Module 7.4: Static Risk Management Common Stocks

Learning objectives

- Review the basics of stock static risk management
- Review the basic properties of stock duration and convexity
- Introduce advanced stock static risk measures
- Review LSC valuation model static risk measures

Executive summary

In this module, we explore various static risk measures related to several stock valuation models. First, we examine the traditional single stage dividend discount model. Specifically, we explore durations and convexities within this model. Second, we sketch various potential measures related to the N-stage dividend discount model. Third, we explore various static risk measures related to the LSC valuation model. Finally, we review selected basic univariate statistics and illustrate results for a couple of financial instruments.

Central finance concepts

The analysis presented in this section is supported by detailed technical explanations in the next section.

Static risk measures related to the Gordon dividend discount model

The Gordon growth model assumes the expected dividends grow as some constant growth rate and these expected dividend payments are discounted at a constant rate.

One approach to stock SRM is assessing how sensitive the stock price is to changes in the investor's required rate of return. Thus, we can derive the first and second derivatives of the Gordon growth model with respect to the assumed discount rate. With these derivatives, we can estimate the modified stock duration, the Macaulay stock duration, and standard stock convexity. We could also perform similar analysis with respect to the assumed growth rate.

Within the Gordon growth model, we could further elaborate with other model, such as the Capital Asset Pricing Model (CAPM). Given the importance of the risk-free interest rate on so many quantitative finance applications, the ability to compute the sensitivity of a particular stock to changes in interest rates is particularly useful.

Static risk measures applied to the *N*-stage dividend discount model

Although tedious, it is straightforward to extend these SRMs to the NDDM. The generality of the NDDM affords the capacity to incorporate analysts' viewpoints within the SRMs.

Applying the LSC model to the present value of expected dividend payments

The present value of expected dividends, regardless of the assumed growth rates and assumed discount rates form a function that is typically rather smooth and asymptotically tends to zero as maturity time tends to infinity.

We now review briefly important concepts related to calculating rates of return, the very foundation of a large portion of financial tasks.

Calculating rates of return[1]

The CFA Institute has taken the lead in establishing the highest standards of ethics within the investment management community, especially when it comes to calculating and presenting investment performance. The CFA Institute has developed the Global Investment Performance Standards (GIPS) and it has become the highest standard in performance presentation.

Selected key features of GIPS include:

[1]Based, in part, on the *Global Investment Performance Standards*, 2010, CFA Institute.

"The GIPS standards are ethical standards for investment performance presentation to ensure fair representation and full disclosure of investment performance. In order to claim compliance, firms must adhere to the requirements included in the GIPS standards.

The GIPS standards rely on the integrity of input data. ..."

Our focus here is on calculating rates of return and understanding the importance of using the correct process to report valid performance numbers and make appropriate investment decisions. GIPS provide explicit details on the required calculation methodology in Section 2. Selected requirements include:

"2.A.1 TOTAL RETURNS MUST be used.

2.A.2 FIRMS MUST calculate TIME-WEIGHTED RATES OF RETURN that adjust for EXTERNAL CASH FLOWS. Both periodic and sub-period returns MUST be geometrically LINKED. ...

2.A.6 COMPOSITE returns MUST be calculated by asset-weighting the individual PORTFOLIO returns using beginning-of-period values or a method that reflects both beginning-of-period values and EXTERNAL CASH FLOWS."

We introduce here the concepts expressed in GIPS required calculation methodology and to provide a rational justification for using it. We start with introducing the notion of interim rate of return and then introduce two methods for computing the average rate of return. With this foundation, we introduce the notion of time-weighting and linking returns. The index method is introduced as a pedagogical tool to appreciate the rationale of linking. We conclude with a brief discussion of different weighting methods for computing indexes or composites.

Interim rate of return

The total rate of return over some period usually requires computing interim rates of return and linking them together in an appropriate manner. The total rate of return is simply the ending market value divided by the beginning market value minus one. Although relatively intuitive to describe, it is often incorrectly computed in practice. The interim rate of return is again simply the ending market value divided by the beginning market value minus one, except this time it only covers a period where there was neither cash inflow nor outflow within the position.

Methods for averaging rates of return

Over multiple interim periods, there are two methods for computing average ex-post rates of return, the arithmetic method, and the geometric method. Each method is useful for different context.

As we will see in the technical section below, the geometric average can be interpreted as reflecting the true growth of the position whereas the arithmetic average does not reflect the true growth of the position. The arithmetic average, however, does accurately reflect the average rate of return across different instruments for a single period and is an unbiased estimate of the expected rate of return based on the observed sample.

Time-weighted rate of return

GIPS require the time-weighted rate of return be computed and external cash flows incorporated. The key insight is that the time-weighted rate of return method assumes all interim cash flows are at once reinvested in the existing position. Specifically, GIPS defines time-weighted rate of return as "(a) method of calculating period-by-period returns that negates the effects of external cash flows" and defines external cash flow as "(c)apital (cash or investments) that enters or exits a portfolio." Early in the development of GIPS, the time-weighted approach was explained in the following way:[2]

If cash flows occur during the period, they must theoretically be used, in effect, "to buy additional units" of the portfolio at the market price on the day that they are received. Thus the most accurate approach is to calculate the market value of the portfolio on the date of each cash flow, calculate an interim rate of return for the subperiod, and then link the subperiod returns to get the return for the month or quarter.

There are two ways to illustrate the appropriate way to compute rates of return that are following GIPS, the linking method and the index method. We examine the details below.

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[2]*Report of the Performance Presentation Standards Implementation Committee*, AIMR, December 1991, p. 28. Note AIMR denotes the Association of Investment Management and Research, the predecessor organization of the CFA Institute.

The time-weighted rates of return is superior to other methods of computing rates of return because it appropriately handles the timing of cash flows and appropriately handles the reinvestment of these cash flows. The key insight is that anything of value received from an investment is reinvested in that same investment. Hence, if an owner of a real estate investment receives a two-week time share in Orlando Florida, then it must be assumed that the time share was immediately sold, and the proceeds reinvested in the real estate investment.

GIPS requires composites to be carefully constructed when investment managers have multiple accounts. The composite is defined by GIPS as "(a)n aggregation of one or more PORTFOLIOS managed according to a similar investment mandate, objective, or strategy." GIPS asset-weighted composite construction is basically taking a value-weighted approach where larger accounts receive greater weight.

Most investment managers have a disclosed benchmark to compare their composite rates of return. These benchmarks are often popular market indexes, such as the S&P 500 or the Russell 1000 Large Cap Growth. Indexes are constructed in a wide variety of ways. We now introduce of the basic index construction methodologies.

Indexes

Indexes are useful for a wide variety of reasons. Stock indexes, such as the Dow Jones Industrial Average and the S&P 500 index, are useful benchmarks for various components of the stock market. There are three main ways to construct indexes, price-weighted, value-weighted, and equally-weighted.

Price-weighted indexes are constructed in such a way that the implied quantity assigned to each position is the same, hence, the higher the price, the more influence the position has on the index.

Value-weighted indexes are constructed in such a way that the implied quantity assigned to each position is based on the market value of underlying instrument.

Equally-weighted indexes in some way ascribes an equal amount to each position. Like averaging, there are two ways to construct an equally-weighted index, the arithmetic method, and the geometric method.

Sample statistics

We now introduce various the population and sample statistics that are illustrated R code in this module.

To compute a rate of return assumes that the investment has positive value initially. Rates of return can be computed either discretely or continuously. Although either approach can be used in quantitative finance, often analysis based on continuous compounded is easier to manage. Finally, analysis can be conducted on dollar profits. Often, financial derivatives positions are easier to analyze using dollar profits as opposed to percentage returns.

We now briefly introduce several univariate statistics.

Mean

Sample arithmetic mean is also often called the mean or average. For the arithmetic average, each outcome is assumed to have equal weight. The geometric average is based on the product ex-post rates of return over the time periods and then takes the n^{th} root to express the average on a per period basis. When severe outliers are a problem, the harmonic mean can be used. The harmonic average is often used with financial multiples like market-to-book ratio and price-to-earnings ratio.

Standard deviation

The standard deviation is a measure of dispersion and is based on the variance. The variance is the expected deviations from the mean. The variance, however, is in units squared. By taking the standard deviation of the variance, we have a dispersion measure in the same units has the underlying observations.

Skewness

Skewness is the third standardized moment and measures asymmetry in the distribution of the sample. Symmetric distributions will have a population skewness of zero and a sample skewness near zero. Negative values for this statistic indicate that the distribution is skewed left, and the left tail of the distribution is longer. Positive values for this statistic indicate that the distribution is skewed right, and the right tail of the distribution is longer.

Kurtosis

Kurtosis is the fourth standardized moment and measures the height and sharpness of the central peak of the distribution relative to the normal distribution. Higher kurtosis implies higher and sharper central peak whereas lower kurtosis implies lower and flatter central peak. The normal distribution has a kurtosis of 3. Therefore, it is customary to subtract three from kurtosis and report what is termed excess kurtosis. Often excess kurtosis is just referred to as kurtosis. One must always know how to interpret the reported kurtosis. That is, whether it has 3 subtracted or not.

Static risk measures related to the LSC valuation model

Recall LSC valuation model presented in Module 4.3 is based on some generic valuation tasks involving annual expected future series of cash flows with known positive initial cash flow. We assumed cash flow growth rates are modeled within LSC model framework. Specifically, we estimate the perpetual (level) growth rate, the short-term (slope) growth rate, and as many growth rate curvature factors as desired. We also assumed forward discount rates can also be modeled within the LSC model framework. Specifically, we estimated the perpetual (level) forward discount rate, the short-term (slope) forward discount rate, and as many forward discount rate curvature factors as desired.

In the technical materials below, we illustrate one possible implementation of SRMs with the LSC valuation model.

Quantitative finance materials

We start the quantitative finance materials analyzing the traditional Gordon dividend discount model.

