Module 7.1: Static Risk Management Centered Differencing

Learning objectives

- Explain how to estimate numerical derivatives using a centered difference approach that can be generalized.
- Contrast the first four orders of accuracy
- Illustrate some of the problems with numerical derivatives with R code

Executive summary

In this module, we illustrate the centered difference technique applied to the computation of mathematical derivatives. Although the analytics are rather technical, the overall objective is clear. We seek to numerically estimate the *n*th derivative of some function without having to analytically derive it. We illustrate centered differencing approach developed here with bond valuation.

Central finance concepts

Many quantitative finance problems require the estimation of numerical derivatives. For example, recall the compound option valuation model (COVM) introduced in Module 5.7. As this is a newly developed model containing an option yield, we need to derive the analytic Greeks, such as delta and gamma. These Greeks are required to successfully use the COVM. The numerical techniques developed here can be used to verify the derived analytical Greeks are correct or alternatively just avoid derivation of the Greeks all together and simply rely on the numerical Greeks.

We introduce a centered difference technique that allows increasing orders of accuracy. Examples of numerical derivatives in finance include:

- Duration and convexity
- Yield volatility estimates from price volatility
- Option pricing model standard Greeks, delta, gamma, theta, vega, and rho
- Option pricing model advanced Greeks, vanna, charm, speed, zomma, color, vomma, DvegaDtime, ultima, and so forth
- Linear model and VaR (delta-VaR, delta-gamma-VaR, and so forth)
- Merton's default probabilities (delta estimate)
- Exotic option Greeks
- Local volatility models (volatility surface estimation)
- Marginal contribution to expected return and risk

We now cover the highly technical issues related to the centered differencing approach to estimating numerical derivatives.

Quantitative finance materials

We now present the first and second derivative approximations for up to order or accuracy 4. We assume a generic function, $y = f(x)$ and $f^d(x)$ denotes the d^{th} derivative of y with respect to x. Also, p denotes the $O(h^p)$ order of accuracy of the approximation from the Taylor series.

Numerical approximations of mathematical derivatives

There are a wide variety of ways to numerically estimate derivatives (mathematics, not finance) available in the literature. We introduce one method that provides some flexibility as to degree of accuracy desired. The method provided here was stimulated from a paper by Eberly (2008).

We first introduce the univariate Taylor series for the purpose of explaining the centered difference theorem that forms the basis for the numerical procedure provided here. We then review selected pieces of source code.

Univariate Taylor series

Assume a continuous function $f(x)$, where $-\infty < x < \infty$ and $-\infty < f(x) < \infty$. Also assume that $f(x_0)$ has derivatives of all orders. Then the Taylor series of *f* about the number x_0 can be expressed as (where $h = x - x_0$, "small positive value")

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} h^n \text{, (**Taylor series**)} \tag{7.1.1}$$

where

$$f^n(x_0) \equiv \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0} . \tag{7.1.2}$$

(The *n*th derivative of $f(x)$ evaluated at x_0.)

The d^{th} Taylor polynomial p_n of f about x_0 is

$$p_d(x) = f(x_0) + f^1(x_0)h + \frac{f^2(x_0)}{2!}h^2 + \cdots + \frac{f^d(x_0)}{2!}h^d \text{. (}d^{th}\text{ Taylor polynomial)} \tag{7.1.3}$$

The d^{th} Taylor remainder r_d of f about x_0 is

$$r_d(x) = f(x) - p_d(x) \text{. (}d^{th}\text{ Taylor remainder)} \tag{7.1.4}$$

It can be shown that

$$r_d(x) = \frac{f^{d+1}(x_z)}{(d+1)!} h^{d+1} = O\left(h^{d+1}\right), \tag{7.1.5}$$

for some $x_0 < x_z < x$.[1]

The univariate Taylor series can be used to estimate numerical derivatives based on the following theorem.

Theorem 1: Numerical derivatives approximation theorem for dth order derivatives

If

$$f^d(x) = \frac{d!}{h^d} \sum_{n=0}^{d+p-1} \left(\sum_{i=i_{\min}}^{i_{\max}} i^n C_i \right) \frac{h^n}{n!} f^n(x_0) \text{, (}\boldsymbol{d}\text{**th order derivative**)} \tag{7.1.6}$$

then

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O\left(h^p\right) \text{, (**Approximation theorem equation**)} \tag{7.1.7}$$

where $h = x - x_0$, > 0 (small), $p > 0$ denotes integer order of error, d denotes the integer derivative order, i_{max}, and i_{min} denote extreme indices, C_i denotes some coefficients where $C = (C_{min}, \ldots, C_{max})$ denotes the template of approximation.

