Module 7.2

Static Risk Measures
US Treasuries

1

Overview

- Review traditional bond risk measures
- Role of compounding
  - Holding period returns (HPR)
  - Bond valuation
- Review selected empirical evidence
- Introduce LSC-based bond risk measures based on HPRs
- Illustrate selected LSC applications

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Central Finance Concepts

- Origins of bond risk management
- Traditional bond SRM definitions
- Immunization

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BRM History

- Lidstone (1895) earliest know writings on duration concept
- Macaulay (1938) defined duration, a measure of 'longness'.
- Redington (1952) introduced immunization

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Traditional Bond Duration

- Macaulay duration: Present value weighted average of time to cash flows
- Modified duration: Percentage change in the bond price (or portfolio) for a given change in the yield to maturity
- Effective duration: Cash flow adjusted percentage change in the bond price (or portfolio) for a given change in the yield to maturity

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Traditional Bond Convexity

- Standard convexity: Measures the curvature of the price-yield relationship
- Effective convexity: Cash flow adjusted measure of the curvature of the price-yield relationship

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Hicks Elasticity Measure(1939)

- Early thoughts related to immunization:

"... is the *average length of time for which the various payments are deferred from the present, when the time of deferment are weighted by the discounted values of the payments*."

Early Immunization Insights

- Samuelson (1945)
  - *Increased interest rates will help any organization whose (weighted) average time period of disbursements is greater than the average time period of its receipts*
  - According to Poitras (2006), Samuelson's work was "... an extension of Hicks (1939) and an anticipation of Redington (1952)."

Redington's Immunization (1952)

- "... investment of the assets in such a way that the existing business is immune to a general change in the rate of interest"
- "... first derivative is the most important for small changes in the rate of interest..."
- In different terms, noted that asset convexity needs to exceed liability convexity

Holding Period Returns

Now consider a four-year, 10% annual coupon-paying bond with a yield to maturity of 8.5%. The current bond price is $104.913, and the duration is 3.5 years. Also suppose that there were two other candidate four-year bonds to purchase, 20% coupon bond priced at $137.669 with a duration of 3.24 years and a zero coupon bond priced at $72.157 with a duration of 4.0 years. The bond that immunizes the portfolio assuming the desired holding period is 3.5 years is the 10% coupon bond. The 20% coupon bond has net reinvestment rate risk and hence will suffer if rates fall, and the zero-coupon bond has net price risk and hence will suffer if rates rise.

Interest Rate	Zero Coupon Bond	10% Coupon Bond	20% Coupon Bond
3%	9.31%	8.54%	8.13%
5%	9.01%	8.52%	8.25%
7%	8.72%	8.50%	8.39%
8.5%	8.50%	8.50%	8.50%
10%	8.29%	8.50%	8.62%
12%	8.01%	8.51%	8.78%
14%	7.74%	8.54%	8.95%

Note that if my horizon is 3.5 years, the 10% coupon bond has virtually no sensitivity to changes in interest rates.

Quantitative Finance Materials

- Traditional valuation and SRMs
- Discrete versus continuous compounding
  - Valuation
  - Holding period returns
- Empirical evidence
- Embedded optionality
- LSC model and performance

Selected Notation

- *Coupon* – annual dollar coupon
- *m* – coupons per year
- *Par* – notional amount (principal)
- *f* – fraction of payment period elapsed since last coupon (*NAD*/*NTD*)
- *N* – number of remaining cash flows
- CF_i – *i*th cash flow, $i = n$: $CF_i = (Coupon/m) + Par$, otherwise $CF_i = (Coupon/m)$

Traditional Valuation

- Fixed rate bond valuation with discretely compounded discounting

$$V_{B,y} = \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right) Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}}$$

$$= \sum_{i=1}^{N} \frac{CF_i}{\left(1+\frac{y}{m}\right)^{i-f}}$$

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Traditional Risk Measures

- Modified duration: $ModDur_{B,y} \equiv -\frac{dV_{B,y}/V_B}{dy}$

$$= \frac{1}{m}\left[\sum_{i=1}^{N} \frac{(i-f)CF_i}{\left(1+\frac{y}{m}\right)^{i+1-f}}\right]\frac{1}{V_B}$$

- Standard convexity

$$Convexity_{B,y} \equiv \frac{1}{V_B}\frac{d^2V_{B,y}}{dy^2}$$

$$= \frac{1}{m^2}\left[\sum_{i=1}^{N} \frac{(i-f)(i+1-f)CF_i}{\left(1+\frac{y}{m}\right)^{i+2-f}}\right]\frac{1}{V_B}$$

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Macaulay Duration Code

```
#
# Duration: Macaulay duration
#
Duration = function(B){
  with(B,{
    DV = 0.0
    RemainingCoupons = CouponsRemaining(B)
    ElapsedTime = FractionElapsed(B)
    for(i in 1:RemainingCoupons){
      DV = DV + (i - ElapsedTime)*( (CouponRate/(Frequency*100.0))*Par ) /
        ((1.0 + (YieldToMaturity/(Frequency*100.0)))^(i + 1 - ElapsedTime))
    }
    DV = DV + ((RemainingCoupons - ElapsedTime) * Par) /
      ((1.0 + (YieldToMaturity /
                  (Frequency*100.0)))^(RemainingCoupons + 1 - ElapsedTime))
    DV = DV / (Frequency*BondValue(B))
    return( DV )
  })
}
```

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Standard Convexity Code

```
#
# Convexity: Standard convexity
#
Convexity = function(B){
  with(B,{
    Convexity = 0.0
    RemainingCoupons = CouponsRemaining(B)
    ElapsedTime = FractionElapsed(B)
    for(i in 1:RemainingCoupons){
      Convexity = Convexity + ( (i + 1 - ElapsedTime)*(i - ElapsedTime) *
                                 ( (CouponRate/(Frequency*100.0))*Par) ) /
        ((1.0 + (YieldToMaturity/(Frequency*100.0)))^(i + 2 - ElapsedTime))
    }
    Convexity = Convexity + ((RemainingCoupons + 1 - ElapsedTime) *
                             (RemainingCoupons - ElapsedTime) * Par) /
      ((1.0 + (YieldToMaturity /
                 (Frequency*100.0)))^(RemainingCoupons + 2 - ElapsedTime))
    Convexity = Convexity/( (Frequency^2) * BondValue(B))
    return( Convexity )
  })
}
```