Static risk measures related to the Gordon dividend discount model

Recall in the Gordon growth model the expected dividends are assumed to be paid annually, are expected to grow as some constant growth rate, *g*, and these expected dividend payments are discounted at a constant rate, *k*. Thus,

$$V_S = \frac{D_0\left(1+g\right)}{k-g}. \tag{7.4.1}$$

The discount rate is the cost of equity capital or the investor's required rate of return.

Static risk measures applied to the required rate of return and growth rate

One approach to stock SRM is assessing how sensitive the stock price is to changes in the investor's required rate of return as well as the dividend growth rate. The first derivative of the stock value with respect to the investor's required rate of return is

$$\begin{aligned}\frac{\partial V_S}{\partial k} &= \frac{\partial}{\partial k}\left[D_0\left(1+g\right)\left(k-g\right)^{-1}\right] \\ &= D_0\left(1+g\right)\left[-\left(k-g\right)^{-2}\right] \\ &= -\frac{V_S}{k-g}\end{aligned}. \tag{7.4.2}$$

Thus, modified stock duration with respect to *k* is

$$\begin{aligned}ModDurk_S &\equiv -\frac{1}{V_S}\frac{\partial V_S}{\partial k} \\ &= -\frac{1}{V_S}\left(-\frac{V_S}{k-g}\right) \\ &= \frac{1}{k-g}\end{aligned}. \tag{7.4.3}$$

In this case, Macaulay duration with discrete compounding is

$$\begin{aligned}MacDurk_S &= ModDurk_S\left(1+k\right) \\ &= \frac{1+k}{k-g}\end{aligned}. \tag{7.4.4}$$

The second derivative of the stock value with respect to the investor's required rate of return is

$$
\begin{aligned}
\frac{\partial^2 V_S}{\partial k^2} &= \frac{\partial}{\partial k}\left[-\frac{D_0(1+g)}{(k-g)^2}\right] \\
&= -D_0(1+g)\frac{\partial}{\partial k}\left[\frac{1}{(k-g)^2}\right]. \\
&= -D_0(1+g)\left[\frac{-2}{(k-g)^3}\right] \\
&= 2\frac{V_S}{(k-g)^2}
\end{aligned}
\tag{7.4.5}
$$

Thus, standard convexity with respect to k is

$$
\begin{aligned}
Convexityk_S &\equiv \frac{1}{V_S}\frac{\partial^2 V_S}{\partial k^2} \\
&= \frac{1}{V_S}\left[2\frac{V_S}{(k-g)^2}\right]. \\
&= \frac{2}{(k-g)^2}
\end{aligned}
\tag{7.4.6}
$$

Given the importance of the dividend growth rate, we apply similar comparative analysis to this growth rate. That is, taking the first derivative of the stock value with respect to the dividend growth rate is

$$
\begin{aligned}
\frac{\partial V_S}{\partial g} &= \frac{\partial}{\partial g}\left[D_0(1+g)(k-g)^{-1}\right] \\
&= D_0\left[(k-g)^{-1} - (-1)(1+g)(k-g)^{-2}\right] \\
&= \frac{D_0(1+k)}{(k-g)^2} \\
&= \frac{V_S}{(1+g)(k-g)}
\end{aligned}
\tag{7.4.7}
$$

Thus, modified stock duration with respect to g is

$$
\begin{aligned}
ModDurg_S &\equiv -\frac{1}{V_S}\frac{\partial V_S}{\partial g} \\
&= -\frac{1}{V_S}\left[\frac{V_S}{(1+g)(k-g)}\right]. \\
&= \frac{1}{(1+g)(k-g)}
\end{aligned}
\tag{7.4.8}
$$

In this case, Macaulay duration with respect to g with discrete compounding is

$$
\begin{aligned}
MacDurg_S &= ModDurg_S(1+k) \\
&= \frac{1+k}{(1+g)(k-g)}
\end{aligned}
\tag{7.4.9}
$$

The second derivative of the stock value with respect to the g is

$$
\begin{aligned}
\frac{\partial^2 V_S}{\partial g^2} &= \frac{\partial}{\partial g}\left[\frac{D_0(1+k)}{(k-g)^2}\right] \\
&= D_0(1+k)\frac{\partial}{\partial g}\left[\frac{1}{(k-g)^2}\right]. \\
&= D_0(1+k)\left[\frac{2}{(k-g)^3}\right] \\
&= 2\frac{V_S(1+k)}{(1+g)(k-g)^2}
\end{aligned}
\tag{7.4.10}
$$

Thus, standard convexity with respect to g is

$$
\begin{aligned}
Convexityg_S &\equiv \frac{1}{V_S}\frac{\partial^2 V_S}{\partial g^2} \\
&= \frac{1}{V_S}\left[2\frac{V_S(1+k)}{(1+g)(k-g)^2}\right]. \\
&= \frac{2(1+k)}{(1+g)(k-g)^2}
\end{aligned}
\tag{7.4.11}
$$

With these results and the multivariate Taylor series, we note

$$
V_S = f(k,g). \tag{7.4.12}
$$

Thus, applying Taylor series with respect to the first and second derivatives, we have

$$
\begin{aligned}
V_S'(k,g) &\cong V_S + \left(\frac{\partial V_S}{\partial k}\right)\Delta k + \left(\frac{\partial V_S}{\partial g}\right)\Delta g \\
&+ \frac{1}{2}\left(\frac{\partial^2 V_S}{\partial k^2}\right)\Delta k^2 + \frac{1}{2}\left(\frac{\partial^2 V_S}{\partial g^2}\right)\Delta g^2
\end{aligned}
. \tag{7.4.13}
$$

The discretely compounded holding period return is approximated as

$$
\begin{aligned}
R_{S,dc} &\equiv \frac{V_S' - V_S}{V_S} \\
&\cong \frac{1}{V_S}\left(\frac{\partial V_S}{\partial k}\right)\Delta k + \frac{1}{V_S}\left(\frac{\partial V_S}{\partial g}\right)\Delta g + \frac{1}{2}\frac{1}{V_S}\left(\frac{\partial^2 V_S}{\partial k^2}\right)\Delta k^2 + \frac{1}{2}\frac{1}{V_S}\left(\frac{\partial^2 V_S}{\partial g^2}\right)\Delta g^2 \quad . \\
&= -ModDurk_S\Delta k - ModDurg_S\Delta g + \frac{1}{2}Convexityk_S\Delta k^2 + \frac{1}{2}Convexityg_S\Delta g^2
\end{aligned}
\tag{7.4.14}
$$

Static risk measures applied to the assumed spot rate and expected growth rate

The typical way the investor's required return is estimated is by using the risk-free rate plus a risk premium. One example is the Capital Asset Pricing Model (CAPM) which can be expressed here as

$$
\begin{aligned}
k &= r + \beta\left[E(r_M) - r\right] \\
&= r(1-\beta) + \beta E(r_M)
\end{aligned}
, \tag{7.4.15}
$$

where r denotes the risk-free rate, r_M denotes the return on the market portfolio, and β denotes the sensitivity of this stock to changes in the excess expected return of the market portfolio over the risk free rate. Within the CAPM framework, we have

$$
V_S = \frac{D_0(1+g)}{r(1-\beta) + \beta E(r_M) - g}. \tag{7.4.16}
$$

Note that

$$\frac{\partial k}{\partial r} = 1 - \beta. \tag{7.4.17}$$

Thus, modified stock duration with respect to r is

$$\begin{aligned} ModDurr_S &\equiv -\frac{1}{V_S}\frac{\partial V_S}{\partial k}\frac{\partial k}{\partial r} \\ &= \frac{1-\beta}{k-g} \end{aligned}. \tag{7.4.18}$$

Standard convexity with respect to r is

$$Convexityr_S \equiv \frac{1}{V_S}\frac{\partial^2 V_S}{\partial r^2} = \frac{2(1-\beta)^2}{(k-g)^2}. \tag{7.4.19}$$

Thus, one alternative approximation for the discretely compounded holding period return is

$$\begin{aligned} R_{S,dc} &\equiv \frac{V_S' - V_S}{V_S} \\ &\cong \frac{1}{V_S}\left(\frac{\partial V_S}{\partial r}\right)\Delta r + \frac{1}{V_S}\left(\frac{\partial V_S}{\partial g}\right)\Delta g + \frac{1}{2}\frac{1}{V_S}\left(\frac{\partial^2 V_S}{\partial r^2}\right)\Delta r^2 + \frac{1}{2}\frac{1}{V_S}\left(\frac{\partial^2 V_S}{\partial g^2}\right)\Delta g^2 \\ &= -ModDurr_S \Delta k - ModDurg_S \Delta g + \frac{1}{2}Convexityr_S \Delta r^2 + \frac{1}{2}Convexityg_S \Delta g^2 \end{aligned}. \tag{7.4.20}$$

Static risk measures applied to the *N*-stage dividend discount model
Recall the NDDM contains three components:

$$Stub = D_{-1}\sum_{q=1}^{H} e^{-f_0 \sum_{k=1}^{q}\Delta\tau_k}, \text{ (Stub Component)} \tag{7.4.21}$$

$$Series = D_{-1}e^{-f_0\sum_{k=1}^{H}\Delta\tau_k}\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j-1} e^{-(f_i - g_i)m_i}\right]\left[e^{f_j\Delta\tau(3)} + e^{f_j\Delta\tau(2)} + e^{f_j\Delta\tau(1)} + 1\right]\frac{1-e^{-(f_j-g_j)m_j}}{e^{(f_j-g_j)}-1}, \text{(Series Component)} \tag{7.4.22}$$

$$Final = D_{-1}e^{-f_0\sum_{k=1}^{H}\Delta\tau_k}\left[\prod_{i=1}^{N-1} e^{-(f_i-g_i)m_i}\right]\left[e^{f_N\Delta\tau(3)} + e^{f_N\Delta\tau(2)} + e^{f_N\Delta\tau(1)} + 1\right]\frac{1}{e^{(f_N-g_N)}-1}. \text{ (Final Component)} \tag{7.4.23}$$

The *N*-Stage DDM can be expressed simply as

$$V_S = Stub + Series + Final. \tag{7.4.24}$$

With this closed-form expression, comparative static analysis can easily be performed either through direct analytic analysis or through numerical analysis.