Note that the d^{th} order derivative equation above holds if and only if

$$\sum_{i=i_{\min}}^{i_{\max}} i^n C_i = \begin{cases} 0 \text{ for } 0 \le n \le d+p+1 \text{ and } n \ne d \\ 1 \text{ for } n = d \end{cases} \text{. (**Template sum equation**)} \tag{7.1.8}$$

Centered differencing implies $i_{max} = i_{min} = (d + p - 1)/2$ where $(d + p - 1)$ is assumed to be even. The template sum equation holds if and only if

$$\underline{C}_{d+px1} = \underline{A}^{-1}_{d+pxd+p} \underline{B}_{d+px1}, \tag{7.1.9}$$

where the A matrix and B vector are defined as follows

[1]The univariate Taylor series can be found in just about any calculus book.

$$\underline{A}_{d+pxd+p} = \begin{bmatrix} i_{\min}^0 & i_{\min+1}^0 & \cdots & i_{\max-1}^0 & i_{\max}^0 \\ i_{\min}^1 & i_{\min+1}^1 & \cdots & i_{\max-1}^1 & i_{\max}^1 \\ \vdots & & \ddots & & \vdots \\ i_{\min}^{d+p-2} & i_{\min+1}^{d+p-2} & \cdots & i_{\max-1}^{d+p-2} & i_{\max}^{d+p-2} \\ i_{\min}^{d+p-1} & i_{\min+1}^{d+p-1} & \cdots & i_{\max-1}^{d+p-1} & i_{\max}^{d+p-1} \end{bmatrix} \text{ and} \tag{7.1.10}$$

$$\underline{B}_{d+px1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} 0 \\ \vdots \\ d \\ \vdots \\ d+p-1 \end{matrix} . \tag{7.1.11}$$

Proof: Expand the approximation theorem equation to order d + p – 1, based on Taylor series.[2]

We now review selected results up to integer derivative order 4.

Numerical derivative order of accuracy

We illustrate order of accuracy from one to four.

Numerical derivative order of accuracy 1

Recall centered differencing implies $i_{max} = i_{min} = (d + p - 1)/2$ where $(d + p - 1)$ is assumed to be even. Therefore, $d + p = 3$ and we have $i_{max} = -i_{min} = 1$.

$$\underline{A}_{3x3} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ \underline{A}_{3x3}^{-1} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 1 & 0 & -1 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$

Identity equation: d = 0, p = 3

$$\underline{B}_{3x1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{3x1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p) = \frac{0!}{h^0} f(x+0h) = f(x).$$

First derivative: d = 1, p = 2

$$\underline{B}_{3x1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \ \underline{C}_{3x1} = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p) = \frac{1!}{h^1}\left[-0.5f(x-h) + 0f(x) + 0.5f(x+h)\right] + O(h^2) .$$

$$f^1(x) \cong \frac{f(x+h) - f(x-h)}{2h}$$

Second derivative: d = 2, p = 1

$$\underline{B}_{3x1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \underline{C}_{3x1} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}, \text{ and}$$

[2]See Eberly (2008).

$$f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p) = \frac{2!}{h^2}\left[0.5f(x-h) - 1f(x) + 0.5f(x+h)\right] + O(h^1)$$

$$f^2(x) \cong \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Numerical derivative order of accuracy 2

In this case, $d + p = 5$ and we have $i_{max} = -i_{min} = 2$.

$$\underline{A}_{5x5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix} \text{ and } \underline{A}_{5x5}^{-1} = \begin{bmatrix} 0 & 1/12 & -1/24 & 1/12 & 1/24 \\ 0 & -2/3 & 2/3 & 1/6 & -1/6 \\ 1 & 0 & -5/4 & 0 & 1/4 \\ 0 & 2/3 & 2/3 & -1/6 & -1/6 \\ 0 & -1/12 & -1/24 & 1/12 & 1/24 \end{bmatrix}.$$

Identity equation: d = 0, p = 5

$$\underline{B}_{5x1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{5x1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p) = \frac{0!}{h^0} f(x+0h) = f(x).$$

