not 'storable,' hence carry arbitrage is very difficult to execute. Thus, interest rates reflect participants risk preferences and expectations of future interest rates. Third, at least theoretically, there are an infinite number of interest rates due to the infinite number of maturities, not to mention the large number of different rates observed (for example, LIBOR, Treasury bills, Treasury notes, Treasury bonds, commercial paper, AAA corporate bonds, AA corporate bonds, and so forth).

Advanced bond static risk measures: Applying the LSC model[9]

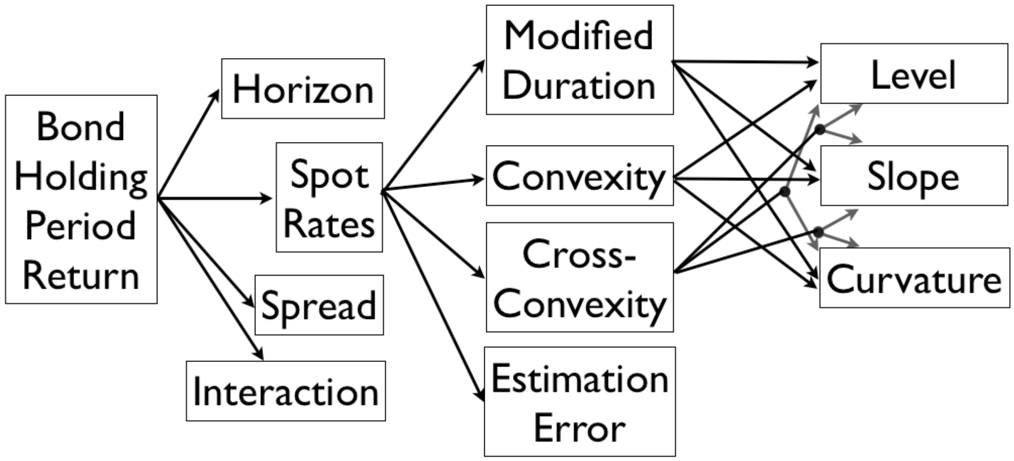
In this section as well as the next several sections, we seek to decompose bond value and bond holding period returns into different components. The ability to attribute investment performance to different factors is very valuable for managing various enterprises from companies to families to municipalities.

The bond HPR is decomposed into four major components, the non-random horizon component, the spread component, the base rate component, and an interaction component. The horizon component captures the bond HPR attributable to the mere passage of time over the holding period horizon based solely on the current information. The base or spot rate component captures movement in the LSC-fitted base rate curve. The spread component captures the bond HPR attributable to any change in the spread over the fitted spot

[9]The next several sections are based on Brooks and Upton (2017) and Brooks (2017).

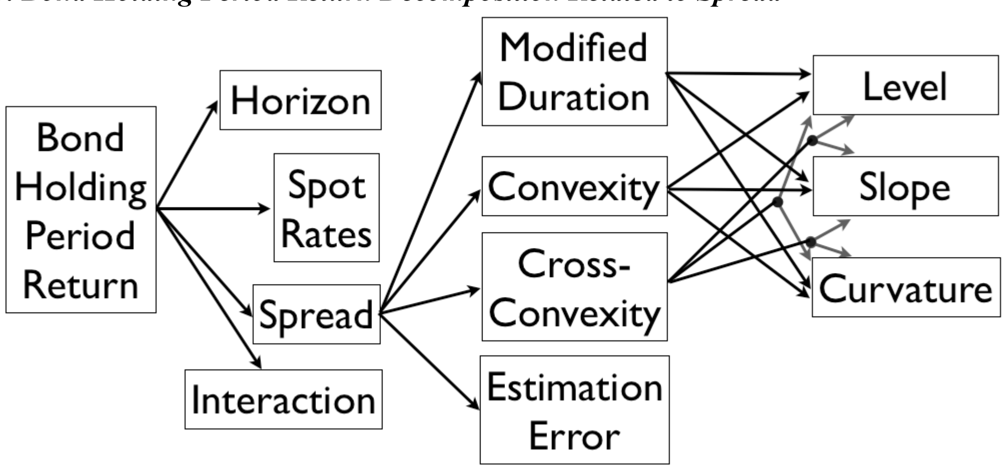
rate curve. The interaction component contains the interaction components. Figure 7.2.2 illustrates the general structure of the bond HPR decomposition.

Figure 7.2.2. Bond Holding Period Return Decomposition Related to Spot Rates



The base rate component of bond returns can be further decomposed into three components attributable to factor duration, convexity, and cross-convexity based on a Taylor series approximation explained in detail below. Thus, the approximation contains an error component. Each of these three rate components can be further decomposed into three subcomponents tied to movement in level, movement in slope, and movement in curvature. As previously covered, the LSC model is general enough to have multiple curvature components. Module 7.3 we explore the LSC model applied to the spread component and a similar decomposition applied to the resultant spread curve as illustrated in Figure 7.2.3.

Figure 7.2.3. Bond Holding Period Return Decomposition Related to Spread



For the purposes needed below, we define the current bond value as

$$V = \sum_{i=1}^{N_t} CF_{i,t} DF_{i,t} = \sum_{i=1}^{N_t} CF_{i,t} e^{-y_{i,t}\tau_{i,t}}, \tag{7.2.27}$$

where $CF_{i,t}$ denotes the promised cash flow at time t for the i^{th} cash flow and $DF_{i,t}$ denotes the appropriate discount factor at time t for the i^{th} cash flow. The yield to maturity at time t for the i^{th} cash flow is denoted $y_{i,t}$ and time to the i^{th} cash flow measured at time t is denoted $t_{i,t}$. Clearly,

$$DF_{i,t} \equiv e^{-y_{i,t}\tau_{i,t}}. \tag{7.2.28}$$

Note we use y here to contrast with the base rate denoted r.