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Effective Duration/Convexity

- Effective duration

$$EffDur = \frac{V_{B-} - V_{B+}}{2V_B S}$$

- Effective convexity

$$EffConv = \frac{V_{B-} + V_{B+} - 2V_B}{V_B S^2}$$

- Cash flow adjusted measures
- Not addressed in this module

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Continuous Comp. Valuation

- Continuous compounding version

$$V_{B,r} = \sum_{i=1}^{N}\left(\frac{CR}{m}\right)Par\left(e^{-r\tau_i}\right) + Par\left(e^{-r\tau_N}\right)$$

$$= \sum_{i=1}^{N} CF_i e^{-r\tau_i}$$

- Discounting based on continuously compounded rate
- Compounding convention arbitrary

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Macaulay Duration

- Present value weighted average of time to cash flows

$$MacDur_{B,r} \equiv \sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B}\tau_i$$
$$= \sum_{i=1}^{N} w_i \tau_i$$

- w_i – proportion of *PV* cash flow *i* of V_B
- Note if zero coupon bond, then *MacDur* equals time to maturity

Hicks Elasticity Measure(1939)

- Recall "... is the *average length of time for which the various payments are deferred from the present, when the time of deferment are weighted by the discounted values of the payments.*"

$$HE_B \equiv \frac{dV_B / V_B}{d\delta / \delta} = \frac{dV_B}{d\delta}\frac{\delta}{V_B} \qquad \delta = e^{-r}$$
$$= -\left(\sum_{i=1}^{N} \tau_i CF_i \delta^{\tau_i - 1}\right)\frac{\delta}{V_B} = -\left(\sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B}\tau_i\right)$$
$$= -\sum_{i=1}^{N} w_i \tau_i = -ModDur_B$$

Redington's Taylor Series

- Taylor Series approximation of bond value for a given change in the continuously compounded discount rate

$$\hat{V}_{B,r} = \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}(\hat{r} - r)^i = \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}\Delta r^i$$
$$\cong V_{B,r} + \left(\frac{\partial V_{B,r}}{\partial r}\right)\Delta r + \frac{1}{2}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right)\Delta r^2$$

SRM, Continuous Comp.

- Modified duration

$$ModDur_{B,r} \equiv -\frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right) = -\frac{\partial V_{B,r} / V_B}{\partial r}$$

Measure of volatility: Percentage change in bond value for given change in rate

- Standard convexity

$$Convexity_B \equiv \frac{1}{V_B}\left(\frac{\partial^2 V_B}{\partial r^2}\right)$$

Discretely Comp. HPRs

- Discretely compounded holding period returns (dc) with continuously compounded yield (r)

$$R_{B,dc} \equiv \frac{\hat{V}_{B,r} - V_B}{V_B} = \frac{1}{V_B}\sum_{i=1}^{\infty}\frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}(\hat{r} - r)^i = \frac{1}{V_B}\sum_{i=1}^{\infty}\frac{1}{i!}\frac{\partial V_{B,r}^i}{\partial r^i}\Delta r^i$$
$$\cong \frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right)\Delta r + \frac{1}{2}\frac{1}{V_B}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right)\Delta r^2$$
$$= -ModDur_{B,r}\Delta r + \frac{1}{2}Convexity_{B,r}\Delta r^2$$

Modified Duration (r)

- Modified duration based on continuously compounded discount rates (r)

$$ModDur_{B,r} \equiv -\frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right)$$
$$= \sum_{i=1}^{N}\frac{CF_i e^{-r\tau_i}}{V_B}\tau_i$$
$$= \sum_{i=1}^{N} w_i \tau_i$$

Standard Convexity (r)

- Standard convexity based on continuously compounded discount rates (r)

$$Convexity_{B,r} \equiv \frac{1}{V_B}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right) = \sum_{i=1}^{N} \frac{CF_i e^{-r\tau_i}}{V_B}\tau_i^2 = \sum_{i=1}^{N} w_i \tau_i^2$$

Log Value Taylor Series

- Taylor Series approximation of the natural log of bond value for a given change in the continuously compounded discount rate (used next)

$$\ln\left(\hat{V}_{B,r}\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\left(\hat{r} - r\right)^i = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\Delta r^i$$

$$\cong \ln\left(V_{B,r}\right) + \frac{\partial \ln\left(V_{B,r}\right)}{\partial r}\Delta r + \frac{1}{2}\frac{\partial \ln^2\left(V_{B,r}\right)}{\partial r^2}\Delta r^2$$

Continuously Comp. HPRs

- Continuously compounded holding period returns (cc) with continuously compounded yield (r)

$$R_{B,cc} \equiv \ln\left(\frac{\hat{V}_{B,r}}{V_B}\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\left(\hat{r} - r\right)^i = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \ln\left(V_{B,r}\right)}{\partial r^i}\Delta r^i$$

$$\cong \frac{1}{V_B}\left(\frac{\partial V_{B,r}}{\partial r}\right)\Delta r + \frac{1}{2}\left[\frac{1}{V_B}\left(\frac{\partial^2 V_{B,r}}{\partial r^2}\right) + \left(\frac{-1}{V_B^2}\right)\left(\frac{\partial V_{B,r}}{\partial r}\right)^2\right]\Delta r^2$$

$$= -ModDur_{B,r}\Delta r + \frac{1}{2}\left(Convexity_{B,r} - ModDur_{B,r}^2\right)\Delta r^2$$