First derivative: $d = 1$, $p = 4$

$$\underline{B}_{5x1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{5x1} = \begin{bmatrix} 1/12 \\ -2/3 \\ 0 \\ 2/3 \\ -1/12 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^1(x) = \frac{1!}{h^1}\left[\frac{f(x-2h)}{12} - \frac{2f(x-h)}{3} + 0f(x) + \frac{2f(x+h)}{3} - \frac{f(x+2h)}{12}\right] + O(h^4).$$

$$f^1(x) \cong \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

Second derivative: $d = 2$, $p = 3$

$$\underline{B}_{5x1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{5x1} = \begin{bmatrix} -1/24 \\ 2/3 \\ -5/4 \\ 2/3 \\ -1/24 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^2(x) = \frac{2!}{h^2} \left[-\frac{f(x-2h)}{24} + \frac{2f(x-h)}{3} - \frac{5f(x)}{4} + \frac{2f(x+h)}{3} - \frac{f(x+2h)}{24} \right] + O(h^3).$$

$$f^2(x) \cong \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2}$$

Third derivative: $d = 3$, $p = 2$

$$\underline{B}_{5x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{C}_{5x1} = \begin{bmatrix} -1/12 \\ 1/6 \\ 0 \\ -1/6 \\ 1/12 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^3(x) = \frac{3!}{h^3} \left[-\frac{f(x-2h)}{12} + \frac{f(x-h)}{6} + 0f(x) - \frac{f(x+h)}{6} + \frac{f(x+2h)}{12} \right] + O(h^2).$$

$$f^3(x) \cong \frac{f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)}{2h^3}$$

Fourth derivative: $d = 4$, $p = 1$

$$\underline{B}_{5x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \underline{C}_{5x1} = \begin{bmatrix} 1/24 \\ -1/6 \\ 1/4 \\ -1/6 \\ 1/24 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^4(x) = \frac{4!}{h^4} \left[\frac{f(x-2h)}{24} - \frac{f(x-h)}{6} + \frac{f(x)}{4} - \frac{f(x+h)}{6} + \frac{f(x+2h)}{24} \right] + O(h^1).$$

$$f^4(x) \cong \frac{f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)}{h^4}$$

Numerical derivative order of accuracy 3

In this case, $d + p = 7$ and we have $i_{max} = -i_{min} = 3$.

$$\underline{A}_{7x7} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 9 & 4 & 1 & 0 & 1 & 4 & 9 \\ -27 & -8 & -1 & 0 & 1 & 8 & 27 \\ 81 & 16 & 1 & 0 & 1 & 16 & 81 \\ -243 & -32 & -1 & 0 & 1 & 32 & 243 \\ 729 & 64 & 1 & 0 & 1 & 64 & 729 \end{bmatrix} \text{ and}$$

$$\underline{A}_{7x7}^{-1} = \begin{bmatrix} 0 & -1/60 & 1/180 & 1/48 & -1/144 & -1/240 & 1/720 \\ 0 & 3/20 & -3/40 & -1/6 & 1/12 & 1/60 & -1/120 \\ 0 & -3/4 & 3/4 & 13/48 & -13/48 & -1/48 & 1/48 \\ 1 & 0 & -49/36 & 0 & 7/18 & 0 & -1/36 \\ 0 & 3/4 & 3/4 & -13/48 & -13/48 & 1/48 & 1/48 \\ 0 & -3/20 & -3/40 & 1/6 & 1/12 & -1/60 & -1/120 \\ 0 & 1/60 & 1/180 & -1/48 & -1/144 & 1/240 & 1/720 \end{bmatrix}.$$

Identity equation: d = 0, p = 7

$$\underline{B}_{7x1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{7x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p) = \frac{0!}{h^0} f(x+0h) = f(x).$$

First derivative: $d = 1$, $p = 6$

$$\underline{B}_{7x1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{7x1} = \begin{bmatrix} -1/60 \\ 3/20 \\ -3/4 \\ 0 \\ 3/4 \\ -3/20 \\ 1/60 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^1(x) = \frac{1!}{h^1}\left[-\frac{f(x-3h)}{60} + \frac{3f(x-2h)}{20} - \frac{3f(x-h)}{4} + 0f(x) + \frac{3f(x+h)}{4} - \frac{3f(x+2h)}{20} + \frac{f(x+3h)}{60}\right] + O(h^6).$$

$$f^1(x) \cong \frac{f(x+3h) - 9f(x+2h) + 45f(x+h) - 45f(x-h) + 9f(x-2h) - f(x-3h)}{60h}$$