Following Module 4.2 closely, we first decompose the bond value into two major components, the base rate function, and the spread function. With the approach taken here, we can easily layer several spread functions rather than just one. For our purposes, we assume only one spread, for example, the spread of BB-yields over UST CMT.

Again, we denote the spread over the fitted LSC spot rate as $sp_{i,t}$. Thus, we can decompose the yield to maturity, $y_{i,t}$, into LSC spot rates, $r_{i,t}^{LSC}$, the LSC spreads $sp_{i,t}^{LSC}$, and an error term $\varepsilon_{i,t}$. Remember that with this approach any error in estimating both LSC models is completely captured in the error term. Therefore, the current bond value can be expressed as

$$\begin{aligned} V_t &= \sum_{i=1}^{N_t} CF_{i,t} DF_{i,t} = \sum_{i=1}^{N_t} CF_{i,t} e^{-y_{i,t}\tau_{i,t}} = \sum_{i=1}^{N_t} CF_{i,t} e^{-\left(r_{i,t}^{LSC}+sp_{i,t}^{LSC}+\varepsilon_{i,t}\right)\tau_i} \\ &= \sum_{i=1}^{N_t} CF_{i,t} e^{-\left(\sum_{j=0}^{N^r} x_{i,j} f_j^r + \sum_{j=0}^{N^{sp}} x_{i,j} f_j^{sp} + \varepsilon_{i,t}\right)\tau_i} \end{aligned} . \qquad (7.2.29)$$

As calendar time will be an important component, we add the subscript t to the LSC model notation.

Repeating from Module 4.2, we apply the LSC model to bond yields, we have

$$y_{i,t} = \sum_{j=0}^{N^r} x_{i,j,t} f_{j,t}^r + \sum_{j=0}^{N^{sp}} x_{i,j,t} f_{j,t}^{sp} + \varepsilon_{i,t}. \qquad (7.2.30)$$

We define the portion of the yield to maturity attributable to the base rate as

$$r_{i,t}^{LSC} \equiv \sum_{j=0}^{N^r} x_{i,j,t} f_{j,t}^r, \qquad (7.2.31)$$

and the portion of the yield to maturity attributable to the spread as

$$sp_{i,t}^{LSC} \equiv \sum_{j=0}^{N^{sp}} x_{i,j,t} f_{j,t}^{sp}. \qquad (7.2.32)$$

where the x variables are as defined before.

Thus, the approximate value of the bond, based on both LSC models (base rate and spread), is expressed as

$$V_t \cong V_t^{LSC} = \sum_{i=1}^{N_t} CF_{i,t} DF_{i,t}^{LSC} = \sum_{i=1}^{N_t} CF_{i,t} e^{-\left(r_{i,t}^{LSC}+sp_{i,t}^{LSC}\right)\tau_i}. \qquad (7.2.33)$$

With enough LSC factors, the residual error will be negligible. Thus, we define the discount factor based solely on the LSC model applied to the base rates and the LSC model applied to spreads as

$$DF_{i,t}^{LSC} \equiv e^{-\left(r_{i,t}^{LSC}+sp_{i,t}^{LSC}\right)\tau_i}. \qquad (7.2.34)$$

With this setup, we define several bond valuation calculations depending on what parameters are allowed to change over the measurement horizon, Δ. The bond value at the horizon based on the two LSC model parameters at time $t + \Delta$ is

$$\tilde{V}_{t+\Delta}^{LSC} \equiv \sum_{i=1}^{N_t} CF_{i,t+\Delta} D\tilde{F}_{i,t+\Delta}^{LSC} = \sum_{i=0}^{N_{t+\Delta}} CF_{i,t+\Delta} e^{-\left(\tilde{r}_{i,t+\Delta}^{LSC}+s\tilde{p}_{i,t+\Delta}^{LSC}\right)(\tau_i-\Delta)}. \text{ (LSC Value at } t+\Delta) \qquad (7.2.35)$$

where

$$D\tilde{F}_{i,t+\Delta}^{LSC} \equiv e^{-\left(\tilde{r}_{i,t+\Delta}^{LSC}+s\tilde{p}_{i,t+\Delta}^{LSC}\right)(\tau_i-\Delta)}. \qquad (7.2.36)$$

The bond value at the horizon based on the two LSC model parameters at time t is

$$V_{t+\Delta}^{LSC} \equiv \sum_{i=1}^{N_t} CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC} = \sum_{i=1}^{N_{t+\Delta}} CF_{i,t+\Delta} e^{-\left(r_{i,t}^{LSC}+sp_{i,t}^{LSC}\right)(\tau_i-\Delta)}. \text{ (Value at } t+\Delta \text{ with Initial Values)} \qquad (7.2.37)$$

where

$$DF_{i,t+\Delta}^{LSC} \equiv e^{-\left(r_{i,t}^{LSC}+sp_{i,t}^{LSC}\right)(\tau_i-\Delta)}. \tag{7.2.38}$$