Applying the LSC model to the present value of expected dividend payments
Applying the LSC model to the present value of dividends, we have

$$PVD_i = \sum_{j=0}^{N} x_{i,j} f_j, \tag{7.4.25}$$

where PVD_i denotes the input present value of dividends corresponding to maturity i, $x_{i,j}$ denotes input LSC coefficients based on some maturity and some factor, and f_j denotes the output factors. Recall the LSC model in general form assumes

$$x_{i,0} = 1,\ x_{i,1} = \frac{s_1}{\tau_i}\left(1 - e^{-\tau_i/s_1}\right), \text{ and } x_{i,j} = \frac{s_j}{\tau_i}\left(1 - e^{-\tau_i/s_j}\right) - e^{-\tau_i/s_j}; j > 1,$$

where the variables are as defined in Module 3.5.

Calculating rates of return[3]

We now explain the technical details of the concepts expressed in GIPS required calculation methodology.

Interim rate of return

We now seek to explain some of the technical details of computing rates of return in a manner consistent with GIPS. The total rate of return over some period usually requires computing interim rates of return and linking them together in an appropriate manner. The total rate of return is simply the ending market value divided by the beginning market value minus one. Although relatively intuitive to describe, it is often incorrectly computed in practice. The interim rate of return is again simply the ending market value divided by the beginning market value minus one, except this time it only covers a period where there was neither cash inflow nor outflow within the position. Mathematically, the interim rate of return (*IRoR*) can be expressed as

$$IRoR = \frac{EMV + Inc - BMV}{BMV}, \textbf{ (Interim rate of return)} \qquad (7.4.26)$$

where EMV denotes the ending market value, Inc denotes any financial benefits received due to owning the position (e.g., cash dividends), and BMV denotes the beginning market value. Note that equation (1) will be in decimal form and often rates of return are reported in percentage form, hence *IRoR* would be multiplied by 100.

For example, suppose you own 1,000 shares of FRM stock that was trading for \$10 per share on January 1st and it was trading for \$11 per share on March 15th when it paid a \$0.10 dividend. The interim rate of return from January 1st to March 15th is

$$IRoR = \frac{EMV + Inc - BMV}{BMV} = \frac{1{,}000(11) + 1{,}000(0.10) - 1{,}000(10)}{1{,}000(10)} = \frac{11{,}000 + 100 - 10{,}000}{10{,}000} = 0.11$$

or 11 percent.

Methods for averaging rates of return

Over multiple interim periods, there are two methods for computing average ex-post rates of return, the arithmetic method, and the geometric method. Each method is useful for different context. Let R_t denote the interim rate of return over some period starting at t. The period could be one year, one quarter, one day, or even one minute. The arithmetic method sums the ex-post rates of return over the time periods and then divides them by the total number of observations denoted n. That is, the arithmetic average rate of return ($\bar{R}_A$) can be expressed as

$$\bar{R}_A \equiv \frac{\sum_{t=1}^{n} R_t}{n}. \textbf{ (Arithmetic average)} \qquad (7.4.27)$$

For example, suppose FRM stock returns over the past five years were 10%, −15%, −5%, 30%, and 20%. The arithmetic average rate of return is

$$\bar{R}_A \equiv \frac{\sum_{t=1}^{n} R_t}{n} = \frac{0.10 - 0.15 - 0.05 + 0.40 + 0.20}{5} = 0.10 \text{ or 10 percent.}$$

The geometric method links the ex-post rates of return over the time periods and then takes the nth root to express the average on a per period basis. That is, the geometric average rate of return ($\bar{R}_G$) can be expressed as

$$\bar{R}_G \equiv \left[\prod_{t=1}^{n}\left(1 + R_t\right)\right]^{1/n} - 1. \textbf{ (Geometric average)} \qquad (7.4.28)$$

Using the numerical example above, we have

[3]Based, in part, on the *Global Investment Performance Standards*, 2010, CFA Institute.

$$\bar{R}_G \equiv \left[\prod_{t=1}^{n}(1+R_t)\right]^{1/n} - 1 = \left[(1+0.10)(1-0.15)(1-0.05)(1+0.40)(1+0.20)\right]^{1/5} - 1$$
$$= 1.49226^{0.2} - 1 = 0.08335$$
or 8.335 percent.

Is the average return 10 percent or 8.335 percent? The answer is that it depends on your context or purpose for using the average. A simple two period case with highly volatile results helps to clarify the issues. Suppose you owned a stock that was trading at \$100 per share initially. After one year, it was worth \$200 per share. After two years, it was worth \$100 per share again. Intuitively, what was your average rate of return? You start with \$100 and end with \$100, so your average rate of return was zero, right? Well, whether that answer is correct depends on your context. The interim rates of return are
$$R_0 = \frac{EMV + Inc - BMV}{BMV} = \frac{200 + 0 - 100}{100} = 1.0 \text{ and}$$
$$R_1 = \frac{EMV + Inc - BMV}{BMV} = \frac{100 + 0 - 200}{200} = -0.5.$$
and so the arithmetic and geometric rates of return are
$$\bar{R}_A \equiv \frac{\sum_{t=1}^{n} R_t}{n} = \frac{1.0 - 0.5}{2} = 0.25 \text{ and}$$
$$\bar{R}_G \equiv \left[\prod_{t=1}^{n}(1+R_t)\right]^{1/n} - 1 = \left[(1+2)(1-0.5)\right]^{1/2} - 1 = 0,$$
or a 25 percent arithmetic average and a 0 percent geometric average. Note that as long as rates of return are volatile, the arithmetic average rate of return will be higher than the geometric average rate of return. Also, the greater the volatility, the greater will be the difference between these two averages. We will return to this issue when we examine the benefits of diversification.

Clearly, the geometric average can be interpreted as reflecting the true growth of the position whereas the arithmetic average does not reflect the true growth of the position. The arithmetic average, however, does accurately reflect the average rate of return across different instruments for a single period of time and is an unbiased estimate of the expected rate of return based on the observed sample.

Time-weighted rate of return

GIPS require the time-weighted rate of return be computed and external cash flows incorporated. The key insight is that the time-weighted rate of return method assumes all interim cash flows are at once reinvested in the existing position. Specifically, GIPS defines time-weighted rate of return as "(a) method of calculating period-by-period returns that negates the effects of external cash flows" and defines external cash flow as "(c)apital (cash or investments) that enters or exits a portfolio." Early in the development of GIPS, the time-weighted approach was explained in the following way:[4]

> *If cash flows occur during the period, they must theoretically be used, in effect, "to buy additional units" of the portfolio at the market price on the day that they are received. Thus the most accurate approach is to calculate the market value of the portfolio on the date of each cash flow, calculate an interim rate of return for the subperiod, and then link the subperiod returns to get the return for the month or quarter.*

There are two ways to illustrate the appropriate way to compute rates of return that are in compliance with GIPS, the linking method and the index method. We illustrate both methods using a simple example of a dividend paying stock. Table 7.4.1 presents the relevant inputs.

[4]*Report of the Performance Presentation Standards Implementation Committee*, AIMR, December 1991, p. 28. Note AIMR denotes the Association of Investment Management and Research, the predecessor organization of the CFA Institute.

Table 7.4.1. Inputs for Computing Time Weighted Rate of Return

Date	Dollar Dividend Per Share	Other Events	Market Price
January 1			$100
March 15	$1		$105
June 15	$1		$110
September 15	$1		$108
October 1		2-for-1 Stock Split	$52 (post split price)
December 15	$0.5		$51
December 31			$50

What is the GIPS-compliant rate of return if the portfolio manager started the year with 100 shares and did not trade this stock? The time-weighted rate of return by the linking method is found by computing the interim rates of return and then linking them together in the same manner as used when computing the geometric average rate of return. The linking method is the method of choice for computational implementation of rates of return calculations. Table 7.4.2 illustrates the computation of the time weighted rate of return by the linking method.

Table 7.4.2. Linking Method Illustrated for Computing Time Weighted Rate of Return

Date	Interim Period	Interim Rate of Return	Time-Weighted Rate of Return
January 1			
March 15	1	0.0600000 (1)	0.0600000
June 15	2	0.0571486 (2)	0.1205775 (7)
September 15	3	-0.0090909 (3)	0.1103905
October 1	4	-0.0370370 (4)	0.0692649
December 15	5	-0.0096154 (5)	0.0589835
December 31	6	-0.0196078 (6)	0.0382192 (8)

(1) (105 + 1 – 100)/100 = 0.06
(2) (110 + 1 – 105)/105 = 0.0571486
(3) (108 + 1 – 110/110 = –0.090909
(4) (2(52) – 108)/108 = –0.0370370
(5) (51 + 0.5 – 52)/52 = –0.0096154
(6) (50 – 51)/51 = –0.0196078
(7) (1 + 0.06)(1 + 0.0571486) – 1 = 0.1205775
(8) (1 + 0.0589835)(1 – 0.0196078) – 1 = 0.0382192

Thus, the time-weighted rate of return over this year was 3.82 percent. Note that this stock paid $4 in dividends on a $100 stock or a 4 percent dividend yield. Also, the stock started and finished the year at the same pre-split price of $100. Why isn't the rate of return 4 percent? The index method of computing rates of return makes it clear why the rate of return was lower in this case. The index method illustrates the reinvestment of all interim cash flows back into the position. The index method highlights the intuition. Thus, an index of the number of shares held is constructed and adjusted over the period. Table 7.4.3 illustrates the computation of the time weighted rate of return by the index method.