Second derivative: $d = 2$, $p = 5$

$$\underline{B}_{7x1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{7x1} = \begin{bmatrix} 1/180 \\ -3/40 \\ 3/4 \\ -49/36 \\ 3/4 \\ -3/40 \\ 1/180 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^2(x) = \frac{2!}{h^2}\begin{bmatrix} \frac{f(x-3h)}{180} - \frac{3f(x-2h)}{40} + \frac{3f(x-h)}{4} \\ -\frac{49f(x)}{36} + \frac{3f(x+h)}{4} - \frac{3f(x+2h)}{40} + \frac{f(x+3h)}{180} \end{bmatrix} + O(h^5).$$

$$f^2(x) \cong \frac{f(x+3h) - 13.5f(x+2h) + 135f(x+h) - 245f(x) + 135f(x-h) - 13.5f(x-2h) + f(x-3h)}{90h^2}$$

Third derivative: $d = 3$, $p = 4$

$$\underline{B}_{7x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{7x1} = \begin{bmatrix} 1/48 \\ -1/6 \\ 13/48 \\ 0 \\ -13/48 \\ 1/6 \\ -1/48 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^3(x) = \frac{3!}{h^3}\begin{bmatrix} \frac{f(x-3h)}{48} - \frac{f(x-2h)}{6} + \frac{13f(x-h)}{48} \\ +0f(x) - \frac{13f(x+h)}{48} + \frac{f(x+2h)}{6} - \frac{f(x+3h)}{48} \end{bmatrix} + O(h^4).$$

$$f^3(x) \cong \frac{-f(x+3h) + 8f(x+2h) - 13f(x+h) + 0f(x) + 13f(x-h) - 8f(x-2h) + f(x-3h)}{8h^3}$$

Fourth derivative: $d = 4$, $p = 3$

$$\underline{B}_{7x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{7x1} = \begin{bmatrix} -1/144 \\ 1/12 \\ -13/48 \\ 7/18 \\ -13/48 \\ 1/12 \\ -1/144 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^4(x) = \frac{4!}{h^4} \begin{bmatrix} -\dfrac{f(x-3h)}{144} + \dfrac{f(x-2h)}{12} - \dfrac{13f(x-h)}{48} \\ +\dfrac{7f(x)}{18} - \dfrac{13f(x+h)}{48} + \dfrac{f(x+2h)}{12} - \dfrac{f(x+3h)}{144} \end{bmatrix} + O(h^3).$$

$$f^4(x) \cong \frac{-f(x+3h)+12f(x+2h)-39f(x+h)+56f(x)-39f(x-h)+12f(x-2h)-f(x-3h)}{6h^4}$$

Fifth derivative: $d = 5$, $p = 2$

$$\underline{B}_{7x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{C}_{7x1} = \begin{bmatrix} -1/240 \\ 1/60 \\ -1/48 \\ 0 \\ 1/48 \\ -1/60 \\ 1/240 \end{bmatrix}, \text{ and}$$

$$f^d(x) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^p)$$

$$f^5(x) = \frac{5!}{h^5} \begin{bmatrix} -\dfrac{f(x-3h)}{240} + \dfrac{f(x-2h)}{60} - \dfrac{f(x-h)}{48} \\ +0f(x) + \dfrac{f(x+h)}{48} - \dfrac{f(x+2h)}{60} + \dfrac{f(x+3h)}{240} \end{bmatrix} + O(h^2).$$

$$f^5(x) \cong \frac{f(x+3h)-4f(x+2h)+5f(x+h)+0f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{2h^5}$$

Sixth derivative: $d = 6$, $p = 1$

$$\underline{B}_{7x1}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\end{bmatrix},\ \underline{C}_{7x1}=\begin{bmatrix}1/720\\-1/120\\1/48\\-1/36\\1/48\\-1/120\\1/720\end{bmatrix}, \text{ and}$$

$$f^d\left(x\right)=\frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}}C_i f\left(x+ih\right)+O\left(h^p\right)$$

$$f^6\left(x\right)=\frac{6!}{h^6}\begin{bmatrix}\frac{f\left(x-3h\right)}{720}-\frac{f\left(x-2h\right)}{120}+\frac{f\left(x-h\right)}{48}\\-\frac{f\left(x\right)}{36}+\frac{f\left(x+h\right)}{48}-\frac{f\left(x+2h\right)}{120}+\frac{f\left(x+3h\right)}{720}\end{bmatrix}+O\left(h^1\right).$$

$$f^6\left(x\right)\cong\frac{f\left(x+3h\right)-6f\left(x+2h\right)+15f\left(x+h\right)-20f\left(x\right)+15f\left(x-h\right)-6f\left(x-2h\right)+f\left(x-3h\right)}{h}$$

Numerical derivative order of accuracy 4

In this case, $d+p=9$ and we have $i_{max}=-i_{min}=4$.