The key distinction is the tilde that indicates the parameters are not know at t but rather in the future at $t + \Delta$. Thus, $V_{t+\Delta}^{LSC}$ expresses the bond value in the future assuming no change in the market parameters whereas $\tilde{V}_{t+\Delta}^{LSC}$ expresses the bond value in the future based on LSC model parameters observe in the future. We now introduce two additional valuations based on whether the base curve or the spread curve is computed from the known values at t or the unknown values at $t + \Delta$. The bond value with just the base curve values at $t + \Delta$ is

$$\tilde{V}_{t+\Delta}^{r} \equiv \sum_{i=1}^{N_{t+\Delta}} CF_{i,t+\Delta} D\tilde{F}_{i,t+\Delta}^{r} = \sum_{i=0}^{N_{t+\Delta}} CF_{i,t+\Delta} e^{-\left(\tilde{r}_{i,t+\Delta}^{LSC}+sp_{i,t+\Delta}^{LSC}\right)(\tau_i-\Delta)}. \text{ (LSC Base Curve at } t+\Delta) \tag{7.2.39}$$

where

$$D\tilde{F}_{i,t+\Delta}^{r} \equiv e^{-\left(\tilde{r}_{i,t+\Delta}^{LSC}+sp_{i,t+\Delta}^{LSC}\right)(\tau_i-\Delta)}. \tag{7.2.40}$$

The bond value with just the spread curve values at $t + \Delta$ is

$$\tilde{V}_{t+\Delta}^{sp} \equiv \sum_{i=1}^{N_{t+\Delta}} CF_{i,t+\Delta} D\tilde{F}_{i,t+\Delta}^{sp} = \sum_{i=0}^{N_{t+\Delta}} CF_{i} e^{-\left(r_{i,t+\Delta}^{LSC}+\tilde{sp}_{i,t+\Delta}^{LSC}\right)(\tau_i-\Delta)}. \text{ (LSC Spread Curve at } t+\Delta) \tag{7.2.41}$$

where

$$D\tilde{F}_{i,t+\Delta}^{sp} \equiv e^{-\left(r_{i,t+\Delta}^{LSC}+\tilde{sp}_{i,t+\Delta}^{LSC}\right)(\tau_i-\Delta)}. \tag{7.2.42}$$

Therefore we have six different valuations for the same bond:

- V_t – the observed market value of the bond at time t.
- V_t^{LSC} – the fitted model value of the bond at time t, both base rate and spreads.
- $\tilde{V}_{t+\Delta}^{LSC}$ – the fitted model value of the bond at time $t + \Delta$, both base rate and spreads.
- $V_{t+\Delta}^{LSC}$ – the fitted model value of the bond at time $t + \Delta$, but base rate and spreads parameters fit at t.
- $\tilde{V}_{t+\Delta}^{r}$ – the fitted model value of the bond at time $t + \Delta$, but base rate at $t + \Delta$ and spreads parameters fit at t.
- $\tilde{V}_{t+\Delta}^{sp}$ – the fitted model value of the bond at time $t + \Delta$, but base rate at t and spreads parameters fit at $t + \Delta$.

The LSC model is run at two points in time, t and $t + \Delta$, for two datasets, the base rate and spreads. The LSC models fit at time t are used at both time t and $t + \Delta$. The base rates are computed in three ways,

$$r_{i,t}^{LSC} \equiv \sum_{j=0}^{N^r} x_{i,j,t} f_{j,t}^{r}, \text{ (Fit at time } t\text{, analyzed at time } t) \tag{7.2.43}$$

$$r_{i,t+\Delta}^{LSC} \equiv \sum_{j=0}^{N^r} x_{i,j,t+\Delta} f_{j,t}^{r}, \text{ (Fit at time } t\text{, analyzed at time } t+\Delta) \tag{7.2.44}$$

$$\tilde{r}_{i,t+\Delta}^{LSC} \equiv \sum_{j=0}^{N^r} x_{i,j,t+\Delta} \tilde{f}_{j,t+\Delta}^{r}. \text{ (Fit at time } t+\Delta\text{, analyzed at time } t+\Delta) \tag{7.2.45}$$

Also, the spreads are computed in three ways,

$$sp_{i,t}^{LSC} \equiv \sum_{j=0}^{N^{sp}} x_{i,j,t} f_{j,t}^{sp}, \text{ (Fit at time } t\text{, analyzed at time } t) \tag{7.2.46}$$

$$sp_{i,t+\Delta}^{LSC} \equiv \sum_{j=0}^{N^{sp}} x_{i,j,t+\Delta} f_{j,t}^{sp}, \text{ (Fit at time } t\text{, analyzed at time } t+\Delta) \tag{7.2.47}$$

$$s\tilde{p}_{i,t+\Delta}^{LSC} \equiv \sum_{j=0}^{N^{sp}} x_{i,j,t+\Delta}\tilde{f}_{j,t+\Delta}^{sp}. \text{ (Fit at time } t+\Delta\text{, analyzed at time } t+\Delta) \tag{7.2.48}$$