Bond Holding Period Returns

- Discretely compounded HPR

$$R_{B,dc} \equiv \frac{\hat{V}_{B,r} - V_B}{V_B}$$

- Continously compounded HPR

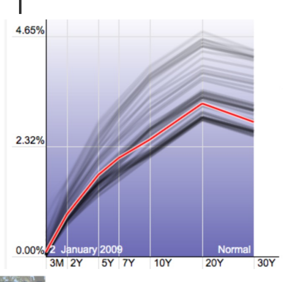
$$R_{B,cc} \equiv \ln\left(\frac{\hat{V}_{B,r}}{V_B}\right)$$

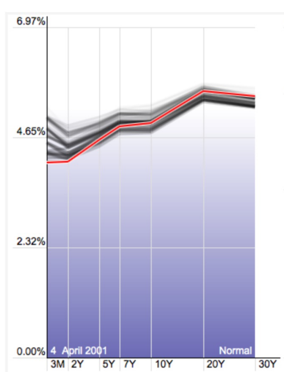
- Both bond valuation (discrete or continuous) and HPR (discrete or continuous) influence BRMs

Empirical Evidence

- Barber (1995) demonstrates that approximating continuously compounded rates of return using duration only or duration and convexity is much more accurate than the more traditional discretely compounded rates of return.
- Curve shifts involve level, slope, and curvature, several authors introduce more complex models.

YC Shifts Often Non-Parallel





Empirical Evidence

- Fisher and Weil (1971) conclude that the reductions in a bond portfolio risk measure "... are so dramatic that we conclude that a properly chosen portfolio of long-term bonds is essentially riskless."
- Several authors introduced more complex models as parallel shifts inadequate

Duration as Risk Measure

Yield to Maturity	9% Coupon Bond Price	11% Coupon Bond Price	9% Coupon % Change in Price	11% Coupon %Change in Price
7.0%	\$121.36	\$142.71	***10.43%****	***10.04%***
8.0%	\$109.90	\$129.69	***9.90%***	***9.53%***
9.0%	\$100.00	\$118.40	9.38%	9.05%
10.0%	\$91.42	\$108.58	***8.89%***	8.58%
11.0%	\$83.95	\$100.00	***8.42%***	8.14%
12.0%	\$77.43	\$92.48		

* 10.43% = (121.36 - 109.90)/109.90

Lower coupon bonds are more sensitive to changes in yield to maturity. Note the dollar change (7-8%) is 11.46 (9%) and 13.02 (11%), but the dollar investment required is 121.36 (9%) and 142.71 (11%). Higher dollar investment results in lower holding period returns.

Bond SRM Calculations

1	2	3	4	5	6
Maturity	**Cash Flow**	**PV(CF)**	**w(t)**	**t*w(t)**	**t*(t+1)*w(t)**
1	10	9.091	0.091	0.091	0.182
2	10	8.264	0.083	0.165	0.496
3	10	7.513	0.075	0.225	0.902
4	110	75.131	0.751	3.005	15.026
	Sum	100.000	1.000	3.487	16.606
				Convexity	***13.724***
				Duration	***3.487***
				Modified Duration	***3.170***

Note: Convexity = $16.606/(1^2(1+0.1)^2) = 13.724$, Duration = sum(t*w(t)) = 3.487, and Modified Duration = $3.487/(1 + 0.1) = 3.170$.

Effective Duration and Convexity

- Cash Flow Adjusted Volatility

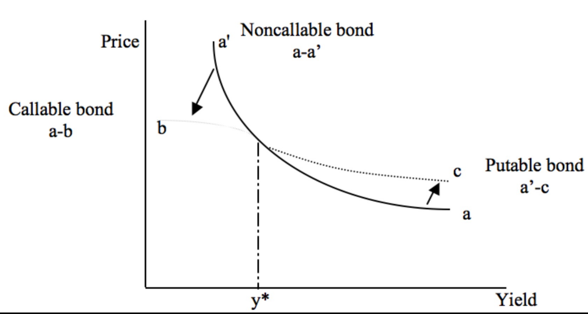
$$\text{Effective Duration} = \frac{(V_{B-}) - (V_{B+})}{2V_B S} \quad \text{Effective Convexity} = \frac{(V_{B-}) + (V_{B+}) - 2V_B}{V_B S^2}$$

Table 1. Effective Duration and Effective Convexity

	Bond X Callable		Bond Y Putable		Bond Z Straight	
Shift	**EffDur***	**EffCon***	**EffDur**	**EffCon**	**EffDur**	**EffCon**
-250	0.5	0.5	4.4	22.5	4.4	22.5
+250	4.2	21.0	0.5	0.5	4.2	21.0

- EffDur denotes effective duration and EffCon denotes effective convexity

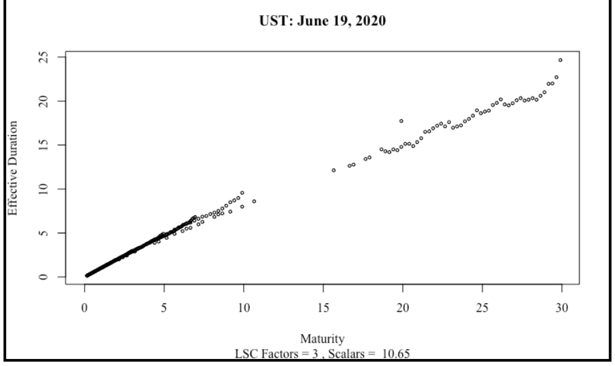
Embedded Optionality



Effective Duration Code

```
#
# Effective Duration
#
EffectiveDuration = function(B){
  OriginalYTM = B$YieldToMaturity
  OriginalBV = BondValue(B)
  B$YieldToMaturity = OriginalYTM + B$ChangeInYTM
  UpBV = BondValue(B)
  B$YieldToMaturity = OriginalYTM - B$ChangeInYTM
  DownBV = BondValue(B)
  B$YieldToMaturity = OriginalYTM
  EffDur = (DownBV - UpBV)/(2.0*OriginalBV*(B$ChangeInYTM/100.0))
  return( EffDur )
}
```


Note that effective duration equals modified duration when there are no cash flow effects. Clearly, modified duration is directly related to maturity with actual bond data.