$$\underline{A}_{9x9}=\begin{bmatrix}1&1&1&1&1&1&1&1&1\\-4&-3&-2&-1&0&1&2&3&4\\16&9&4&1&0&1&4&9&16\\-64&-27&-8&-1&0&1&8&27&64\\256&81&16&1&0&1&16&81&256\\-1{,}024&-243&-32&-1&0&1&32&243&1{,}024\\4{,}096&729&64&1&0&1&64&729&4{,}096\\-16{,}384&-2{,}187&-128&-1&0&1&128&2{,}187&16{,}384\\65{,}536&6{,}561&256&1&0&1&256&6{,}561&65{,}536\end{bmatrix} \text{ and}$$

$$\underline{A}_{9x9}^{-1}=\begin{bmatrix}0&1/280&-1/1{,}120&-7/1{,}440&7/5{,}760&1/720&-1/2{,}880&-1/10{,}080&1/40{,}320\\0&-4/105&4/315&1/20&-1/60&-1/80&1/240&1/1{,}680&-1/5{,}040\\0&1/5&-1/10&-169/720&169/1{,}440&13/360&-13/720&-1/720&1/1{,}440\\0&-4/5&4/5&61/180&-61/180&-29/720&29/720&1/720&-1/720\\1&0&-205/144&0&91/192&0&-5/96&0&1/576\\0&4/5&4/5&-61/180&-61/180&29/720&29/720&-1/720&-1/720\\0&-1/5&-1/10&169/720&169/1{,}440&-13/360&-13/720&1/720&1/1{,}440\\0&4/105&4/315&-1/20&-1/60&1/80&1/240&-1/1{,}680&-1/5{,}040\\0&-1/280&-1/1{,}120&7/1{,}440&7/5{,}760&-1/720&-1/2{,}880&1/10{,}080&1/40{,}320\end{bmatrix}.$$

Identity equation: $d=0$, $p=9$

$$\underline{B}_{9x1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{9x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } f^{d}(x) = \frac{d!}{h^{d}} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^{p}) = \frac{0!}{h^{0}} f(x+0h) = f(x).$$

First derivative: $d = 1$, $p = 8$

$$\underline{B}_{9x1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{9x1} = \begin{bmatrix} 1/280 \\ -4/105 \\ 1/5 \\ -4/5 \\ 0 \\ 4/5 \\ -1/5 \\ 4/105 \\ -1/280 \end{bmatrix}, \text{ and}$$

$$f^{d}(x) = \frac{d!}{h^{d}} \sum_{i=i_{\min}}^{i_{\max}} C_i f(x+ih) + O(h^{p})$$

$$f^{1}(x) = \frac{1!}{h^{1}} \begin{bmatrix} \frac{f(x-4h)}{280} - \frac{4f(x-3h)}{105} + \frac{f(x-2h)}{5} - \frac{4f(x-h)}{5} + 0f(x) \\ + \frac{4f(x+h)}{5} - \frac{f(x+2h)}{5} + \frac{4f(x+3h)}{105} - \frac{f(x+4h)}{280} \end{bmatrix} + O(h^{8}).$$

$$f^{1}(x) \cong \frac{1}{h} \begin{bmatrix} \frac{f(x-4h)}{280} - \frac{4f(x-3h)}{105} + \frac{f(x-2h)}{5} - \frac{4f(x-h)}{5} + 0f(x) \\ + \frac{4f(x+h)}{5} - \frac{f(x+2h)}{5} + \frac{4f(x+3h)}{105} - \frac{f(x+4h)}{280} \end{bmatrix}$$

Second derivative: $d = 2$, $p = 7$

$$\underline{B}_{9x1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{9x1} = \begin{bmatrix} -1/1{,}120 \\ 4/315 \\ -1/10 \\ 4/5 \\ -205/144 \\ 4/5 \\ -1/10 \\ 4/315 \\ -1/1{,}120 \end{bmatrix}, \text{ and}$$

$$f^{d}\left(x\right) = \frac{d!}{h^{d}} \sum_{i=i_{\min}}^{i_{\max}} C_{i} f\left(x+ih\right) + O\left(h^{p}\right)$$

$$f^{2}\left(x\right) = \frac{2!}{h^{2}} \left[\begin{array}{l} -\frac{f\left(x-4h\right)}{1{,}120} + \frac{4f\left(x-3h\right)}{315} - \frac{f\left(x-2h\right)}{10} + \frac{4f\left(x-h\right)}{5} - \frac{205f\left(x\right)}{144} \\ + \frac{4f\left(x+h\right)}{5} - \frac{f\left(x+2h\right)}{10} + \frac{4f\left(x+3h\right)}{315} - \frac{f\left(x+4h\right)}{1{,}120} \end{array} \right] + O\left(h^{7}\right).$$