Bond holding period return decomposition

With the various bond values identified in the previous section, we are now ready to decompose a bond's holding period return (HPR) into various components. Although there are several ways to perform this task, we adopt an approach based on periodic discrete compounding to afford the possibility that the bond value is zero in the future. Thus, we will use the following set of HPRs,

$$\tilde{R}_\Delta \equiv \frac{\tilde{V}_{t+\Delta} - V_t}{V_t}, \text{ (Bond HPR)} \tag{7.2.49}$$

$$\tilde{R}_\Delta^{Unknown} \equiv \tilde{R}_\Delta^{LSC} \equiv \frac{\tilde{V}_{t+\Delta}^{LSC} - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ (Unknown HPR)} \tag{7.2.50}$$

$$R_\Delta^{Known} \equiv \frac{V_{t+\Delta}^{LSC} - V_t^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ (Known HPR)} \tag{7.2.51}$$

$$R_\Delta^h \equiv \frac{V_{t+\Delta}^{LSC} - V_t^{LSC}}{V_t^{LSC}}, \text{ (Bond Horizon HPR)} \tag{7.2.52}$$

$$\tilde{R}_\Delta^r \equiv \frac{\tilde{V}_{t+\Delta}^r - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ (Bond Base Rate HPR)} \tag{7.2.53}$$

$$\tilde{R}_\Delta^{sp} \equiv \frac{\tilde{V}_{t+\Delta}^{sp} - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ and (Bond Spread HPR)} \tag{7.2.54}$$

$$\tilde{I}_\Delta \equiv \frac{\tilde{V}_{t+\Delta}^{LSC} - \tilde{V}_{t+\Delta}^r - \left(\tilde{V}_{t+\Delta}^{sp} - V_{t+\Delta}^{LSC}\right)}{V_{t+\Delta}^{LSC}}. \text{ (Interaction term)} \tag{7.2.55}$$

We suggest that the traditional definition of bond HPR is biased due to the known HPR component. Thus, rather than appraise $\tilde{R}_\Delta$, we suggest a more general return by dividing the bond horizon total HPR by one plus the horizon HPR. That is,

$$\tilde{R}_\Delta^G \equiv \frac{\tilde{R}_\Delta}{\left(1+R_\Delta^h\right)} = R_\Delta^{Known} + \tilde{R}_\Delta^{Unknown}. \tag{7.2.56}$$

From the definitions above, we have the following identity

$$\tilde{R}_\Delta^G = \frac{R_\Delta^h}{1+R_\Delta^h} + \tilde{R}_\Delta^r + \tilde{R}_\Delta^{sp} + \tilde{I}_\Delta. \tag{7.2.57}$$

Further, we assume the estimation error related to the difference between actual HPRs and HPRs based on the LSC model is negligible.

We now have the capacity to attribute the bond HPR to four components, horizon, base rate, spread, and interaction. Notice that the horizon component contains no randomness. That is, the horizon component is known at the beginning of the holding period, whereas the return attributable to the base rate, spread, and interaction term are unknown at the beginning of the period. Thus, the unknown component can be expressed as

$$\tilde{R}_\Delta^{Unknown} \equiv \frac{\tilde{R}_\Delta - R_\Delta^h}{\left(1+R_\Delta^h\right)} = \tilde{R}_\Delta^r + \tilde{R}_\Delta^{sp} + \tilde{I}_\Delta. \tag{7.2.58}$$

This important insight will result in different measures of bond risk. Since the horizon return varies over time, a portion of the measured variance of a bond position's holding period return is actually just the time series variation of known horizon component. This component of the bond's holding period return perhaps should not be included in time series measures of bond risk. We now focus on the unknown component and seek various decompositions.

Bond LSC return decomposition and bond static risk measures

We now turn to decomposing the bond holding period return based on the LSC model into its components based on modified duration, convexity, and cross-convexity. Specifically, we focus solely on the return attributable to the LSC base rates and LSC spreads.

The N-dimensional Taylor series is applied to identify the appropriate equivalent to traditional bond static risk measures within the LSC framework with discretely compounded HPRs. Taylor series is a well-known approximating technique and can be found in many mathematics books. See, for example, Aramanovich (1965).[10]

N-dimensional Taylor series

Assume a continuous function $f\left(\underline{x}\right)$, where $\underline{x}=\left(x_1,x_2,\ldots,x_n\right)$ is a vector with *n* elements, and $-\infty<x_k<\infty, k=1,\ldots,n$. Also assume at $f\left(\underline{x}^0\right)$ has derivatives of all orders. Let $D_k=\partial/\partial x_k$ be the operators of partial differentiation where $D_k f=\partial f/\partial x_k$, $D_k^m f=\partial^m f/\partial x_k^m$, and in the multidimensional case

$$D_{k_1}D_{k_2}\ldots D_{k_K}f=\frac{\partial^K f}{\partial x_{k_1}\partial x_{k_2}\ldots\partial x_{k_K}}, \tag{7.2.59}$$

where the required partial derivatives are assumed to exist. Then the Taylor series of *f* about the point $\underline{x}^0$ is

$$f\left(\underline{x}\right)=\sum_{k=0}^{\infty}\frac{\left[\sum_{l=1}^{n}\left(x_l-x_l^0\right)D_l\right]^k}{k!}f\left(\underline{x}^0\right). \tag{7.2.60}$$

For example, suppose $\underline{x}=\left(x_1,x_2\right)$, then

$$f\left(x_1,x_2\right)=\sum_{k=0}^{\infty}\frac{\left[\left(x_1-x_1^0\right)\frac{\partial}{\partial x_1}+\left(x_2-x_2^0\right)\frac{\partial}{\partial x_2}\right]^k}{k!}f\left(x_1^0,x_2^0\right). \tag{7.2.61}$$