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Effective Convexity Code

```
#
# Effective Convexity
#
EffectiveConvexity = function(B){
  OriginalYTM = B$YieldToMaturity
  OriginalBV = BondValue(B)
  B$YieldToMaturity = OriginalYTM + B$ChangeInYTM
  UpBV = BondValue(B)
  B$YieldToMaturity = OriginalYTM - B$ChangeInYTM
  DownBV = BondValue(B)
  B$YieldToMaturity = OriginalYTM
  Num = log((DownBV - OriginalBV) - (OriginalBV - UpBV))
  Den = -log(OriginalBV) - 2.0*log(B$ChangeInYTM/100.0)
  EffConv = exp(Num + Den)
  return( EffConv )
}
```


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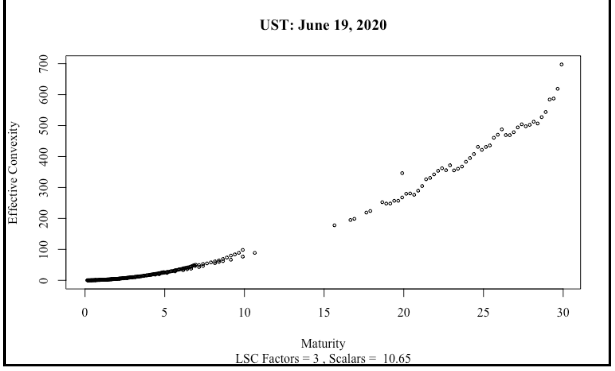
Again note that effective convexity equals standard convesity when there are no cash flow effects. Clearly, standard convexity is directly related to maturity with actual bond data and have positive convexity.



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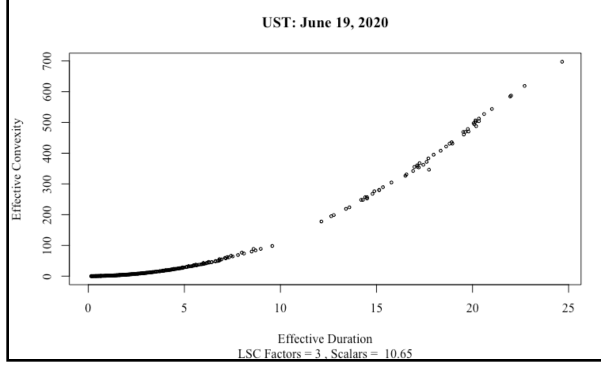
Amazing smooth relationship between duration and convexity with market data.



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Yield Curve Changes

- Observe numerous different types of changes in yield curve over time
- Changes are not just parallel as assumed by traditional risk measures
  - Duration
  - Convexity
- Next several slide illustrate actual changes

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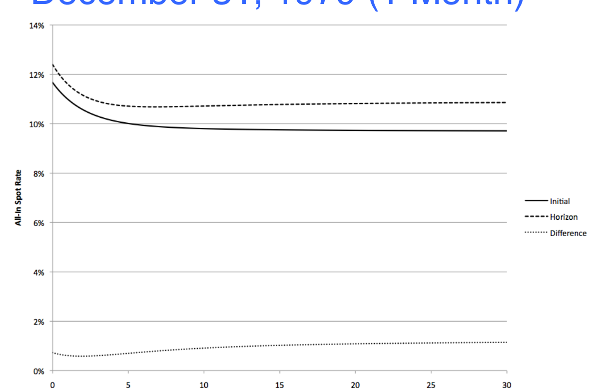
December 31, 1979 (1 Month)



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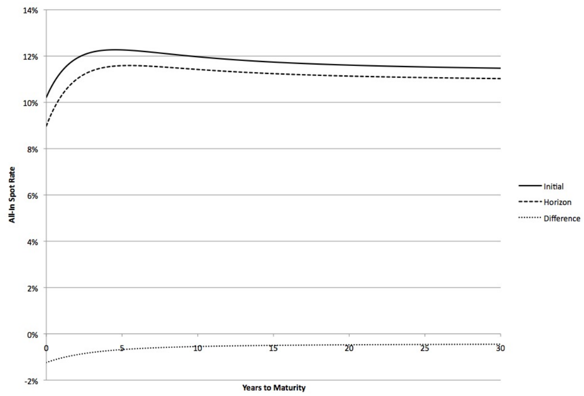
September 28, 1984 (1 Month)



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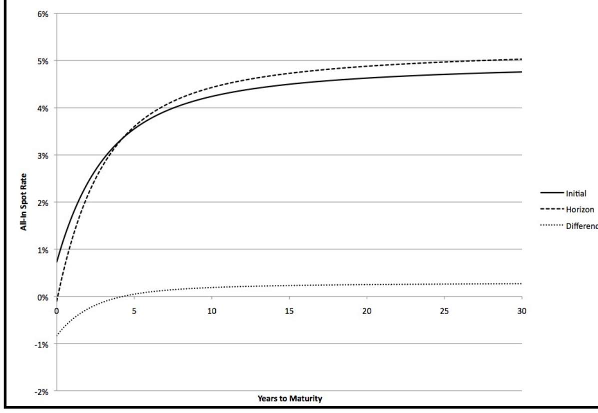
September 30, 2008 (1 Month)