$$f^{2}\left(x\right) \cong \frac{2}{h^{2}} \left[\begin{array}{l} -\frac{f\left(x-4h\right)}{1{,}120} + \frac{4f\left(x-3h\right)}{315} - \frac{f\left(x-2h\right)}{10} + \frac{4f\left(x-h\right)}{5} - \frac{205f\left(x\right)}{144} \\ + \frac{4f\left(x+h\right)}{5} - \frac{f\left(x+2h\right)}{10} + \frac{4f\left(x+3h\right)}{315} - \frac{f\left(x+4h\right)}{1{,}120} \end{array} \right]$$

Third derivative: $d = 3$, $p = 6$

$$\underline{B}_{9x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{C}_{9x1} = \begin{bmatrix} -7/1{,}440 \\ 1/20 \\ -169/720 \\ 61/180 \\ 0 \\ -61/180 \\ 169/720 \\ -1/20 \\ 7/1{,}440 \end{bmatrix}, \text{ and}$$

$$f^{d}\left(x\right)=\frac{d!}{h^{d}}\sum_{i=i_{\min}}^{i_{\max}}C_{i}f\left(x+ih\right)+O\left(h^{p}\right)$$

$$f^{3}\left(x\right)=\frac{3!}{h^{3}}\left[\begin{array}{l}-\frac{7f\left(x-4h\right)}{1,440}+\frac{f\left(x-3h\right)}{20}-\frac{169f\left(x-2h\right)}{720}+\frac{61f\left(x-h\right)}{180}+0f\left(x\right)\\-\frac{61f\left(x+h\right)}{180}+\frac{169f\left(x+2h\right)}{720}-\frac{f\left(x+3h\right)}{20}+\frac{7f\left(x+4h\right)}{1,440}\end{array}\right]+O\left(h^{6}\right).$$

$$f^{3}\left(x\right)\cong\frac{6}{h}\left[\begin{array}{l}-\frac{7f\left(x-4h\right)}{1,440}+\frac{f\left(x-3h\right)}{20}-\frac{169f\left(x-2h\right)}{720}+\frac{61f\left(x-h\right)}{180}+0f\left(x\right)\\-\frac{61f\left(x+h\right)}{180}+\frac{169f\left(x+2h\right)}{720}-\frac{f\left(x+3h\right)}{20}+\frac{7f\left(x+4h\right)}{1,440}\end{array}\right]$$

Fourth derivative: $d = 4$, $p = 5$

$$\underline{B}_{9x1}=\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix},\ \underline{C}_{9x1}=\begin{bmatrix}7/5,760\\-1/60\\169/1,440\\-61/180\\91/192\\-61/180\\169/1,440\\-1/60\\7/5,760\end{bmatrix}, \text{ and}$$

$$f^{d}\left(x\right)=\frac{d!}{h^{d}}\sum_{i=i_{\min}}^{i_{\max}}C_{i}f\left(x+ih\right)+O\left(h^{p}\right)$$

$$f^{4}\left(x\right)=\frac{4!}{h^{4}}\left[\begin{array}{l}\frac{7f\left(x-4h\right)}{5,760}-\frac{f\left(x-3h\right)}{60}+\frac{169f\left(x-2h\right)}{1,440}-\frac{61f\left(x-h\right)}{180}+\frac{91f\left(x\right)}{180}\\-\frac{61f\left(x+h\right)}{180}+\frac{169f\left(x+2h\right)}{1,440}-\frac{f\left(x+3h\right)}{60}+\frac{7f\left(x+4h\right)}{5,760}\end{array}\right]+O\left(h^{5}\right).$$

$$f\;\left(x\right)\cong\frac{24}{h}\left[\begin{array}{l}\frac{7f\left(x-4h\right)}{5,760}-\frac{f\left(x-3h\right)}{60}+\frac{169f\left(x-2h\right)}{1,440}-\frac{61f\left(x-h\right)}{180}+\frac{91f\left(x\right)}{180}\\-\frac{61f\left(x+h\right)}{180}+\frac{169f\left(x+2h\right)}{1,440}-\frac{f\left(x+3h\right)}{60}+\frac{7f\left(x+4h\right)}{5,760}\end{array}\right]$$