Further, suppose we are interested in the second order Taylor polynomial, we have

$$\begin{aligned}
f\left(x_1,x_2\right)&=\sum_{k=0}^{2}\frac{\left[\left(x_1-x_1^0\right)\frac{\partial}{\partial x_1}+\left(x_2-x_2^0\right)\frac{\partial}{\partial x_2}\right]^k}{k!}f\left(x_1^0,x_2^0\right)\\
&=f\left(x_1^0,x_2^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_1}\left(x_1-x_1^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_2}\left(x_2-x_2^0\right)\\
&\quad+\frac{1}{2}\frac{\partial f^2\left(x_1^0,x_2^0\right)}{\partial x_1^2}\left(x_1-x_1^0\right)^2+\frac{1}{2}\frac{\partial f^2\left(x_1^0,x_2^0\right)}{\partial x_2^2}\left(x_2-x_2^0\right)^2\\
&\quad+\frac{\partial f^2\left(x_1^0,x_2^0\right)}{\partial x_1\partial x_2}\left(x_1-x_1^0\right)\left(x_2-x_2^0\right)+\varepsilon_2
\end{aligned}, \tag{7.2.62}$$

where ε_2 denotes the approximation error.

[10]Aramanovich, I. G., R. S. Guter, L. A. Lyusternik, I. L. Raukhvarger, M. I. Skanavi, and Yanpol'skii, A. R. (1965). Mathematical Analysis Differentiation and Integration. New York: Pergamon Press.

N-dimensional Taylor series applied to LSC factors

The focus now is on estimating the portion of the HPR attributable to moves in LSC factors related to the base rate and spreads. Thus, the HPR portion attributable to the holding period or horizon, Δ, is already addressed. Hence, we address the HPR based on the underlying instrument's value at time $\underline{t} + \Delta$ but based on the LSC factor values at time t or $V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)$, where $\underline{f}_t$ denotes the vector of LSC factors including both the base rate factors ($\underline{f}_t^r$) and the spread factors ($\underline{f}_t^{sp}$).

The K^{th} partial derivatives of the bond value with respect to LSC factor j, either related to base rates or spread, at time t (denoted generically as $f_{j,t}$) evaluated at time $t + \Delta$, is

$$\frac{\partial^K V_{t+\Delta}^{LSC}}{\partial f_{j,t}{}^K} = \left(-1\right)^K \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^K x_{i,j}{}^K CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC} \text{ and} \tag{7.2.63}$$

$$\frac{\partial^K V_{t+\Delta}^{LSC}}{\partial f_{j,t}{}^L \partial f_{j',t}{}^{K-L}} = \left(-1\right)^K \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^K x_{i,j}{}^L x_{i,j'}{}^{K-L} CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC}. \tag{7.2.64}$$

Note that j and j' denote particular LSC model factors, such as base rate level and spread slope. Also, K and L denote the derivative order, where $L < K$. Further, we assume N_t remaining instrument cash flows.

We now apply the Taylor series approximation to the discretely compounded bond HPR attributable to the LSC factors or

$$\tilde{R}_\Delta^{LSC} = \frac{\tilde{V}_{t+\Delta}^{LSC}\left(\tilde{\underline{f}}_{t+\Delta}\right) - V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}. \tag{7.2.65}$$

We define the change in the generic LSC factor as

$$\Delta\tilde{f}_j = \tilde{f}_{j,t+\Delta} - f_{j,t}. \tag{7.2.66}$$

Thus, the Taylor series is applied to the underlying value,

$$\tilde{V}_{t+\Delta}^{LSC}\left(\tilde{\underline{f}}_{t+\Delta}\right) \cong \sum_{k=0}^{\infty} \frac{\left(\sum_{j=1}^{N_F} \Delta\tilde{f}_j D_j\right)^k}{k!} V_{t+\Delta}^{LSC}\left(\underline{f}_t\right) \text{ or (Sum begins at 0)} \tag{7.2.67}$$

$$\tilde{V}_{t+\Delta}^{LSC}\left(\tilde{\underline{f}}_{t+\Delta}\right) - V_{t+\Delta}^{LSC}\left(\underline{f}_t\right) \cong \sum_{k=1}^{\infty} \frac{\left(\sum_{j=1}^{N_F} \Delta\tilde{f}_j D_j\right)^k}{k!} V_{t+\Delta}^{LSC}\left(\underline{f}_t\right). \text{ (Sum begins at 1)} \tag{7.2.68}$$

Note that N_F denotes the total number of factors including the number of base curve factors and the number of spread curve factors.

Approximating the discretely compounded bond returns based on the first derivatives, we have

$$\tilde{R}_\Delta^{LSC} = \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)} \left(\sum_{j=1}^{N_F} \frac{\partial V_{t+\Delta}^{LSC}}{\partial f_{j,t}} \Delta\tilde{f}_j + \tilde{\eta}_{t+\Delta}\right). \tag{7.2.69}$$

Approximating the discretely compounded bond returns based on the first and second derivatives, we have

$$\tilde{R}_\Delta^{LSC} = \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)} \left[\sum_{j=1}^{N_F} \frac{\partial V_{t+\Delta}^{LSC}}{\partial f_{j,t}} \Delta\tilde{f}_j + \frac{1}{2}\left(\sum_{j=1}^{N_F} \Delta\tilde{f}_j D_j\right)^2 + \tilde{\eta}_{t+\Delta}\right]. \tag{7.2.70}$$

We define LSC factor durations generically as

$$FD_j \equiv -\frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)} \frac{\partial V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{\partial f_{j,t}}. \tag{7.2.71}$$

