Module 7.1

Static Risk Measures

Centered Differencing

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Derivatives Review

- Suppose y is a functions of other variables

$$y = f\left(x_1, x_2, \ldots, x_n\right)$$

- Total derivative with respect to x_1

$$\frac{dy}{dx_1} = \frac{df\left(x_1, x_2, \ldots, x_n\right)}{dx_1}$$

- Total derivative of y with respect to x_1. Thus, if any other variables, say x_2, is a function of x_1, then the total derivative must incorporate this embedded functionality

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Partial Derivatives

- Partial derivative with respect to x_1

$$\frac{\partial y}{\partial x_1} = \frac{\partial f\left(x_1, x_2, \ldots, x_n\right)}{\partial x_1}$$

- We assume all other embedded functionality is ignored.

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Derivatives Review

- Suppose

$$y = f\left[x_1, x_2\left(x_1\right), \ldots, x_n\right]$$

- Total derivative

$$\frac{dy}{dx_1} = \frac{\partial f\left[x_1, x_2\left(x_1\right), \ldots, x_n\right]}{\partial x_1} + \frac{\partial f\left[x_1, x_2\left(x_1\right), \ldots, x_n\right]}{\partial x_2}\frac{dx_2}{dx_1}$$

- Partial derivative

$$\frac{\partial y}{\partial x_1} = \frac{\partial f\left[x_1, x_2\left(x_1\right), \ldots, x_n\right]}{\partial x_1}$$

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Embedded Functionality

- Many finance-related functions have embedded functionality
- A stock price may be a function of volatility
  - In many cases, if volatility rises, then a particular stock price will decline
  - option valuation models depend on both the stock price as well as volatility
  - Embedded functionality is often ignored

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Central Finance Concepts

- Many quant tasks involve computing numerical derivatives
- Review applications of numerical derivatives in finance

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Compound OVM Challenge

$$CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right)=\iota_C\iota_U S_t B_{t,T_2,\delta}B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{11},\iota_U d_{12};\iota_C\rho\right)$$
$$-\iota_C\iota_U X_U B_{t,T_2,r}B_{T_1,T_2,-\hat{q}}N_2\left(\iota_C\iota_U d_{21},\iota_U d_{22};\iota_C\rho\right)-\iota_C X_C B_{t,T_1,r}N\left(\iota_C\iota_U d_{21}\right)$$

$$N_2\left(a,b;\rho\right)\equiv\int_{-\infty}^{a}\int_{-\infty}^{b}\frac{\exp\left\{-\frac{z_1^2-2\rho z_1 z_2+z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}}dz_1\,dz_2 \quad d_{21}\equiv\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S^*_{T_1}}\right)-\frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}} \quad d_{22}\equiv\frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right)-\frac{\sigma^2_{t,T_2}}{2}}{\sigma_{t,T_2}}$$

$$d_{11}\equiv\frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S^*_{T_1}}\right)+\frac{\sigma^2_{t,T_1}}{2}}{\sigma_{t,T_1}}=d_{21}+\sigma_{t,T_1} \quad d_{12}\equiv\frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right)+\frac{\sigma^2_{t,T_2}}{2}}{\sigma_{t,T_2}}=d_{22}+\sigma_{t,T_2}$$

Estimating Greeks

- Need to validate candidate solution
- Need estimates:
  - Delta (dCO/dS ?)
  - Gamma
  - Theta
  - Vega
  - Rho
- Analytic or numerical?

Other Numerical Derivatives

- Duration and convexity
- Yield volatility estimates from price volatility
- Option pricing model standard ‘greeks,’ delta, gamma, theta, vega, and rho
- Option pricing model advanced ‘greeks,’ vanna, charm, speed, zomma, color, vomma, DvegaDtime, ultima, and so forth
- Linear model and VaR (delta-VaR, delta-gamma-VaR, and so forth)
- Merton’s default probabilities (delta estimate)
- Exotic option ‘greeks’
- Local volatility models (volatility surface estimation)
- Marginal contribution to expected return and risk

Quantitative Finance Materials

- Univariate Taylor series
- Centered differencing (Eberly)
  - Review formulas
  - Applications
- Illustrated with R code
- Application to other derivatives

Univariate Taylor Series

- Assume $f(x)$ is a continuous function
  - $-\infty < x < \infty$
  - $-\infty < f\left(x\right) < \infty$
  - $f(x_0)$ has derivatives of all orders
- Taylor series of f about the number x_0 is