Fifth derivative: $d = 5$, $p = 4$

$$\underline{B}_{9x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{9x1} = \begin{bmatrix} 1/720 \\ -1/80 \\ 13/360 \\ -29/720 \\ 0 \\ 29/720 \\ -13/360 \\ 1/80 \\ -1/720 \end{bmatrix}, \text{ and}$$

$$f^d\left(x\right) = \frac{d!}{h^d} \sum_{i=i_{\min}}^{i_{\max}} C_i f\left(x+ih\right) + O\left(h^p\right)$$

$$f^5\left(x\right) = \frac{5!}{h^5} \left[\begin{array}{l} \frac{f\left(x-4h\right)}{720} - \frac{f\left(x-3h\right)}{80} + \frac{13f\left(x-2h\right)}{360} - \frac{29f\left(x-h\right)}{720} + 0f\left(x\right) \\ + \frac{29f\left(x+h\right)}{720} - \frac{13f\left(x+2h\right)}{360} + \frac{f\left(x+3h\right)}{80} - \frac{f\left(x+4h\right)}{720} \end{array} \right] + O\left(h^4\right).$$

$$f^5\left(x\right) \cong \frac{120}{h} \left[\begin{array}{l} \frac{f\left(x-4h\right)}{720} - \frac{f\left(x-3h\right)}{80} + \frac{13f\left(x-2h\right)}{360} - \frac{29f\left(x-h\right)}{720} + 0f\left(x\right) \\ + \frac{29f\left(x+h\right)}{720} - \frac{13f\left(x+2h\right)}{360} + \frac{f\left(x+3h\right)}{80} - \frac{f\left(x+4h\right)}{720} \end{array} \right]$$

Sixth derivative: $d = 6$, $p = 3$

$$\underline{B}_{9x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{C}_{9x1} = \begin{bmatrix} -1/2,880 \\ 1/240 \\ -13/720 \\ 29/720 \\ -5/96 \\ 29/720 \\ -13/720 \\ 1/240 \\ -1/2,880 \end{bmatrix}, \text{ and}$$

$$f^{d}\left(x\right)=\frac{d!}{h^{d}}\sum_{i=i_{\min}}^{i_{\max}}C_{i}f\left(x+ih\right)+O\left(h^{p}\right)$$

$$f^{6}\left(x\right)=\frac{6!}{h^{6}}\left[\begin{array}{l}-\frac{f\left(x-4h\right)}{2,880}+\frac{f\left(x-3h\right)}{240}-\frac{13f\left(x-2h\right)}{720}+\frac{29f\left(x-h\right)}{720}-\frac{5f\left(x\right)}{96}\\+\frac{29f\left(x+h\right)}{720}-\frac{13f\left(x+2h\right)}{720}+\frac{f\left(x+3h\right)}{240}-\frac{f\left(x+4h\right)}{2,880}\end{array}\right]+O\left(h^{3}\right).$$

$$f^{6}\left(x\right)\cong\frac{720}{h}\left[\begin{array}{l}-\frac{f\left(x-4h\right)}{2,880}+\frac{f\left(x-3h\right)}{240}-\frac{13f\left(x-2h\right)}{720}+\frac{29f\left(x-h\right)}{720}-\frac{5f\left(x\right)}{96}\\+\frac{29f\left(x+h\right)}{720}-\frac{13f\left(x+2h\right)}{720}+\frac{f\left(x+3h\right)}{240}-\frac{f\left(x+4h\right)}{2,880}\end{array}\right]$$

Seventh derivative: $d = 7$, $p = 2$

$$\underline{B}_{9x1}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix},\ \underline{C}_{9x1}=\begin{bmatrix}-1/10,080\\1/1,680\\-1/720\\1/720\\0\\-1/720\\1/720\\-1/1,680\\1/10,080\end{bmatrix}, \text{and}$$

$$f^{d}\left(x\right)=\frac{d!}{h^{d}}\sum_{i=i_{\min}}^{i_{\max}}C_{i}f\left(x+ih\right)+O\left(h^{p}\right)$$

$$f^{7}\left(x\right)=\frac{7!}{h^{7}}\left[\begin{array}{l}-\frac{f\left(x-4h\right)}{10,080}+\frac{f\left(x-3h\right)}{1,680}-\frac{f\left(x-2h\right)}{720}+\frac{f\left(x-h\right)}{720}+0f\left(x\right)\\-\frac{f\left(x+h\right)}{720}+\frac{f\left(x+2h\right)}{720}-\frac{f\left(x+3h\right)}{1,680}+\frac{f\left(x+4h\right)}{10,080}\end{array}\right]+O\left(h^{2}\right).$$

$$f^{7}\left(x\right)\cong\frac{5,040}{h}\left[\begin{array}{l}-\frac{f\left(x-4h\right)}{10,080}+\frac{f\left(x-3h\right)}{1,680}-\frac{f\left(x-2h\right)}{720}+\frac{f\left(x-h\right)}{720}+0f\left(x\right)\\-\frac{f\left(x+h\right)}{720}+\frac{f\left(x+2h\right)}{720}-\frac{f\left(x+3h\right)}{1,680}+\frac{f\left(x+4h\right)}{10,080}\end{array}\right]$$