Let the present value cash flow weights for a given bond cash flow be defined as

$$w_{i,t+\Delta}^{LSC} = \frac{CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC}}{V_{t+\Delta}}, \tag{7.2.72}$$

Substituting from Equation (7.2.63), we have the j^{th} factor duration is

$$\begin{aligned} FD_j &= -\frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}\left[\left(-1\right)^1 \sum_{i=0}^{N_t}\left(\tau_i - \Delta\right)^1 x_{i,j}{}^1 CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC}\right] \\ &= \sum_{i=0}^{N_t}\left(\tau_i - \Delta\right) x_{i,j} CF_{i,t+\Delta} w_{i,t+\Delta}^{LSC} \end{aligned}. \tag{7.2.73}$$

Thus, focusing solely on LSC factor durations, the discretely compounded rate of return can be approximated with LSC factor durations and LSC factor changes as

$$\tilde{R}_{\Delta}^{LSC} \cong -\sum_{j=0}^{N_F} FD_j \Delta\tilde{f}_j \,. \tag{7.2.74}$$

Factor convexity is defined and expressed as

$$FC_j \equiv \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)} \frac{\partial^2 V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{\partial f_{j,t}^2}. \tag{7.2.75}$$

Substituting from Equation (7.2.64), we have the n^{th} factor convexity is

$$\begin{aligned} FC_j &= \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}\left[\left(-1\right)^2 \sum_{i=0}^{N_t}\left(\tau_i - \Delta\right)^2 x_{i,j}{}^2 CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC}\right] \\ &= \sum_{i=0}^{N_t}\left(\tau_i - \Delta\right)^2 x_{i,j}{}^2 w_{i,t+\Delta}^{LSC} \end{aligned}. \tag{7.2.76}$$

Similarly, factor cross convexity is defined and expressed as

$$FCC_{j,j'} \equiv \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)} \frac{\partial^2 V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{\partial f_{j,t}\,\partial f_{j',t}}. \tag{7.2.77}$$

Substituting again from Equation (7.2.64), we can express a factor cross convexity generically as

$$\begin{aligned} FCC_{j,j'} &= \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}\left[\left(-1\right)^2 \sum_{i=0}^{N_t}\left(\tau_i - \Delta\right)^2 x_{i,j} x_{i,j'} CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC}\right] \\ &= \sum_{i=0}^{N_t}\left(\tau_i - \Delta\right)^2 x_{i,j} x_{i,j'} w_{i,t+\Delta}^{LSC} \end{aligned}. \tag{7.2.78}$$

Let $A_j \equiv \Delta\tilde{f}_j D_j$, then

$$\begin{aligned} \left(\sum_{j=0}^{N_F}\Delta\tilde{f}_j D_j\right)^2 &= \left(\sum_{j=0}^{N_F} A_j\right)^2 = \sum_{j=0}^{N_F}\sum_{j'=0}^{N_F} A_j A_{j'} = \sum_{j=0}^{N_F} A_j^2 + 2\sum_{j=0}^{N_F}\sum_{j'=n+1}^{N_F} A_j A_{j'} \\ &= \sum_{j=0}^{N_F}\Delta\tilde{f}_j^2 D_j^2 + 2\sum_{j=0}^{N_F}\sum_{j'=j+1}^{N_F}\Delta\tilde{f}_j\,\Delta\tilde{f}_{j'} D_j D_{j'} \\ &= \sum_{j=0}^{N_F}\frac{\partial^2 V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{\partial f_{j,t}^2}\Delta\tilde{f}_j^2 + 2\sum_{j=0}^{N_F}\sum_{j'=j+1}^{N_F}\frac{\partial^2 V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{\partial f_{j,t}\,\partial f_{j',t}}\Delta\tilde{f}_j\,\Delta\tilde{f}_{j'} \end{aligned}. \tag{7.2.79}$$

Thus, the bond HPR can be approximated by factor risk measures along with changes in the factors. Specifically, based on first and second order Taylor series, we substitute in Equation (7.2.70) and have

$$
\begin{aligned}
\tilde{R}_{\Delta}^{LSC} &= \frac{1}{V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}\left[\sum_{j=0}^{N_F}\frac{\partial V_{t+\Delta}^{LSC}\left(\underline{f}_t\right)}{\partial f_{j,t}}\Delta\tilde{f}_j + \frac{1}{2}\left(\sum_{j=0}^{N_F}\Delta\tilde{f}_j D_j\right)^2 + \tilde{\eta}_{2,t+\Delta}\right]. \\
&\cong -\sum_{j=0}^{N_F} FD_j\Delta\tilde{f}_j + \frac{1}{2}\sum_{j=0}^{N_F} FC_j\Delta\tilde{f}_j^2 + \sum_{j=0}^{N_F}\sum_{j'=j+1}^{N_F} FCC_{j,j'}\Delta\tilde{f}_j\,\Delta\tilde{f}_{j'}
\end{aligned}
\tag{7.2.80}
$$

Note that the LSC factors include both base rate and spread factors, where

$$\Delta\tilde{f}_{j,t+\Delta}^{r} = \tilde{f}_{j,t+\Delta}^{r} - f_{j,t}^{r} \text{ or (Base rate)} \tag{7.2.81}$$

$$\Delta\tilde{f}_{j,t+\Delta}^{sp} = \tilde{f}_{j,t+\Delta}^{sp} - f_{j,t}^{sp} \text{ or (Spread)} \tag{7.2.82}$$