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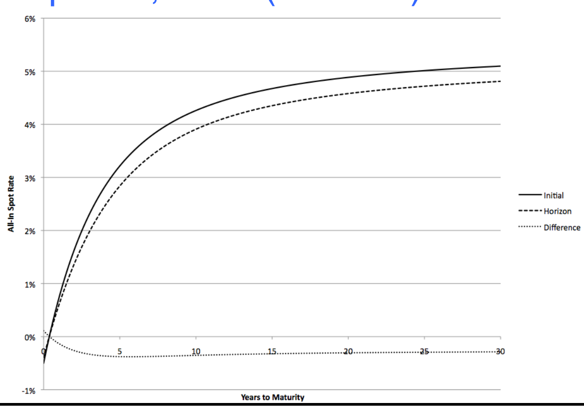
April 30, 2010 (1 Month)



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LSC Model and Performance

- Need mechanism to compare similar debt instruments that vary solely by maturity
- Yield to maturity and parallel shifts have proven inadequate

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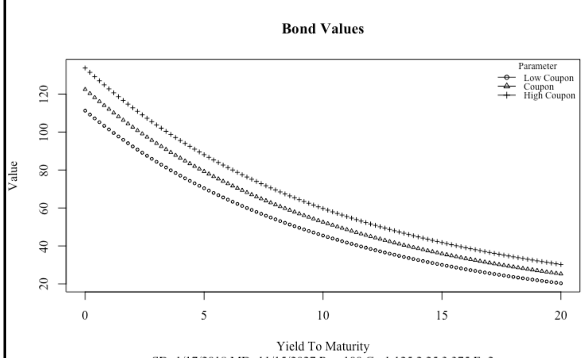

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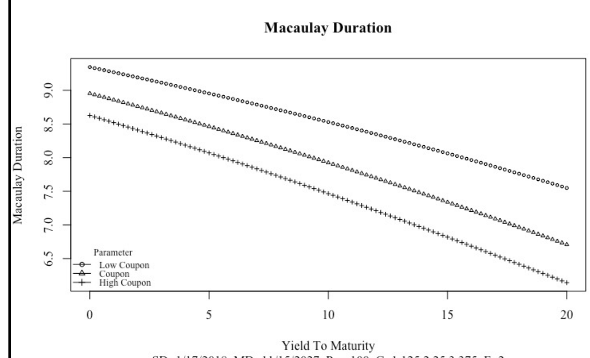

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HPR Decomposition w/ LSC

- Introduce new tools for clarifying interest rate-related financial performance
- Tools can be used either for ex-ante or ex-post analysis
- Illustrate with U. S. Treasury data

Source: Brooks and Upton, "Bond Portfolio Holding Period Return Decomposition," *Journal of Investing*, 2017, 78-90.

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Overview

- Bond holding period returns (HPRs) are decomposed into four main components
  - **Horizon component:** Passage of time
  - **Spread component:** Change in the fitted spread curve
  - **Base-rate component:** Change in the fitted base spot curve
  - **Interaction component** contains the residual HPRs

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Base-Rate Decomposition

- Modified duration
  - Level, Slope, Curvature
- Convexity
  - Level, Slope, Curvature
- Cross-convexity
  - Level-Slope
  - Slope-Curvature
  - Level-Curvature

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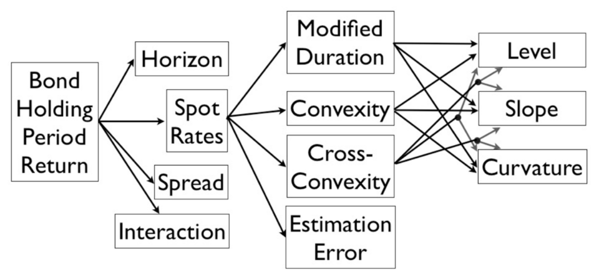
Spot Rate Decomposition



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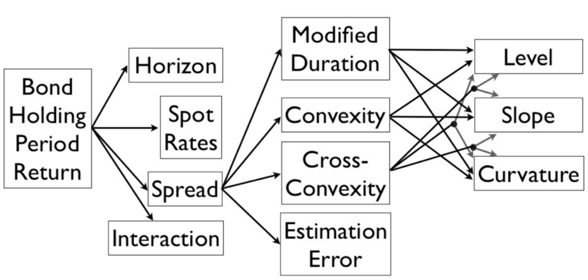
Spread Decomposition



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Conceptual Illustration

- Two-year bond, UST + 100 basis points (level = 4%, slope = –3.5%, curvature = –6%)
- Investment horizon is one month
- Bond HPR decomposed assuming at horizon:
  - UST + 200 basis points over the horizon spot rate curve
  - Level = 3%, slope = 1.5%, curvature = 1%

Based on U.S. Treasury data in February 2012

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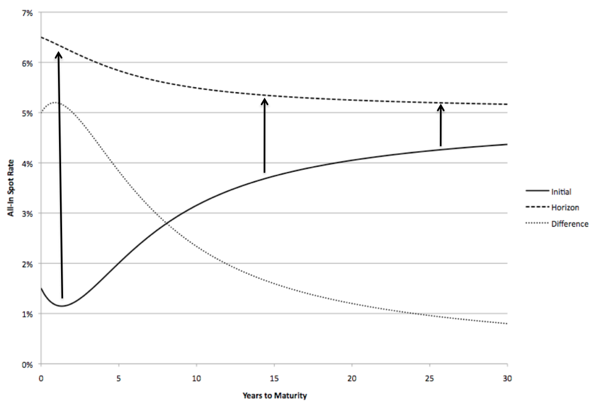