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{f^n\left(x_0\right)}{n!}h^n \qquad f^n\left(x_0\right)\equiv\left.\frac{d^n f\left(x\right)}{dx^n}\right|_{x=x_0}$$

where $h = x - x_0$, “small positive value”

Centered Differencing (Eberly)

Theorem 1: Numerical derivatives approximation theorem for dth order derivatives

If

$$f^d\left(x\right)=\frac{d!}{h^d}\sum_{n=0}^{d+p-1}\left(\sum_{i=i_{min}}^{i_{max}}i^n C_i\right)\frac{h^n}{n!}f^n\left(x_0\right) \quad \textbf{(dth order derivative)}$$

then

$$f^d\left(x\right)=\frac{d!}{h^d}\sum_{i=i_{min}}^{i_{max}}C_i f\left(x+ih\right)+O\left(h^p\right) \quad \textbf{(Approximation theorem equation)}$$

where $h = x - x_0$, > 0 (small), $p > 0$ denotes integer order of error, d denotes the integer derivative order, i_{max}, and i_{min} denote extreme indices, C_i denotes some coefficients where $C = \{C_{min}, \ldots, C_{max}\}$ denotes the template of approximation.

Order 1

Numerical derivative order of accuracy 1

First derivative: d = 1, p = 2

$$f^1(x) \cong \frac{f(x+h) - f(x-h)}{2h}$$

Second derivative: d = 2, p = 1

$$f^2(x) \cong \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Order 2

Numerical derivative order of accuracy 2

First derivative: d = 1, p = 4

$$f^1(x) \cong \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

Second derivative: d = 2, p = 3

$$f^2(x) \cong \frac{-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)}{12h^2}$$

Order 3

Numerical derivative order of accuracy 3

First derivative: d = 1, p = 6

$$f^1(x) \cong \frac{f(x+3h) - 9f(x+2h) + 45f(x+h) - 45f(x-h) + 9f(x-2h) - f(x-3h)}{60h}$$

Second derivative: d = 2, p = 5

$$f^2(x) \cong \frac{f(x+3h) - 13.5f(x+2h) + 135f(x+h) - 245f(x) + 135f(x-h) - 13.5f(x-2h) + f(x-3h)}{90h^2}$$

Order 4

Numerical derivative order of accuracy 4

First derivative: d = 1, p = 8

$$f^1(x) \cong \frac{1}{h}\left[\begin{array}{l} \frac{f(x-4h)}{280} - \frac{4f(x-3h)}{105} + \frac{f(x-2h)}{5} - \frac{4f(x-h)}{5} + 0f(x) \\ + \frac{4f(x+h)}{5} - \frac{f(x+2h)}{5} + \frac{4f(x+3h)}{105} - \frac{f(x+4h)}{280} \end{array}\right]$$

Second derivative: d = 2, p = 7

$$f^2(x) \cong \frac{2}{h^2}\left[\begin{array}{l} -\frac{f(x-4h)}{1{,}120} + \frac{4f(x-3h)}{315} - \frac{f(x-2h)}{10} + \frac{4f(x-h)}{5} - \frac{205f(x)}{144} \\ + \frac{4f(x+h)}{5} - \frac{f(x+2h)}{10} + \frac{4f(x+3h)}{315} - \frac{f(x+4h)}{1{,}120} \end{array}\right]$$

Order 4, 8th Derivative

- Just for fun ...

$$f^8(x) \cong \frac{40{,}320}{h}\left[\begin{array}{l} \frac{f(x-4h)}{40{,}320} - \frac{f(x-3h)}{5{,}040} + \frac{f(x-2h)}{1{,}440} - \frac{f(x-h)}{720} + \frac{f(x)}{576} \\ -\frac{f(x+h)}{720} + \frac{f(x+2h)}{1{,}440} - \frac{f(x+3h)}{5{,}040} + \frac{f(x+4h)}{40{,}320} \end{array}\right]$$

Illustration with Modified Duration (First Derivative, 1%)

```
> BONDInputData$ChangeInYTM = 1.0
> FD1 <- BondFD(BONDInputData)
> BONDInputData$Order <- 2
> FD2 <- BondFD(BONDInputData)
> BONDInputData$Order <- 3
> FD3 <- BondFD(BONDInputData)
> BONDInputData$Order <- 4
> FD4 <- BondFD(BONDInputData)
> MDDur <- Duration(BONDInputData)
> FD <- -MDDur*BondValue(BONDInputData)/100
> MDDur
[1] 8.710204
> Error1 <- FD1 - FD; Error2 <- FD2 - FD; Error3 <- FD3 - FD; Error4 <- FD4 - FD
> Error1; Error2; Error3; Error4
[1] -146.9193
[1] 0.3773835
[1] -0.001237588
[1] 4.925503e-06
```