Interpreting LSC factor static risk measures

We focus now on interpreting the static risk measures defined above within the context of bond HPRs. From the definition of bond HPR based on the LSC framework, we have

$$\tilde{R}_{\Delta}^{Unknown} = \tilde{R}_{\Delta}^{LSC} = \frac{\tilde{V}_{t+\Delta}^{LSC} - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}} \cong R\tilde{C}_{FD}^{r} + R\tilde{C}_{FC}^{r} + R\tilde{C}_{FCC}^{r} + R\tilde{C}_{FD}^{sp} + R\tilde{C}_{FC}^{sp} + R\tilde{C}_{FCC}^{sp} + R\tilde{C}_{FCC}^{r,sp}, \tag{7.2.83}$$

where *RC* denotes the return contribution, *r* denotes the base rate, *sp* denotes the spread, *FD* denotes the LSC factor durations, *FC* denotes the LSC factor convexities, and *FCC* denotes the LSC factor cross-convexities.

Each of the above return contributions can be further decomposed in the following manner,

$$R\tilde{C}_{FD}^{r} \equiv R\tilde{C}_{FD}^{r,L} + R\tilde{C}_{FD}^{r,S} + R\tilde{C}_{FD}^{r,C}, \tag{7.2.84}$$

$$R\tilde{C}_{FC}^{r} \equiv R\tilde{C}_{FC}^{r,L} + R\tilde{C}_{FC}^{r,S} + R\tilde{C}_{FC}^{r,C}, \tag{7.2.85}$$

$$R\tilde{C}_{FCC}^{r} = R\tilde{C}_{FCC}^{r,L,S} + R\tilde{C}_{FCC}^{r,S,C} + R\tilde{C}_{FCC}^{r,L,C}, \tag{7.2.86}$$

$$R\tilde{C}_{FD}^{sp} = R\tilde{C}_{FD}^{sp,L} + R\tilde{C}_{FD}^{sp,S} + R\tilde{C}_{FD}^{sp,C}, \tag{7.2.87}$$

$$R\tilde{C}_{FC}^{sp} = R\tilde{C}_{FC}^{sp,L} + R\tilde{C}_{FC}^{sp,S} + R\tilde{C}_{FC}^{sp,C}, \tag{7.2.88}$$

$$R\tilde{C}_{FCC}^{sp} = R\tilde{C}_{FCC}^{sp,L,S} + R\tilde{C}_{FCC}^{sp,S,C} + R\tilde{C}_{FCC}^{sp,L,C}, \text{ and} \tag{7.2.89}$$

$$
\begin{aligned}
R\tilde{C}_{FCC}^{r,sp} &= R\tilde{C}_{FCC}^{r(L),sp(L)} + R\tilde{C}_{FCC}^{r(L),sp(S)} + R\tilde{C}_{FCC}^{r(L),sp(C)} \\
&+ R\tilde{C}_{FCC}^{r(S),sp(L)} + R\tilde{C}_{FCC}^{r(S),sp(S)} + R\tilde{C}_{FCC}^{r(S),sp(C)} \\
&+ R\tilde{C}_{FCC}^{r(C),sp(L)} + R\tilde{C}_{FCC}^{r(C),sp(S)} + R\tilde{C}_{FCC}^{r(C),sp(C)}
\end{aligned}
. \tag{7.2.90}
$$

For completeness, we formally document each return contribution formula for the base rate. The spread results are the same, just replace *r* with *sp*. Thus,

$$R\tilde{C}_{FD}^{r,L} \equiv -FD_L^r\Delta\tilde{f}_L^r, \tag{7.2.91}$$

$$R\tilde{C}_{FD}^{r,S} \equiv -FD_S^r\Delta\tilde{f}_S^r, \tag{7.2.92}$$

$$R\tilde{C}_{FD}^{r,C} \equiv -FD_C^r\Delta\tilde{f}_C^r, \tag{7.2.93}$$

$$R\tilde{C}_{FC}^{r,L} \equiv \frac{1}{2}FC_{FC}^{r,L}\left(\Delta\tilde{f}_L^r\right)^2, \tag{7.2.94}$$

$$R\tilde{C}_{FC}^{r,S} \equiv \frac{1}{2}FC_{FC}^{r,S}\left(\Delta\tilde{f}_S^r\right)^2, \tag{7.2.95}$$

$$R\tilde{C}_{FC}^{r,C} \equiv \frac{1}{2}FC_{FC}^{r,C}\left(\Delta\tilde{f}_C^r\right)^2, \tag{7.2.96}$$

$$R\tilde{C}_{FCC}^{r,L,S} \equiv FCC_{FCC}^{r,L,S}\Delta\tilde{f}_L^r\Delta\tilde{f}_S^r, \tag{7.2.97}$$

$$R\tilde{C}_{FCC}^{r,L,C} \equiv FCC_{FCC}^{r,L,C}\Delta\tilde{f}_L^r\Delta\tilde{f}_C^r, \text{ and} \tag{7.2.98}$$

$$R\tilde{C}_{FCC}^{r,S,C} \equiv FCC_{FCC}^{r,S,C}\Delta\tilde{f}_S^r\Delta\tilde{f}_C^r. \tag{7.2.99}$$