Table 7.4.3. Index Method Illustrated for Computing Time Weighted Rate of Return

Date	Dividend and Other Events	Additional Shares Purchased	Number of Shares Owned
January 1			100
March 15	$1	0.952381 (1)	100.952381 (2)
June 15	$1	0.917749 (3)	101.870130 (4)
September 15	$1	0.943242 (5)	102.813372 (6)
October 1	2-for-1 Split		205.626744 (7)
December 15	$0.5	2.015948 (8)	207.642692 (9)
December 31			207.642692

(1) 1(100)/105 = 0.952381
(2) 100 + 0.952381 = 100.0952381
(3) 1(100.952381)/110 = 0.917749
(4) 100.952381 + 0.917749 = 101.870130
(5) 1(101.870130)/108 = 0.943242
(6) 101.870130 + 0.943242 = 102.813372
(7) 2(102.813372) = 205.626744
(8) 0.5(205.626744) = 2.015948
(9) 205.626744 + 2.015948 = 207.642692

Thus, the rate of return is

$$R_0 = \frac{EMV + Inc - BMV}{BMV} = \frac{207.642692(50) - 100(100)}{100(100)} = 0.038213.$$

The difference between the linking method and the index method is solely rounding error.

The time-weighted rates of return is superior to other methods of computing rates of return because it appropriately handles the timing of cash flows and appropriately handles the reinvestment of these cash flows. The key insight is that anything of value received from an investment is reinvested in that same investment. Hence, if an owner of a real estate investment receives a two-week time share in Orlando Florida, then it must be assumed that the time share was immediately sold and the proceeds reinvested in the real estate investment.

GIPS requires composites to be carefully constructed when investment managers have multiple accounts. The composite is defined by GIPS as "(a)n aggregation of one or more PORTFOLIOS managed according to a similar investment mandate, objective, or strategy." GIPS asset-weighted composite construction is basically taking a value-weighted approach where larger accounts receive greater weight.

Most investment managers have a disclosed benchmark to compare their composite rates of return. These benchmarks are often popular market indexes, such as the S&P 500 or the Russell 1000 Large Cap Growth. Indexes are constructed in a wide variety of ways. We now review of the basic index construction methodologies.

Indexes

Indexes are useful for a wide variety of reasons. Stock indexes, such as the Dow Jones Industrial Average and the S&P 500 index, are useful benchmarks for various components of the stock market. There are three main ways to construct indexes, price-weighted, value-weighted, and equally-weighted.

Most indexes are built based on the following general approach:

$$I_t = \sum_{i=1}^{n} Q_{i,t} P_{i,t}, \tag{7.4.29}$$

where $P_{i,t}$ denotes the price of instrument i observed at time t, $Q_{i,t}$ denotes some quantity related to instrument i based on the index construction methodology, and n denotes the total number of instruments in the index. Price-weighted indexes, such as the Dow Jones Industrial Average, are constructed in such a way that the implied quantity assigned to each position is the same, hence, the higher the price, the more influence the position has on the index.

$$I_{PW,t} = \sum_{i=1}^{n} Q_{i,t-1}P_{i,t} = \sum_{i=1}^{n} Q_{t-1}P_{i,t} = Q_{t-1}\sum_{i=1}^{n} P_{i,t}\text{. } \textbf{(Price-weighted index)} \tag{7.4.30}$$

Note that the quantity of each instrument is the same for all instruments. Typically, this quantity is expressed in terms of a divisor, or Q_{t-1} = 1/divisor. The divisor is a number that encapsulates all changes to the index, such as stock splits or revision of the instruments in the portfolio.

Value-weighted indexes, such as Russell indexes, are constructed in such a way that the implied quantity assigned to each position is based on the market value of underlying instrument. For example, if a particular stock has no liabilities, then the market value of the underlying instrument is the market capitalization of the firm.

$$I_{VW,t} = \sum_{i=1}^{n} Q_{i,t-1}P_{i,t} = c_{t-1}\sum_{i=1}^{n} N_{i,t-1}P_{i,t}\text{. } \textbf{(Value-weighted index)} \tag{7.4.31}$$

where $c_{t-1}N_{i,t-1}$ reflects the proportion of position i within the market or portfolio. Thus, the higher the market cap of stocks within this index, the more influence the stock has on the index. The constant c_{t-1} often reflects the initial value of the index. For example, if a stock index was initially set to 100, thus we have

$$c_0 = 100\left(\frac{1}{\sum_{i=1}^{n} N_{i,0}P_{i,0}}\right). \tag{7.4.32}$$

Equally-weighted indexes, such as CRSP and Value Line, in some way ascribes an equal amount to each position. Like averaging, there are two ways to construct an equally-weighted index, the arithmetic method and the geometric method. Mathematically, these indexes can be represented as

$$I_{EW,AM,t} = I_{EW,AM,t-1}\left(1 + \frac{\sum_{i=1}^{n} R_{i,t}}{n}\right)\text{. } \textbf{(Arithmetic method)} \tag{7.4.33}$$

$$I_{EW,GM,t} = I_{EW,GM,t-1}\left[\prod_{i=1}^{n}\left(1 + R_{i,t}\right)\right]^{1/n}\text{. } \textbf{(Geometric method)} \tag{7.4.34}$$

The Center for Research in Security Prices (CRSP) offers two version of the arithmetic method for their equally-weighted indexes, with and without dividend adjustment. Historically, the geometric method was used in Value Line indexes.

Sample statistics

We now contrast the population and sample statistics that are illustrated in this program. Let R denotes the holding period rate of return. Before turning to these statistics, we explore three measures of return.

Periodic rates of return

To compute a rate of return assumes that the investment has positive value initially. Let R_{dc} denote the discretely compounded periodic holding period return. Let V_t denote the financial instrument value observed at some initial time t ($V_t > 0$). Thus, the discretely compounded periodic holding period return between time t and time $t+\Delta t$ is

$$\begin{aligned} R_{dc} &\equiv \frac{Profit}{Investment} \\ &= \frac{\Delta V}{V} \\ &= \frac{V_{t+\Delta t} - V_t}{V_t} \end{aligned}. \tag{7.4.35}$$

Rearranging, we can express the future value as

$$V_{t+\Delta t} = V_t\left(1+R_{dc}\right). \tag{7.4.36}$$

The continuously compounded periodic holding period return between time t and time $t+\Delta t$ is

$$R_{cc} \equiv \ln\left(\frac{V_{t+\Delta t}}{V_t}\right). \tag{7.4.37}$$

Rearranging, we can express the future value as

$$V_{t+\Delta t} = V_t e^{R_{cc}}. \tag{7.4.38}$$

Annualized discretely compounded holding period returns

There are several ways to annualize periodic returns ($R_{a,dc}$). Let m denote the compounding frequency per year. With this notation, the relationship between discretely compounded periodic holding period returns and annualized discretely compounded holding period returns can be expressed as

$$\left(1+R_{dc}\right) = \left(1+\frac{R_{a,dc}}{m}\right)^{(m)\Delta t}. \tag{7.4.39}$$

Thus, the annualized compounded holding period returns can be expressed as

$$R_{a,dc} = m\left[\left(1+R_{dc}\right)^{1/(m)\Delta t} - 1\right]. \tag{7.4.40}$$

Annualized continuously compounded holding period returns

With this notation, the relationship between the continuously compounded periodic holding period returns and annualized continuously compounded holding period returns is simply

$$\begin{aligned} R_{a,cc} &= \ln\left(\frac{V_{t+\Delta t}}{V_t}\right)\Big/\Delta t. \\ &= R_{cc}\Delta t \end{aligned} \tag{7.4.41}$$

Thus, the annualized continuously compounded holding period returns can be expressed as

$$V_{t+\Delta t} = V_t e^{R_{cc}\Delta t}. \tag{7.4.42}$$

Dollar returns

Dollar profit or loss is useful especially when the initial financial instrument is zero or negative. The change in instrument value or dollar return is

$$\begin{aligned} \Delta V &\equiv Profit \\ &= V_{t+\Delta t} - V_t \end{aligned}. \tag{7.4.43}$$

With this setup, we are ready to compute various statistics related to returns.

Mean

Sample arithmetic mean is also often called the mean or average. The population mean is the first moment of the distribution, μ_1.

$\mu_1 = E(R)$ and (**First moment, population mean, or population average**) (7.4.44)

$\bar{R}_A = \frac{1}{n}\sum_{i=1}^{n} R_i$. (**Sample average**) (7.4.45)

Geometric average is based on the product ex-post rates of return over the time periods and then takes the nth root to express the average on a per period basis or

$\bar{R}_G = \left[\prod_{i=1}^{n}\left(1+R_i\right)\right]^{\frac{1}{n}} - 1$. (**Geometric average**) (7.4.46)

When severe outliers are a problem, the harmonic mean can be used. The harmonic average is often used with financial multiples like market-to-book ratio and price-to-earnings ratio. For illustration purposes only, we can compute the harmonic average of total return and then subtract one.

$\bar{R}_H = n\left[\sum_{i=1}^{n}\frac{1}{\left(1+R_i\right)}\right]^{-1} - 1$. (**Harmonic average**) (7.4.47)

Note that the arithmetic mean is greater than the geometric mean. The geometric mean is also greater than the harmonic mean.