Illustration with Modified Duration (FD, 0.1%)

```
> #
> # Change YTM by 10 basis points
> #
> BONDInputData$ChangeInYTM = 0.1
> FD1 <- BondFD(BONDInputData)
> BONDInputData$Order <- 2
> FD2 <- BondFD(BONDInputData)
> BONDInputData$Order <- 3
> FD3 <- BondFD(BONDInputData)
> BONDInputData$Order <- 4
> FD4 <- BondFD(BONDInputData)
> MDDur <- Duration(BONDInputData)
> FD <- -MDDur*BondValue(BONDInputData)/100
> Error1 <- FD1 - FD; Error2 <- FD2 - FD; Error3 <- FD3 - FD; Error4 <- FD4 - FD
> Error1; Error2; Error3; Error4
[1] 1.449371e-08
[1] 3.768281e-05
[1] 1.263106e-08
[1] 1.449371e-08
```

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Illustration with Modified Duration (FD, 0.01%)

```
> #
> # Change YTM by 1 basis points
> #
> BONDInputData$ChangeInYTM = 0.01
> FD1 <- BondFD(BONDInputData)
> BONDInputData$Order <- 2
> FD2 <- BondFD(BONDInputData)
> BONDInputData$Order <- 3
> FD3 <- BondFD(BONDInputData)
> BONDInputData$Order <- 4
> FD4 <- BondFD(BONDInputData)
> MDDur <- Duration(BONDInputData)
> FD <- -MDDur*BondValue(BONDInputData)/100
> Error1 <- FD1 - FD; Error2 <- FD2 - FD; Error3 <- FD3 - FD; Error4 <- FD4 - FD
> Error1; Error2; Error3; Error4
[1] 1.212175e-08
[1] 5.558832e-09
[1] 1.67347e-09
[1] 1.212175e-08
```

Machine error appears impacts order 3 and 4

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Illustration with Modified Duration (FD, 0.000001%)

```
> #
> # Change YTM by 0.0000001 basis points
> #
> BONDInputData$ChangeInYTM = 0.00000001
> FD1 <- BondFD(BONDInputData)
> BONDInputData$Order <- 2
> FD2 <- BondFD(BONDInputData)
> BONDInputData$Order <- 3
> FD3 <- BondFD(BONDInputData)
> BONDInputData$Order <- 4
> FD4 <- BondFD(BONDInputData)
> MDDur <- Duration(BONDInputData)
> FD <- -MDDur*BondValue(BONDInputData)/100
> Error1 <- FD1 - FD; Error2 <- FD2 - FD; Error3 <- FD3 - FD; Error4 <- FD4 - FD
> Error1; Error2; Error3; Error4
[1] -0.01492698
[1] -0.0141236
[1] -0.02304877
[1] -0.01492698
```

Machine error dominates

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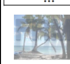
Other 'greeks'

Parameter	S	t	σ	r	X	δ
First Order Derivatives						
Value (O)	Delta (Δ)	Theta (θ)	Vega (ν)	Rho (ρ)	dOdX	dOdδ
Elasticity	Lambda (λ)	O%t%	O%σ%	O%r%	O%X%	O%δ%
Selected Second Order Derivatives						
Delta (Δ)	Gamma (Γ)	Charm	Vanna	dΔdr	dΔdX	dΔdδ
Theta (θ)		dθdt	Veta	dθdr	dθdX	dθdδ
Vega (ν)			Vomma	Ver	dνdX	dνdδ
Rho (ρ)				dρdr	dρdX	dρdδ
dOdX					(dOdX)dX	(dOdX)dδ
DdOδ						(dOδd)dδ
Selected Third Order Derivatives						
Gamma (Γ)	Speed ()	Color ()	Zomma ()	dΓdr	dΓdX	dΓdδ
dθdt		(dθdt)dt	(dθdt)dσ	(dθdt)dr	(dθdt)dX	(dθdt)dδ
Vomma			Ultima ()	dVommadr	dVommadX	dVommadδ
dρdr				(dρdr)dr	(dρdr)dX	(dρdr)dδ
…						



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Summary

- Numerical derivative estimation techniques useful in finance
- Several finance applications of numerical derivatives
- Introduced centered differencing
  - Review formulas
  - Applications

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