For completeness, we also provide the explicit formulas for factor durations, factor convexities, and factor cross convexities for the base rate curve. Assuming a three factor model (Level, Slope, and Curvature), we denote $x_{i,0} = x_{i,L} = 1$ (Level), $x_{i,1} = x_{i,S}$ (Slope), and $x_{i,2} = x_{i,C}$ (Curvature).We will present the corresponding formulas for spreads in the next module.

$$FD_L^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right) w_{i,t+\Delta}^{LSC}, \tag{7.2.100}$$

$$FD_S^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right) x_{i,S} w_{i,t+\Delta}^{LSC}, \tag{7.2.101}$$

$$FD_C^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right) x_{i,C} w_{i,t+\Delta}^{LSC}, \tag{7.2.102}$$

$$FC_L^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 w_{i,t+\Delta}^{LSC}, \tag{7.2.103}$$

$$FC_S^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,S}{}^2 w_{i,t+\Delta}^{LSC}, \tag{7.2.104}$$

$$FC_C^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,C}{}^2 w_{i,t+\Delta}^{LSC}, \tag{7.2.105}$$

$$FCC_{L,S}^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,S} w_{i,t+\Delta}^{LSC}, \tag{7.2.106}$$

$$FCC_{S,C}^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,S} x_{i,C} w_{i,t+\Delta}^{LSC}, \text{ and} \tag{7.2.107}$$

$$FCC_{L,C}^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,C} w_{i,t+\Delta}^{LSC}. \tag{7.2.108}$$

Note that for most applications, the vast majority of return contributions will be negligible. Thus, the actual analysis will be more straightforward. When developing software solutions however, it is better to have a thorough design that can be simplified by the user.

We now explore bond expected HPRs and related variances.

Bond expected HPRs and variance

Recall the grossed up bond HPR can be expressed as

$$\tilde{R}_\Delta^G = R_\Delta^{Known} + \tilde{R}_\Delta^{Unknown}. \tag{7.2.109}$$

Based on the properties of expected return, we know

$$E\left(\tilde{R}_\Delta^G\right) = R_\Delta^{Known} + E\left(\tilde{R}_\Delta^{Unknown}\right). \tag{7.2.110}$$

Based on the properties of variance, we know

$$\mathrm{var}\left(\tilde{R}_\Delta^G\right) = \mathrm{var}\left(\tilde{R}_\Delta^{Unknown}\right). \tag{7.2.111}$$

Substituting from Equation (7.2.58), we have

$$\mathrm{var}\left(\tilde{R}_\Delta^G\right) = \mathrm{cov}\left(\tilde{R}_\Delta^{Unknown}, \tilde{R}_\Delta^r + \tilde{R}_\Delta^{sp} + \tilde{I}_\Delta\right). \tag{7.2.112}$$

Thus, the bond variance can be decomposed based on properties of covariance as

$$\mathrm{var}\left(\tilde{R}_\Delta^G\right) = \mathrm{cov}\left(\tilde{R}_\Delta^{Unknown}, \tilde{R}_\Delta^r\right) + \mathrm{cov}\left(\tilde{R}_\Delta^{Unknown}, \tilde{R}_\Delta^{sp}\right) + \mathrm{cov}\left(\tilde{R}_\Delta^{Unknown}, \tilde{I}_\Delta\right). \tag{7.2.113}$$

Dividing both sides by the bond variance, we have the percentage marginal contribution to variance as beta coefficients or

$$1=\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{Unknown},\tilde{R}_{\Delta}^{r}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{Unknown},\tilde{R}_{\Delta}^{sp}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{Unknown},\tilde{I}_{\Delta}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}=\beta_{r}+\beta_{sp}+\beta_{I}. \qquad (7.2.114)$$

Alternatively, the bond variance can be decomposed based on return contributions or

$$\begin{aligned}\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)&=\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FD}^{r}\right)+\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FC}^{r}\right)+\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FCC}^{r}\right)\\&+\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FD}^{sp}\right)+\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FC}^{sp}\right)+\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FCC}^{sp}\right)+\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FCC}^{r,sp}\right)\end{aligned}. \qquad (7.2.115)$$

Again, we can estimate the percentage marginal contribution to variance as

$$\begin{aligned}1&=\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FD}^{r}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FC}^{r}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FCC}^{r}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}\\&+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FD}^{sp}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FC}^{sp}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FCC}^{sp}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}+\frac{\operatorname{cov}\left(\tilde{R}_{\Delta}^{G},R\tilde{C}_{FCC}^{r,sp}\right)}{\operatorname{var}\left(\tilde{R}_{\Delta}^{G}\right)}.\\&=\beta_{FD}^{r}+\beta_{FC}^{r}+\beta_{FCC}^{r}+\beta_{FD}^{sp}+\beta_{FC}^{sp}+\beta_{FCC}^{sp}+\beta_{FCC}^{r,sp}\end{aligned} \qquad (7.2.116)$$

Summary

We reviewed various aspects of traditional bond static risk measure. We reviewed traditional bond static risk measures include various forms of duration (Macaulay, modified, and effective) and a couple of forms of convexity (standard and effective). Further, we explored the important role of compounding for both taking the present value of maturity varying cash flows as well as compounding methods for calculating holding period returns. After a brief tour of the history, we review several important aspects of these measures.

With this foundation, we then moved to advanced bond static risk measures based on an application of the LSC model. Within a detailed bond holding period return decomposition, we reviewed numerous new measures of bond static risk. With these advanced measures, we explored bond expected holding period returns as well as its related variance. The module concludes with selected explanations of selected R code.

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