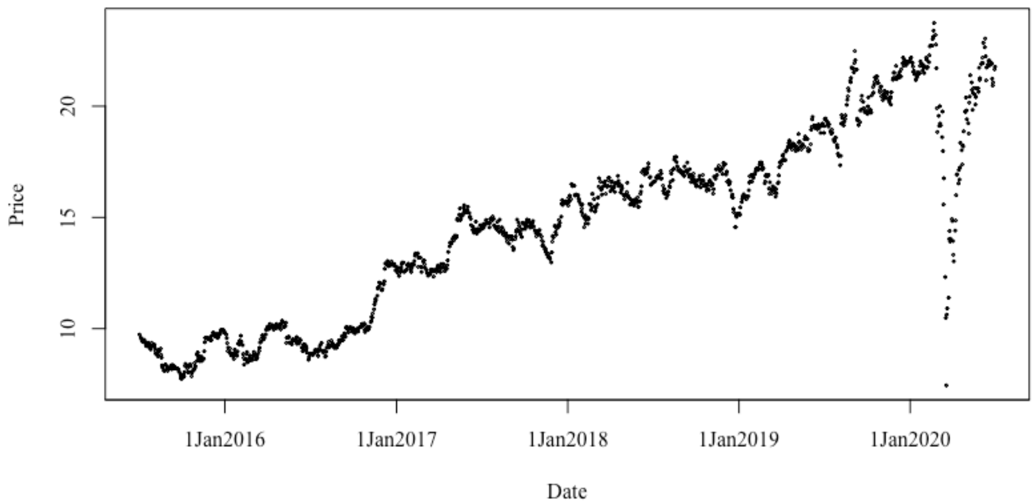
The harmonic average of five P/E ratios, say 10/1, 50/2, 40/0, 70/5, and 30/4, is

$$P\bar{E}_H = n\left(\sum_{i=1}^{n}\frac{1}{P/E}\right)^{-1} = 5\left(\frac{1}{10}+\frac{2}{50}+\frac{0}{40}+\frac{5}{70}+\frac{4}{30}\right)^{-1} = 5\left(0.1+0.04+0+0.07143+0.13333\right)^{-1}.$$

$$= 5\left(0.34476\right)^{-1} = 14.5$$

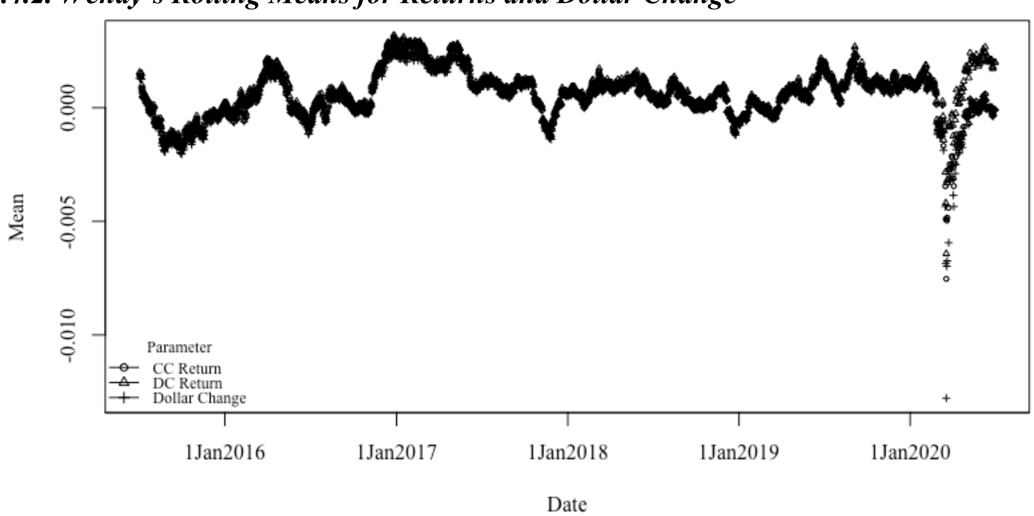
Figure 7.4.1 illustrates the dividend adjusted market price of Wendy's over a 4.5 year period incorporating the pandemic of 2020.

Figure 7.4.1. Dividend Adjusted Market Stock Price of Wendy's over a 4.5 Year Period



We compute three daily variables with this price series, the continuously compounded rate of return, the discretely compounded rate of return, and the dollar change. These daily variables are based on weekdays, where holidays contain forward filled data from the previous trading day. Forward filling is a standard practice although it introduces some mild bias as there will be more zero values than there ought to be. Figure 7.4.2 presents the rolling mean based on the last 130 observations of these variables. Note that there are about 260 calendar weekdays in a year; hence, we illustrate the rolling means over the last one half of a year.

Figure 7.4.2. Wendy's Rolling Means for Returns and Dollar Change



There are several important observations. First, rolling means are not constant. Thus, historical estimates of means are not likely to be very predictive of future means. Second, how we measure returns or changes does not change the general assessment, except under highly stressed markets. Third, extreme outcomes could not have been predicted with data analytics applied to this information. It simply does not show up in this prior history.

Standard deviation

The standard deviation is a measure of dispersion and is based on the variance. The variance is the expected deviations from the mean. The variance, however, is in units squared. By taking the standard deviation of the variance, we have a dispersion measure in the same units has the underlying observations.

The population variance is the second moment about the mean, μ_2.

$$\begin{aligned}\mu_2 &= \sigma_p^2 \\ &= E\left(R-\bar{R}_A\right)^2\end{aligned}. \textbf{ (Population variance)} \tag{7.4.48}$$

With the entire discrete population, the population variance can be expressed as

$$\begin{aligned}\mu_2 &= \frac{1}{n}\sum_{i=1}^{n}\left(R_i-\bar{R}_A\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^2 - \frac{2}{n}\bar{R}_A\sum_{i=1}^{n}R_i + \frac{n}{n}\bar{R}_A^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^2 - \frac{2}{n}\bar{R}_A n\bar{R}_A + \bar{R}_A^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\end{aligned}. \tag{7.4.49}$$

With limited sample data, the unbiased sample variance can be expressed as

$$\begin{aligned}\sigma_s^2 &= \frac{n}{n-1}\sigma_p^2 \\ &= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right) \\ &= \frac{1}{n-1}\sum_{i=1}^{n}R_i^2 - \frac{n}{n-1}\bar{R}_A^2\end{aligned}. \textbf{ (Sample variance)} \tag{7.4.50}$$

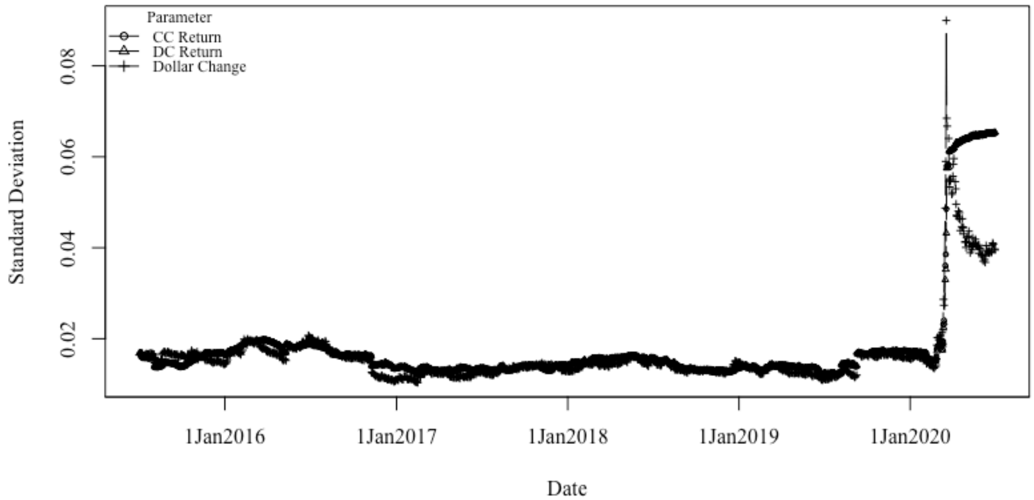
The square root of the variance, known as standard deviation, can therefore be expressed as:

$$\begin{aligned}\sigma_p &= \left[E\left(R-\bar{R}_A\right)^2\right]^{1/2} \\ &= \left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right)^{1/2}\end{aligned}. \textbf{ (Population standard deviation)} \tag{7.4.51}$$

$$\sigma_s = \left(\frac{1}{n-1}\sum_{i=1}^{n}R_i^2 - \frac{n}{n-1}\bar{R}_A^2\right)^{1/2}. \textbf{ (Sample standard deviation)} \tag{7.4.52}$$

Using the same returns and dollar change, Figure 7.4.3 presents the rolling standard deviation again based on the last 130 observations. Again, note that the rolling standard deviations are not constant, though difficult to see given the significant spike during the pandemic. Second, how we measure returns or changes does not generally change the assessment, but there are notable differences. For example, once the stock price fell significantly, then the dollar change standard deviation was much lower. Remember for lower prices, percentage changes would result in lower dollar changes. Third, again extreme outcomes could not have been predicted with data analytics applied to this information. It simply does not show up in this prior history. Finally, there appears to be a negative relationship between the Wendy's stock price and standard deviation. This negative relationship will be an important insight when we explore option valuation.

Figure 7.4.3. Wendy's Rolling Standard Deviations for Returns and Dollar Change



Skewness

Skewness is the third standardized moment and measures asymmetry in the distribution of the sample. Symmetric distributions will have a population skewness of zero and a sample skewness near zero. Negative values for this statistic indicates that the distribution is skewed left and the left tail of the distribution is longer. Positive values for this statistic indicates that the distribution is skewed right and the right tail of the distribution is longer.