Eighth derivative: $d = 8$, $p = 1$

$$\underline{B}_{9x1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \underline{C}_{9x1} = \begin{bmatrix} 1/40{,}320 \\ -1/5{,}040 \\ 1/1{,}440 \\ -1/720 \\ 1/576 \\ -1/720 \\ 1/1{,}440 \\ -1/5{,}040 \\ 1/40{,}320 \end{bmatrix}, \text{ and}$$

$$f^d\left(x\right) = \frac{d!}{h^d}\sum_{i=i_{\min}}^{i_{\max}} C_i f\left(x+ih\right) + O\left(h^p\right)$$

$$f^8\left(x\right) = \frac{8!}{h^8}\begin{bmatrix} \dfrac{f\left(x-4h\right)}{40{,}320} - \dfrac{f\left(x-3h\right)}{5{,}040} + \dfrac{f\left(x-2h\right)}{1{,}440} - \dfrac{f\left(x-h\right)}{720} + \dfrac{f\left(x\right)}{576} \\ -\dfrac{f\left(x+h\right)}{720} + \dfrac{f\left(x+2h\right)}{1{,}440} - \dfrac{f\left(x+3h\right)}{5{,}040} + \dfrac{f\left(x+4h\right)}{40{,}320} \end{bmatrix} + O\left(h^1\right).$$

$$f^8\left(x\right) \cong \frac{40{,}320}{h}\begin{bmatrix} \dfrac{f\left(x-4h\right)}{40{,}320} - \dfrac{f\left(x-3h\right)}{5{,}040} + \dfrac{f\left(x-2h\right)}{1{,}440} - \dfrac{f\left(x-h\right)}{720} + \dfrac{f\left(x\right)}{576} \\ -\dfrac{f\left(x+h\right)}{720} + \dfrac{f\left(x+2h\right)}{1{,}440} - \dfrac{f\left(x+3h\right)}{5{,}040} + \dfrac{f\left(x+4h\right)}{40{,}320} \end{bmatrix}$$

It is possible to extent this analysis to cross derivatives as there is a theorem related to numerical derivatives approximation theorem for *d*th order partial cross derivatives. Although not covered here, the Table 7.1.1 identifies some other partial derivatives that may be considered.

Table 7.1.1. Selected Derivatives

Parameter	**S**	**t**	**σ**	**r**	**X**	**δ**
			First Order Derivatives			
Value (O)	Delta (Δ)	Theta (θ)	Vega (ν)	Rho (ρ)	dOdX	dOdδ
Elasticity	Lambda (λ)	O%t%	O%σ%	O%r%	O%X%	O%δ%
			Selected Second Order Derivatives			
Delta (Δ)	Gamma (Γ)	Charm	Vanna	dΔdr	dΔdX	dΔdδ
Theta (θ)		dθdt	Veta	dθdr	dθdX	dθdδ
Vega (ν)			Vomma	Ver	dνdX	dνdδ
Rho (ρ)				dρdr	dρdX	dρdδ
dOdX					(dOdX)dX	(dOdX)dδ
DdOδ						(dOδd)dδ
			Selected Third Order Derivatives			
Gamma (Γ)	Speed ()	Color ()	Zomma ()	dΓdr	dΓdX	dΓdδ
dθdt		(dθdt)dt	(dθdt)dσ	(dθdt)dr	(dθdt)dX	(dθdt)dδ
Vomma			Ultima ()	dVommadr	dVommadX	dVommadδ
dρdr				(dρdr)dr	(dρdr)dX	(dρdr)dδ
…						

Summary

In this module, we illustrated the centered difference technique applied to the computation of mathematical derivatives. As you have seen, the analytics were rather technical, however, the overall objective was clear. We sought to numerically estimate the *n*th derivative of some function without having to analytically derive it. We illustrated centered differencing approach developed here with bond valuation.

References

Eberly, D. 2008. Derivative Approximation by Finite Differences. http://www.geometrictools.com/, March 2, 2008.