Exhibit 3. Illustration of Bond Holding Period Return Decomposition

2-Year Bond	**Decomposition**	**Spot Rates**	**Total**	**Level**	**Slope**	**Curvature**
Horizon	0.1174%	Duration	-7.6762%	1.9026%	-6.1252%	-3.4535%
Spot Rates	-7.6737%	Convexity	0.0001%	0.0001%	0.0000%	0.0000%
Spread	-1.9023%	Cross-Convexity	0.0001%	-0.0003%	0.0006%	-0.0002%
Interaction	0.0008%	Estimation Error	0.0022%			
Total	-9.4578%		-7.6738%	1.9024%	-6.1246%	-3.4537%
10-Year Bond	**Decomposition**	**Spot Rates**	**Total**	**Level**	**Slope**	**Curvature**
Horizon	0.3442%	Duration	-12.9484%	8.6311%	-9.4448%	-12.1347%
Spot Rates	-12.8795%	Convexity	0.0355%	0.0342%	0.0004%	0.0010%
Spread	-8.5246%	Cross-Convexity	-0.0340%	-0.0162%	0.0210%	-0.0388%
Interaction	0.0478%	Estimation Error	0.0673%			
Total	-21.0122%		-12.8796%	8.6491%	-9.4234%	-12.1725%
30-Year Bond	**Decomposition**	**Spot Rates**	**Total**	**Level**	**Slope**	**Curvature**
Horizon	0.3433%	Duration	-3.7654%	17.8514%	-9.3861%	-12.2307%
Spot Rates	-4.1976%	Convexity	0.5628%	0.5610%	0.0005%	0.0013%
Spread	-16.1595%	Cross-Convexity	-0.1536%	-0.0483%	0.0261%	-0.1314%
Interaction	-0.7387%	Estimation Error	-0.8414%			
Total	-20.7525%		-4.1976%	18.3641%	-9.3595%	-12.3608%

Fitted Term Structure Models

- Calendar time
  - Crack and Nawalkha [2000]: "Up to 95 percent of the returns to U. S. Treasury security portfolios are explained by term-structure level shifts, slope shifts, and curvature shifts" (34)
- Maturity time
  - Non-stochastic shape of the term structure at a particular point in calendar time

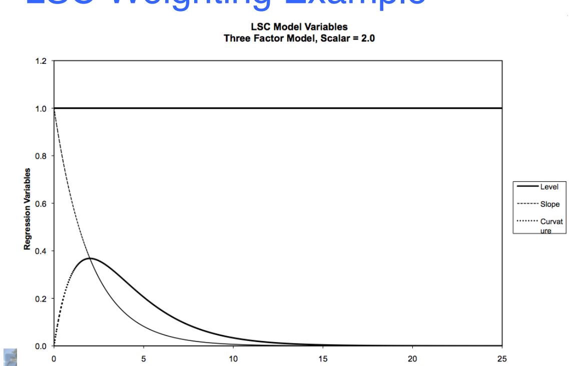
LSC Model

- Willner (1996) posits that the desirable properties of a curve fitting routine must address the bond "portfolio manager's need for *intuitive*, *descriptive*, and *comprehensive* risk exposure information."
- Generalized and parsimonious model

LSC Model

- Linear factors: $y_i = \sum_{j=0}^{N} x_{i,j} f_j$
- Level: $x_{i,0} = 1$ Slope: $x_{i,1} = \frac{s_1}{\tau_i}\left(1 - e^{-\tau_i/s_1}\right)$
- Curvatures: $x_{i,j} = \frac{s_j}{\tau_i}\left(1 - e^{-\tau_i/s_j}\right) - e^{-\tau_i/s_j}; j > 1$
- LSC model has the lowest "average (across the sample) mean (across the curve) absolute yield error" (Steeley) when compared with splines, polynomials and Vasicek's model

LSC Weighting Example



LSC Model Properties

- As maturity goes to infinity, then $y_i = f_0$
- As maturity goes to zero, then $y_i = f_0 + f_1$
- If the interest rate term structure is upward sloping, then f_1 is negative.
- Spot rate factors greater than one measure the curvature. Higher values lead to flatter slopes and lower values lead to steeper slopes.

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Bond Return Decomposition

- Bond Value Today: $V_t = \sum_{i=1}^{N_t} CF_{i,t} e^{-\left(\sum_{j=0}^{N^r} x_{i,j} f_j^r + \sum_{j=0}^{N^{sp}} x_{i,j} f_j^{sp} + \varepsilon_{i,t}\right)\tau_i}$
- Bond Value Later: $\tilde{V}_{t+\Delta}^{LSC} = \sum_{i=0}^{N_{t+\Delta}} CF_{i,t+\Delta} e^{-\left(\tilde{r}_{i,t+\Delta}^{LSC} + s\tilde{p}_{i,t+\Delta}^{LSC}\right)(\tau_i - \Delta)}$
- Bond Holding Period Return Decomposition:

$$\tilde{R}_\Delta = \ln\left[\frac{\tilde{V}_\Delta(\tilde{r}_\Delta, \tilde{s}_\Delta)}{V(r,s)}\right] = \ln\left[\frac{V_\Delta(r,s)}{V(r,s)}\right] + \ln\left[\frac{\tilde{V}_\Delta(r,\tilde{s}_\Delta)}{V_\Delta(r,s)}\right] + \ln\left[\frac{\tilde{V}_\Delta(\tilde{r}_\Delta, s)}{V_\Delta(r,s)}\right] + \ln\left[\frac{\tilde{V}_\Delta(\tilde{r}_\Delta,\tilde{s}_\Delta)/\tilde{V}_\Delta(\tilde{r}_\Delta,s)}{\tilde{V}_\Delta(r,\tilde{s}_\Delta)/\tilde{V}_\Delta(r,s)}\right]$$
$$= R_\Delta^h + \tilde{R}_\Delta^{sp} + \tilde{R}_\Delta^r + \tilde{I}_\Delta$$

- Unknown HPR Decomposition:

$$\tilde{R}_\Delta^{Unknown} = \tilde{R}_\Delta - R_\Delta^h = \tilde{R}_\Delta^{sp} + \tilde{R}_\Delta^r + \tilde{I}_\Delta^I$$

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Bond HPR Measures

$$\tilde{R}_\Delta \equiv \frac{\tilde{V}_{t+\Delta} - V_t}{V_t}, \text{ (Bond HPR)}$$
$$\tilde{R}_\Delta^{Unknown} \equiv \tilde{R}_\Delta^{LSC} \equiv \frac{\tilde{V}_{t+\Delta}^{LSC} - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ (Unknown HPR)}$$
$$R_\Delta^{Known} \equiv \frac{V_{t+\Delta}^{LSC} - V_t^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ (Known HPR)}$$
$$R_\Delta^h \equiv \frac{V_{t+\Delta}^{LSC} - V_t^{LSC}}{V_t^{LSC}}, \text{ (Bond Horizon HPR)}$$