The third moment about the mean, μ_3, can be expressed as

$$\mu_3 = E\left(R - \bar{R}_A\right)^3. \text{ (\textbf{Third moment about the mean})} \tag{7.4.53}$$

With the entire discrete population, the population third moment about the mean can be expressed as

$$\begin{aligned}\mu_3 &= \frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^3 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^3 - \frac{3}{n}\bar{R}_A\sum_{i=1}^{n}R_i^2 + \frac{3}{n}\bar{R}_A^2\sum_{i=1}^{n}R_i + \frac{n}{n}\bar{R}_A^3. \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^3 - \frac{3\bar{R}_A}{n}\sum_{i=1}^{n}R_i^2 - (n-3)\bar{R}_A^3\end{aligned} \tag{7.4.54}$$

Population skewness can be expressed as

$$\gamma_{S,p} = \frac{\mu_3}{\sigma_p^3}. \text{ (\textbf{Population skewness})} \tag{7.4.55}$$

With limited sample data, the unbiased sample skewness can be expressed as

$$\gamma_{S,s} = \frac{\sqrt{n(n-1)}}{n-2}\frac{\mu_3}{\sigma_p^3}$$

$$= \frac{\sqrt{n(n-1)}}{n-2}\frac{\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^3}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^2\right]^{3/2}} \text{. } \textbf{(Sample skewness)} \qquad (7.4.56)$$

$$= \frac{\sqrt{n(n-1)}}{n-2}\frac{\frac{1}{n}\sum_{i=1}^{n}R_i^3 - \frac{3\bar{R}_A}{n}\sum_{i=1}^{n}R_i^2 - (n-3)\bar{R}_A^3}{\left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right)^{3/2}}$$

If skewness is positive, then typically mean > median > mode. If skewness is negative, then typically mean < median < mode. See Stuart and Ord [1987], p. 107.

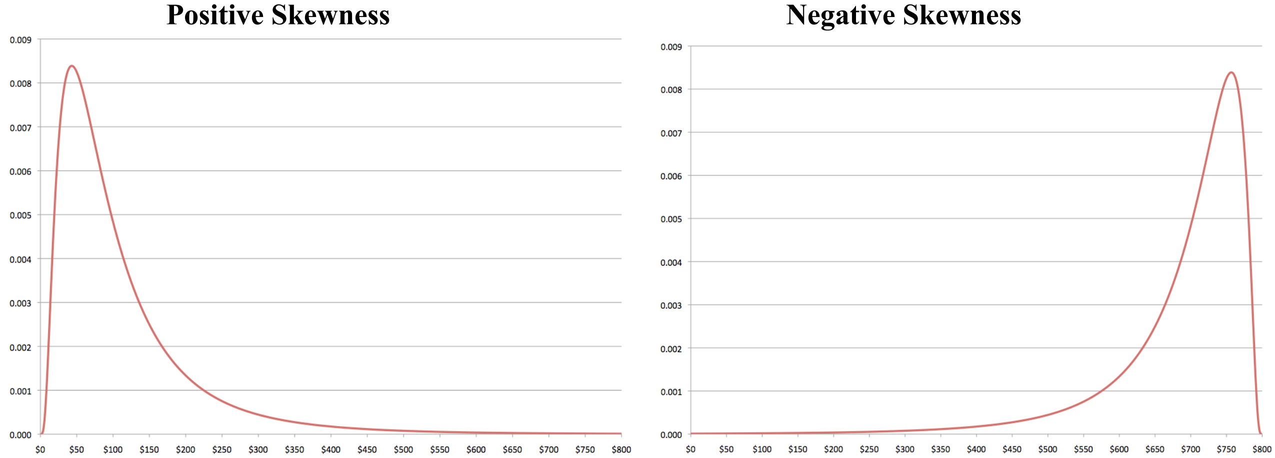
Some have categorized skewness in the following manner:

- Highly skewed: $\mu_3 < -1$ or $\mu_3 > +1$
- Moderately skewed: $-1 < \mu_3 < -1/2$ or $1/2 > \mu_3 > +1$
- Approximately symmetric: $-1/2 < \mu_3 < 1/2$

Figure 7.4.4 illustrates positive and negative skewness. The positive skewness is a lognormal distribution (stock price = $100, expected return = 12%, and standard deviation = 30%). The negative skewness is a shifted lognormal distribution of $800 minus the previous distribution.

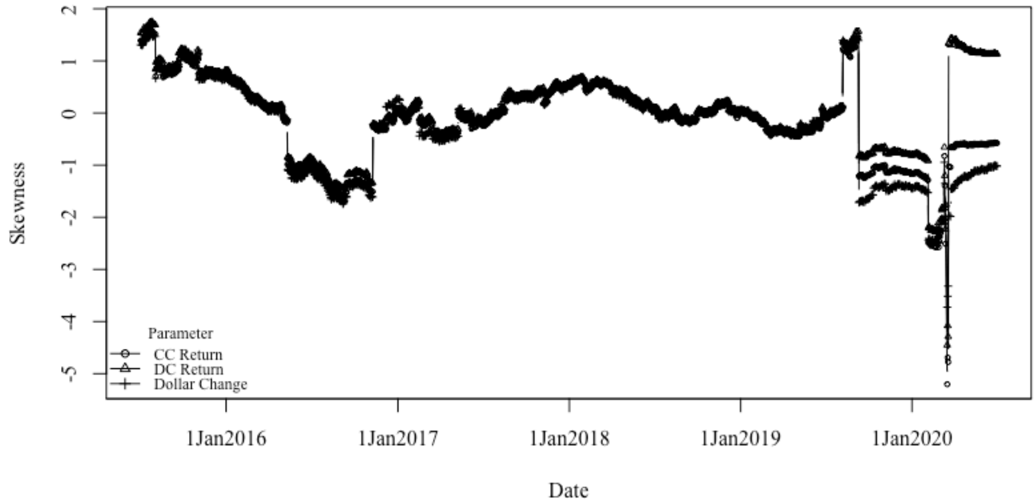
Figure 7.4.4. Illustration of Positive and Negative Skewness

Positive Skewness **Negative Skewness**



Again using the same returns and dollar change, Figure 7.4.5 presents the rolling skewness based on the last 130 observations. So would you say that the latest measure of Wendy's skewness was positive or negative? The last measured skewness is negative for the dollar change and continuous return, but the discrete return skewness is positive. As before, note that the rolling skewness is not constant. Second, how we measure returns or changes does not generally change the assessment except when significant changes occur. Third, again extreme outcomes could not have been predicted with data analytics applied to this information. It simply does not show up in this prior history. Finally, during a crisis, skewness dropped precipitously.

Figure 7.4.5. Wendy's Rolling Skewness for Returns and Dollar Change



Kurtosis

Kurtosis is the fourth standardized moment and measures the height and sharpness of the central peak of the distribution relative to the normal distribution. Higher kurtosis implies higher and sharper central peak whereas lower kurtosis implies lower and flatter central peak. The normal distribution has a kurtosis of 3. Therefore, it is customary to subtract three from kurtosis and report what is termed excess kurtosis. Often excess kurtosis is just referred to as kurtosis. One must always know how to interpret the reported kurtosis. That is, whether it has 3 subtracted or not.

The fourth moment about the mean, μ_4, can be expressed as

$$\mu_4 = E\left(R - \bar{R}_A\right)^4. \text{ (**Fourth moment about the mean**)} \tag{7.4.57}$$

With the entire discrete population, the population fourth moment about the mean can be expressed as

$$\begin{aligned} \mu_4 &= \frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^4 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n}R_i^3 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n}R_i^2 - \frac{4\bar{R}_A^3}{n}\sum_{i=1}^{n}R_i + \frac{n}{n}\bar{R}_A^4. \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n}R_i^2 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n}R_i^2 + (n-4)\bar{R}_A^4 \end{aligned} \tag{7.4.58}$$

Excess kurtosis is fourth standardized moment minus three.

$$\gamma_{K,p} = \frac{\mu_4}{\sigma_p^4} - 3. \text{ (**Population excess kurtosis**)} \tag{7.4.59}$$

With limited sample data, the unbiased sample excess kurtosis can be expressed as

$$\gamma_{K,s} = \frac{n-1}{(n-2)(n-3)}\left\{\frac{\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^4}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^2\right]^2} - 3\right\}$$

$$= \frac{n-1}{(n-2)(n-3)}\left[(n+1)\frac{\frac{1}{n}\sum_{i=1}^{n}R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n}R_i^3 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n}R_i^2 + (n-4)\bar{R}_A^4}{\left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right)^2} + 6\right]$$

. **(Sample excess kurtosis)**

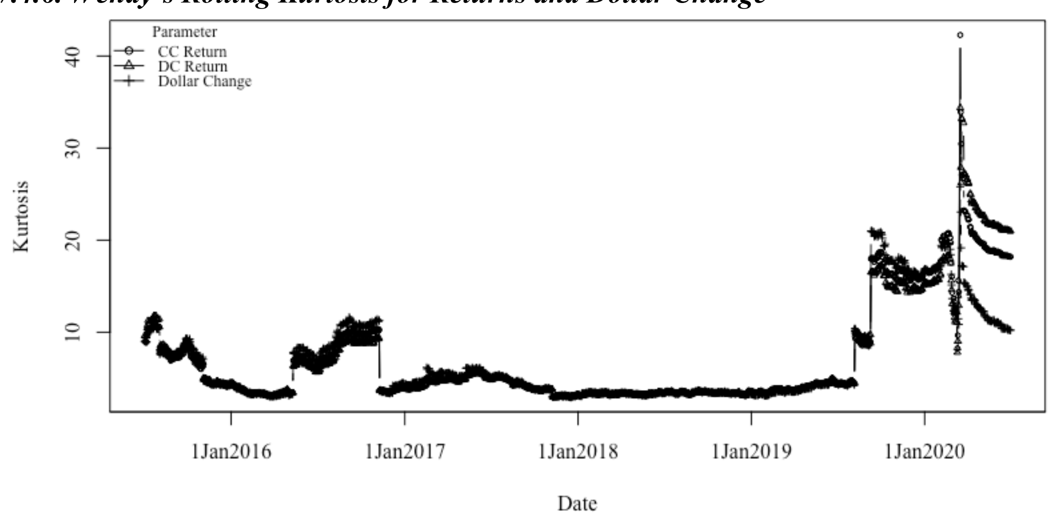
(7.4.60)

Recall if excess kurtosis is zero, then the distribution is said to be mesokurtic. The normal distribution and binomial distributions are mesokurtic. If excess kurtosis is positive, then the distribution is said to be leptokurtic. Leptokurtic distributions have sharper peaks and fatter tails. The lognormal distribution is leptokurtic as well as the Laplace distribution and the logistic distribution. If excess kurtosis is negative, then the distribution is said to be platykurtic. Platykurtic distributions have flatter peaks and thinner tails. The uniform distribution and Bernoulli distribution (p=1/2) are platykurtic.

The lowest value of excess kurtosis is –2 and the highest value of excess kurtosis is positive infinity. A coin flip has a –2 excess kurtosis, it does not have a central peak at all. A student-t distribution with four degrees of freedom has an infinite excess kurtosis.

Using the same returns and dollar change, Figure 7.4.6 presents the rolling kurtosis. As before, note that the rolling kurtosis is not constant. Second, how we measure returns or changes does not generally change the assessment except when significant changes occur. Third, again extreme outcomes could not have been predicted with data analytics applied to this information. Finally, during a crisis, kurtosis rose significantly and then began to taper off. Rates of return kurtosis remained much higher than dollar change as the value of kurtosis fell after the start of the pandemic.

Figure 7.4.6. Wendy's Rolling Kurtosis for Returns and Dollar Change



Contrast of Wendy's and McDonalds

Figure 7.4.7 shows the normalized price for Wendy's and McDonalds. Clearly, these two stocks are highly correlated but far from perfectly correlated.

Figure 7.4.7. Wendy's and McDonalds Normalized Price

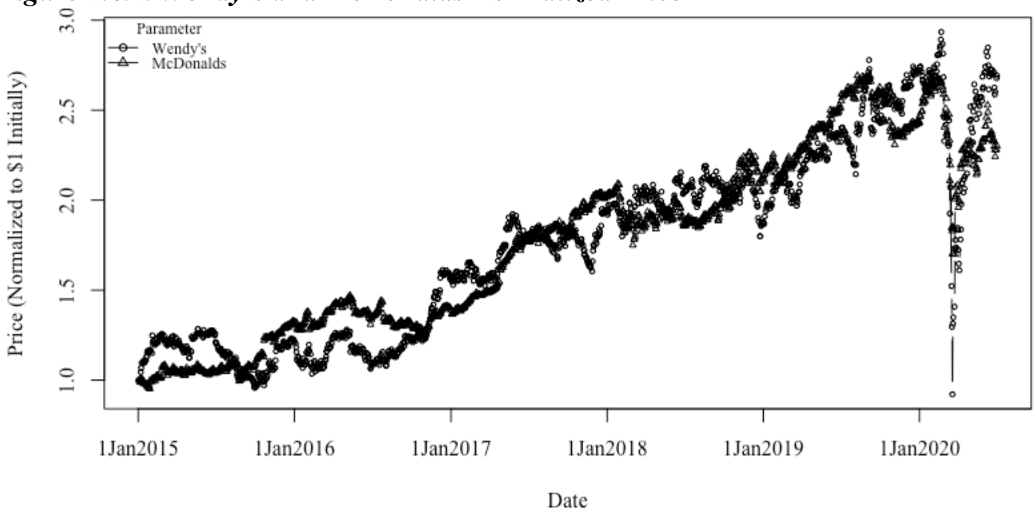
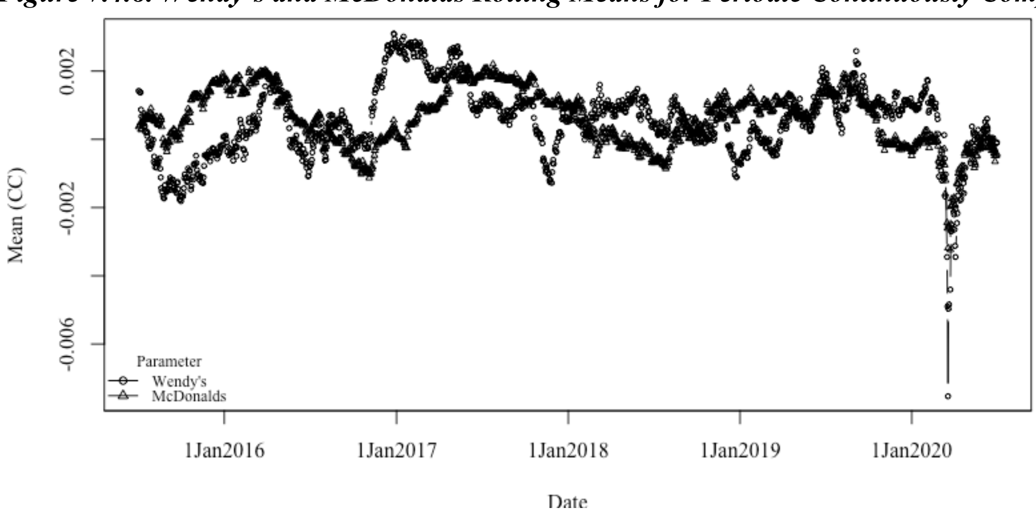


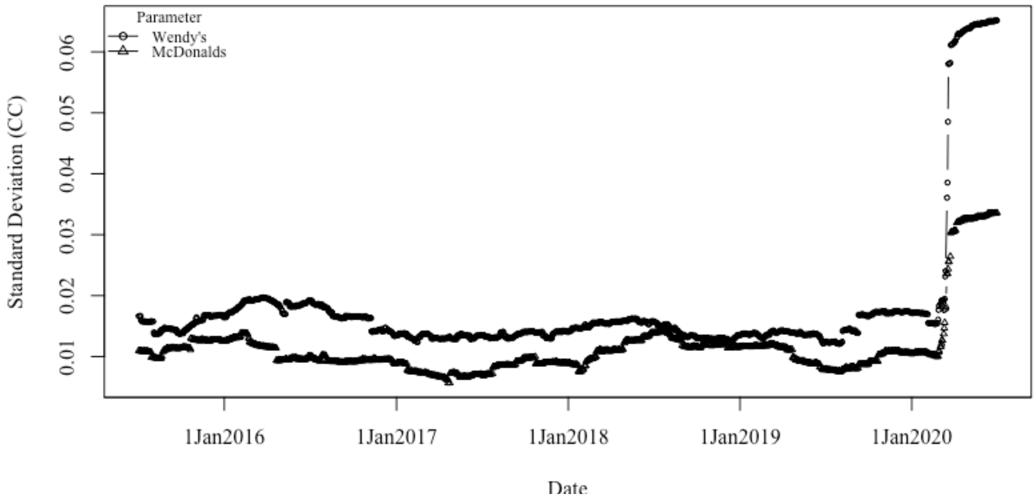
Figure 7.4.8 illustrates the rolling means for both Wendy's and McDonalds. Again, we see positive co-movement but not perfect correlation. At times McDonalds has a higher rolling mean but most of the time Wendy's has a higher rolling mean.

Figure 7.4.8. Wendy's and McDonalds Rolling Means for Periodic Continuously Compounded Returns



An interesting key to understanding this difference in rolling means is perhaps found in an analysis of rolling standard deviations presented in Figure 7.4.9. Note that Wendy's standard deviation is always higher than McDonalds. Empirically, Wendy's is riskier than McDonalds and we would naturally expect higher means on average over the long run.

Figure 7.4.9. Wendy's and McDonalds Rolling Standard Deviations for Periodic Continuously Compounded Returns



To emphasize the dominance, Figure 7.4.10 presents the difference between Wendy's and McDonalds standard deviations. Notice that this difference is always positive.

Figure 7.4.10. Wendy's Less McDonalds Rolling Standard Deviations for Periodic Continuously Compounded Returns

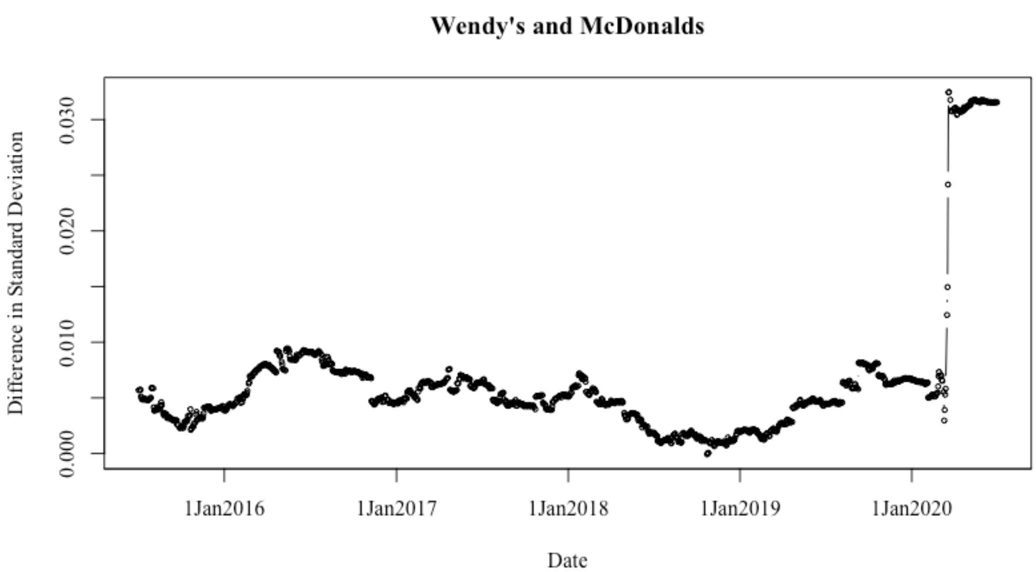


Figure 7.4.11 and 7.4.12 presents the difference between Wendy's and McDonalds skewness and kurtosis, respectively. Notice that there is no clear pattern.