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Bond HPR Measures (Cont')

$$\tilde{R}_\Delta^r \equiv \frac{\tilde{V}_{t+\Delta}^r - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ (Bond Base Rate HPR)}$$
$$\tilde{R}_\Delta^{sp} \equiv \frac{\tilde{V}_{t+\Delta}^{sp} - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}}, \text{ and (Bond Spread HPR)}$$
$$\tilde{I}_\Delta \equiv \frac{\tilde{V}_{t+\Delta}^{LSC} - \tilde{V}_{t+\Delta}^r - \left(\tilde{V}_{t+\Delta}^{sp} - V_{t+\Delta}^{LSC}\right)}{V_{t+\Delta}^{LSC}}. \text{ (Interaction term)}$$

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Unknown HPR

- LSC estimate: $\tilde{R}_\Delta^{LSC} \cong -\sum_{j=0}^{N_F} FD_j \Delta\tilde{f}_j + \frac{1}{2}\sum_{j=0}^{N_F} FC_j \Delta\tilde{f}_j^2 + \sum_{j=0}^{N_F}\sum_{j'=j+1}^{N_F} FCC_{j,j'} \Delta\tilde{f}_j \Delta\tilde{f}_{j'}$
  - Factor duration
  - Factor convexity
  - Factor cross-convexity

$$\Delta\tilde{f}_{j,t+\Delta}^r = \tilde{f}_{j,t+\Delta}^r - f_{j,t}^r \text{ or (Base rate)}$$
$$\Delta\tilde{f}_{j,t+\Delta}^{sp} = \tilde{f}_{j,t+\Delta}^{sp} - f_{j,t}^{sp} \text{ or (Spread)}$$

- HPR decomposition into return contributions:

$$\tilde{R}_\Delta^{Unknown} = \tilde{R}_\Delta^{LSC} = \frac{\tilde{V}_{t+\Delta}^{LSC} - V_{t+\Delta}^{LSC}}{V_{t+\Delta}^{LSC}} \cong R\tilde{C}_{FD}^r + R\tilde{C}_{FC}^r + R\tilde{C}_{FCC}^r + R\tilde{C}_{FD}^{sp} + R\tilde{C}_{FC}^{sp} + R\tilde{C}_{FCC}^{sp} + R\tilde{C}_{FCC}^{r,sp},$$

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Return Contributions (RC)

- RC related to the base curve

$$R\tilde{C}_{FD}^r \equiv R\tilde{C}_{FD}^{r,L} + R\tilde{C}_{FD}^{r,S} + R\tilde{C}_{FD}^{r,C},$$
$$R\tilde{C}_{FC}^r \equiv R\tilde{C}_{FC}^{r,L} + R\tilde{C}_{FC}^{r,S} + R\tilde{C}_{FC}^{r,C},$$
$$R\tilde{C}_{FCC}^r = R\tilde{C}_{FCC}^{r,L,S} + R\tilde{C}_{FCC}^{r,S,C} + R\tilde{C}_{FCC}^{r,L,C},$$

- RC related to the spread curve

$$R\tilde{C}_{FD}^{sp} = R\tilde{C}_{FD}^{sp,L} + R\tilde{C}_{FD}^{sp,S} + R\tilde{C}_{FD}^{sp,C},$$
$$R\tilde{C}_{FC}^{sp} = R\tilde{C}_{FC}^{sp,L} + R\tilde{C}_{FC}^{sp,S} + R\tilde{C}_{FC}^{sp,C},$$
$$R\tilde{C}_{FCC}^{sp} = R\tilde{C}_{FCC}^{sp,L,S} + R\tilde{C}_{FCC}^{sp,S,C} + R\tilde{C}_{FCC}^{sp,L,C}, \text{ and}$$

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Return Contributions (RC)

- RC related to base and spread curves

$$R\tilde{C}_{FCC}^{sp} = R\tilde{C}_{FCC}^{sp,L,S} + R\tilde{C}_{FCC}^{sp,S,C} + R\tilde{C}_{FCC}^{sp,L,C}, \text{ and}$$

$$\begin{aligned} R\tilde{C}_{FCC}^{r,sp} = & R\tilde{C}_{FCC}^{r(L),sp(L)} + R\tilde{C}_{FCC}^{r(L),sp(S)} + R\tilde{C}_{FCC}^{r(L),sp(C)} \\ & + R\tilde{C}_{FCC}^{r(S),sp(L)} + R\tilde{C}_{FCC}^{r(S),sp(S)} + R\tilde{C}_{FCC}^{r(S),sp(C)} \\ & + R\tilde{C}_{FCC}^{r(C),sp(L)} + R\tilde{C}_{FCC}^{r(C),sp(S)} + R\tilde{C}_{FCC}^{r(C),sp(C)} . \end{aligned}$$

- Goal: Eliminate as many RCs as possible as they contribute very little

Base Curve RC Estimates

- Factor duration return contributions

$$R\tilde{C}_{FD}^{r,L} \equiv -FD_L^r \Delta\tilde{f}_L^r,$$

$$R\tilde{C}_{FD}^{r,S} \equiv -FD_S^r \Delta\tilde{f}_S^r,$$

$$R\tilde{C}_{FD}^{r,C} \equiv -FD_C^r \Delta\tilde{f}_C^r,$$

- Factor convexity return contributions

$$R\tilde{C}_{FC}^{r,L} \equiv \frac{1}{2} FC_{FC}^{r,L} \left(\Delta\tilde{f}_L^r\right)^2,$$

$$R\tilde{C}_{FC}^{r,S} \equiv \frac{1}{2} FC_{FC}^{r,S} \left(\Delta\tilde{f}_S^r\right)^2,$$

$$R\tilde{C}_{FC}^{r,C} \equiv \frac{1}{2} FC_{FC}^{r,C} \left(\Delta\tilde{f}_C^r\right)^2,$$