Figure 7.4.10. Wendy's Less McDonalds Rolling Skewness for Periodic Continuously Compounded Returns

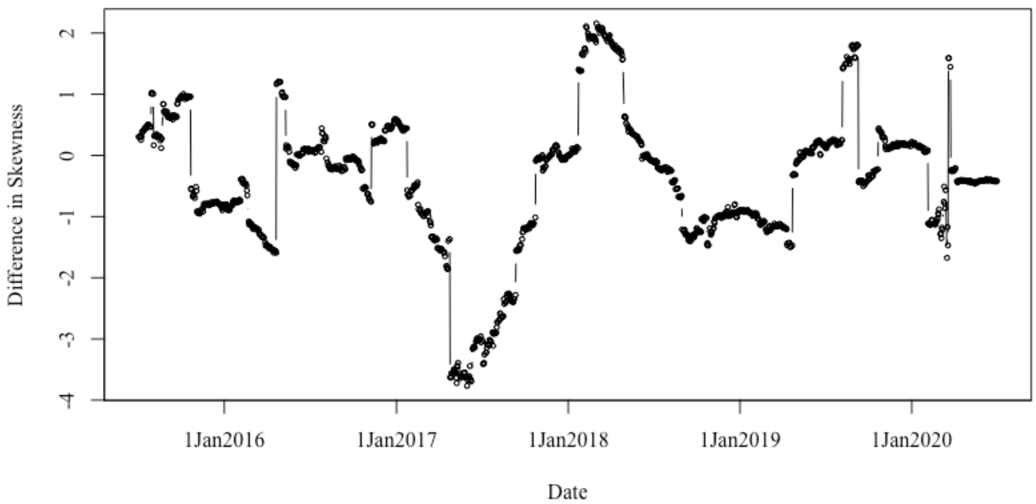
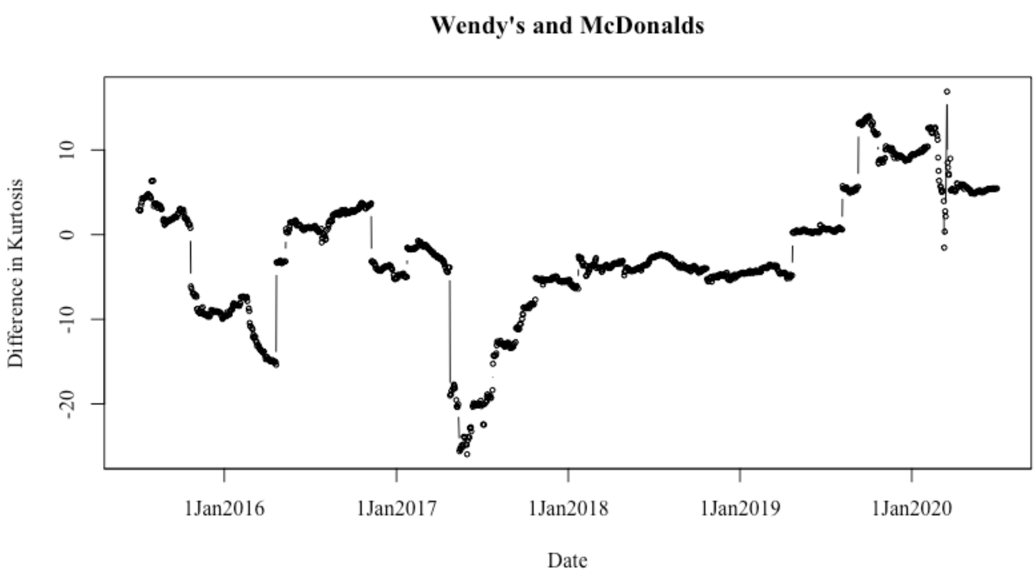


Figure 7.4.11. Wendy's Less McDonalds Rolling Kurtosis for Periodic Continuously Compounded Returns



We now turn to static risk measures related to the LSC valuation model.

Static risk measures related to the LSC valuation model
Recall LSC valuation model presented in Module 4.3 is based on some generic valuation task involving annual expected future series of cash flows (CF_i) with known positive initial cash flow (CF_0). We assumed cash flow rates (g_i) are modeled within LSC model framework. Specifically, we estimate the perpetual (level) growth rate, the short-term (slope) growth rate, and as many growth rate curvature factors as desired. We also assumed forward discount rates (f_i) can also be modeled within the LSC model framework. Specifically, we estimated the perpetual (level) forward discount rate, the short-term (slope) forward discount rate, and as many forward discount rate curvature factors as desired. Recall the asset value (V) can be expressed as

$$V = CF_0 \sum_{i=1}^{\infty} \frac{e^{\sum_{j=1}^{i} g_j \tau_j}}{e^{\sum_{j=1}^{i} f_j \tau_j}}$$

$$= CF_0 \sum_{i=1}^{\infty} e^{\sum_{j=1}^{i} (g_j - f_j)\tau_j} . \quad (7.4.61)$$

$$= CF_0 \sum_{i=1}^{\infty} e^{\sum_{j=1}^{i} -(f_j - g_j)\tau_j}$$

Although straightforward to have multiple factors, we focus here on the two factor LSC model for both the growth rate and forward discount rate thus each rate is estimated based on the LSC model as

$$g_j = L_g + sc_{g,j} S_g \text{ and} \quad (7.4.62)$$

$$f_j = L_f + sc_{f,j} S_f, \quad (7.4.63)$$

where the scalars are provided here again for convenience,

$$sc_{g,j} = \frac{s_g}{\tau_i}\left(1 - e^{-\tau_i / s_g}\right) \text{ and} \quad (7.4.64)$$

$$sc_{f,j} = \frac{s_f}{\tau_i}\left(1 - e^{-\tau_i / s_f}\right). \quad (7.4.65)$$

Thus, the fully calibrated two factor LSC valuation model from Module 4.3 is

$$V_0 = CF_0 \sum_{i=1}^{\infty} e^{\sum_{j=1}^{i} -\left[\hat{L}_f + sc_{f,j}\hat{S}_f - \left(\hat{L}_g + sc_{g,j}\hat{S}_g\right)\right]} . \quad (7.4.66)$$

Or the value per unit of cash flow

$$VCF = \sum_{i=1}^{\infty} e^{\sum_{j=1}^{i} -\left[\hat{L}_f + sc_{f,j}\hat{S}_f - \left(\hat{L}_g + sc_{g,j}\hat{S}_g\right)\right]} . \quad (7.4.67)$$

The first derivatives are provided below for the LSC parameters whether they are calibrated or not,

$$\frac{\partial V}{\partial L_f} = -CF_0 \sum_{i=1}^{\infty} i e^{-i\left(L_f - L_g\right) - \sum_{j=1}^{i}\left(sc_{f,j}S_f - sc_{g,j}S_g\right)}, \quad (7.4.68)$$

$$\frac{\partial V}{\partial L_g} = CF_0 \sum_{i=1}^{\infty} i e^{-i\left(L_f - L_g\right) - \sum_{j=1}^{i}\left(sc_{f,j}S_f - sc_{g,j}S_g\right)}, \quad (7.4.69)$$

$$\frac{\partial V}{\partial S_f} = -CF_0 \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} sc_{f,j}\right) e^{-i\left(L_f - L_g\right) - \sum_{j=1}^{i}\left(sc_{f,j}S_f - sc_{g,j}S_g\right)}, \text{ and} \quad (7.4.70)$$

$$\frac{\partial V}{\partial S_g} = CF_0 \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} sc_{g,j}\right) e^{-i\left(L_f - L_g\right) - \sum_{j=1}^{i}\left(sc_{f,j}S_f - sc_{g,j}S_g\right)} . \quad (7.4.71)$$

Figure 7.4.7 reports the first derivative of different parameters from the LSC valuation model. The LSC valuation model first derivatives include the growth rate level (DeltaLg), the forward discount rate level (DeltaLf), the growth rate slope (DeltaSg), and the forward discount rate slope (DeltaSf). Again note that the nature of the perpetual growth rate and forward discount rates results in identical first derivatives except for sign change.

Figure 7.4.7. First Derivatives (Expressed Assuming a One Percentage Point Move)

Industry	Ticker	DeltaLg	DeltaLf	DeltaSg	DeltaSf	DeltaVCFLg	DeltaVCFLf	DeltaVCFSg	DeltaVCFSf
Broad Market	SPY	2.480	-2.480	0.311	-0.711	0.420	-0.420	0.053	-0.120
Technology	XLK	0.657	-0.657	0.092	-0.201	0.529	-0.529	0.074	-0.162
Financial	XLF	0.044	-0.044	0.013	-0.023	0.071	-0.071	0.020	-0.038
Industrials	XLI	0.294	-0.294	0.054	-0.113	0.186	-0.186	0.034	-0.072
Consumer Discretionary	XLY	0.905	-0.905	0.118	-0.264	0.566	-0.566	0.074	-0.165
Materials	XLB	0.185	-0.185	0.040	-0.079	0.149	-0.149	0.032	-0.064
Healthcare	XLV	0.473	-0.473	0.083	-0.176	0.201	-0.201	0.035	-0.075
Utilities	XLU	0.199	-0.199	0.043	-0.087	0.098	-0.098	0.021	-0.043
Consumer Stables	XLP	0.233	-0.233	0.045	-0.094	0.139	-0.139	0.027	-0.056
Energy	XLE	0.081	-0.081	0.023	-0.042	0.054	-0.054	0.015	-0.029

Summary

In this module, we explored various static risk measures related to several stock valuation models. First, we examined the traditional single stage dividend discount model. Specifically, we explored durations and convexities within this model. Second, we sketched various potential measures related to the N-stage dividend discount model. Third, we explored various static risk measures related to the LSC valuation model. Finally, we reviewed selected basic univariate statistics and illustrate results for a couple of financial instruments.

References

Balanda, Kevin P. and H. L. MacGillivray, "Kurtosis: A Critical Review," *American Statistician* 42, 111–119

Joanes, D. N. and C. A. Gill. "Comparing Measures of Sample Skewness and Kurtosis". *The Statistician* 47(1): 183–189.

Stuart, Alan and J. Keith Ord, *Kendall's Advanced Theory of Statistics Volume 1 Distribution Theory*, Fifth Edition (New York: Oxford University Press, 1987).