Base Curve RC Estimates

- Factor cross-convexity return contributions

$$R\tilde{C}_{FCC}^{r,L,S} \equiv FCC_{FCC}^{r,L,S} \Delta\tilde{f}_L^r \Delta\tilde{f}_S^r,$$

$$R\tilde{C}_{FCC}^{r,L,C} \equiv FCC_{FCC}^{r,L,C} \Delta\tilde{f}_L^r \Delta\tilde{f}_C^r, \text{ and}$$

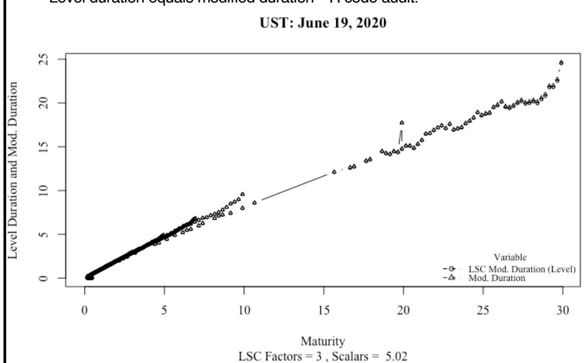
$$R\tilde{C}_{FCC}^{r,S,C} \equiv FCC_{FCC}^{r,S,C} \Delta\tilde{f}_S^r \Delta\tilde{f}_C^r.$$

- Spread curve RCs comparable

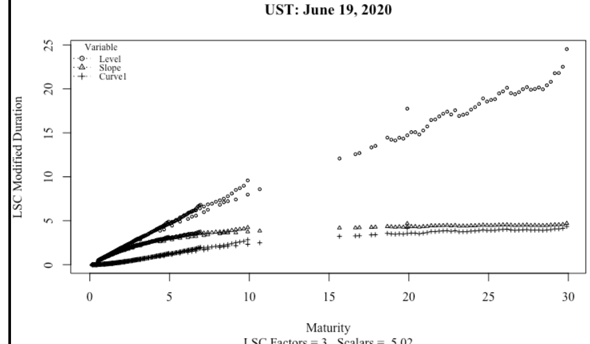
Base Curve Factor Durations

- LSC factor durations $\quad w_{i,t+\Delta}^{LSC} = \dfrac{CF_{i,t+\Delta} DF_{i,t+\Delta}^{LSC}}{V_{t+\Delta}},$
  - FD level $\quad FD_L^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right) w_{i,t+\Delta}^{LSC},$
  - FD slope $\quad FD_S^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right) x_{i,S} w_{i,t+\Delta}^{LSC},$
  - FD curvature $\quad FD_C^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right) x_{i,C} w_{i,t+\Delta}^{LSC},$

Level duration equals modified duration—R code audit.



Rank order of importance is level duration, slope duration, and curvature duration. Note, however, that slope and curvature duration will be much more important after neutralizing level duration (assets and liabilities).



Base Curve Factor Convexities

- LSC factor convexities
  - FC level $FC_L^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 w_{i,t+\Delta}^{LSC}$,
  - FC slope $FC_S^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,S}{}^2 w_{i,t+\Delta}^{LSC}$,
  - FC curvature $FC_C^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,C}{}^2 w_{i,t+\Delta}^{LSC}$,

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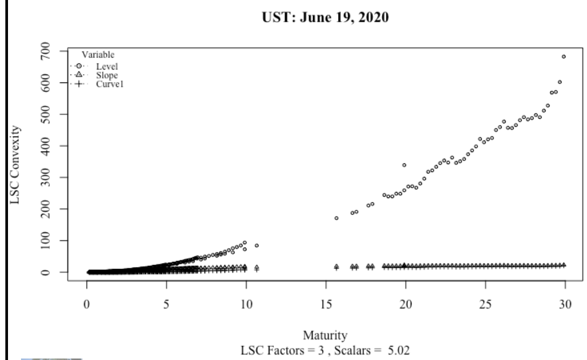
Rank order of importance is level convexity, slope convexity, and curvature convexity. Not sure slope and curvature convexities ever rank significant.



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Base Curve Factor Cross-Convexities

- LSC factor cross-convexities
  - FCC level $FCC_{L,S}^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,S} w_{i,t+\Delta}^{LSC}$,
  - FCC slope $FCC_{S,C}^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,S} x_{i,C} w_{i,t+\Delta}^{LSC}$, and
  - FCC curvature $FCC_{L,C}^r = \sum_{i=0}^{N_t} \left(\tau_i - \Delta\right)^2 x_{i,C} w_{i,t+\Delta}^{LSC}$.

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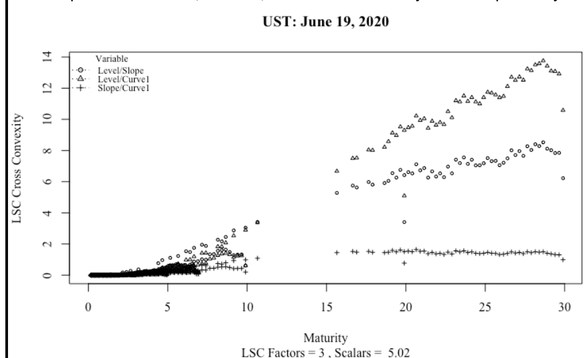
Rank order of cross convexity importance is level/curvature, level/slope, and slope/curvature. Note, however, all values are relatively small comparatively.



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Summary

- Reviewed traditional bond risk measures
- Explored role of compounding
  - Holding period returns (HPR)
  - Bond valuation
- Reviewed selected empirical evidence
- Introduced LSC-based bond risk measures based on HPRs
- Illustrated selected LSC applications

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