Module 5.7: Geometric Brownian Motion-Based Compound Option Valuation Models

Learning objectives

- Explain how to value compound options based on the geometric Brownian motion
- Review the compound option assumptions and boundary conditions

Executive summary[1]

We review the assumptions underlying the compound option valuation model proposed by Geske (1979) and extended by Brooks (2019). Geske's model is based on the underlying instrument following geometric Brownian motion (GBM).[2] Brooks provides an extensive and detailed derivation along with extending the model to handle both cash flows on the underlying instrument as well as cash flows on the underlying option. Next, we review the compound option boundary conditions based on static arbitrage. We review one representation of the GBM compound option valuation model. Finally, the R code is discussed.

Central finance concepts

The first known use of the phrase "compound options" in a financial context is found in *The Bankers Magazine* (Volume 59) published in 1895. The phrase "compound options" is found in the definition of options. Compound options in the late 1800s and early 1900s denoted merely complex option-based strategies, such as purchasing both a call and a put, purchasing the underlying along with purchasing a call option (call of more), or purchasing the underlying along with purchasing a put option (put of more). Charles Castelli describes call of mores and put of mores in detail in this 1877 book titled, *The theory of "options" in stock and shares*.

Black and Scholes (1973) appear to have coined the phrase "compound option" in the context of modern option theory. They note, "If the company has coupon bonds rather than pure discount bonds out- standing, then we can view the common stock as a 'compound option.' The common stock is an option on an option on . . . an option on the firm. After making the last interest payment, the stockholders have an option to buy the company from the bondholders for the face value of the bonds. Call this "option 1." After making the next-to-the-last interest payment, but before making the last interest payment, the stockholders have an option to buy option 1 by making the last interest payment. Call this "option 2." Before making the next-to-the-last interest payment, the stock- holders have an option to buy option 2 by making that interest payment. This is "option 3." The value of the stockholders' claim at any point in time is equal to the value of option n + 1, where n is the number of interest payments remaining in the life of the bond" (651-652)

Geske (1979) provides the first detailed treatment of modern compound options. He specifically focuses on the compound option related to stock and the underlying firm. Rubinstein (1991) generalizes Geske's model for other compound options as well as dividends on the underlying instrument, but not dividends on the underlying option. We now introduce several basic concepts related to compound options.

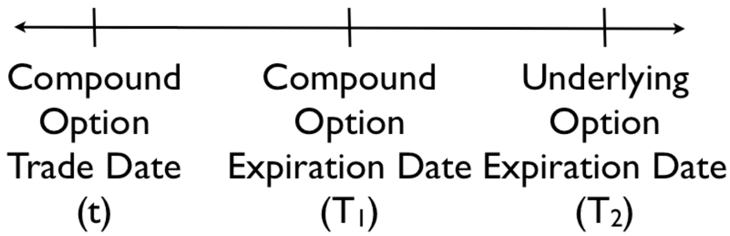
Compound option basics

A compound option is an option on an option. Figure 5.7.1 illustrates three important dates for compound options in calendar time.

[1]This module is based on Brooks (2019).

[2]There is an arithmetic Brownian motion version being developed but will not be covered in this book.

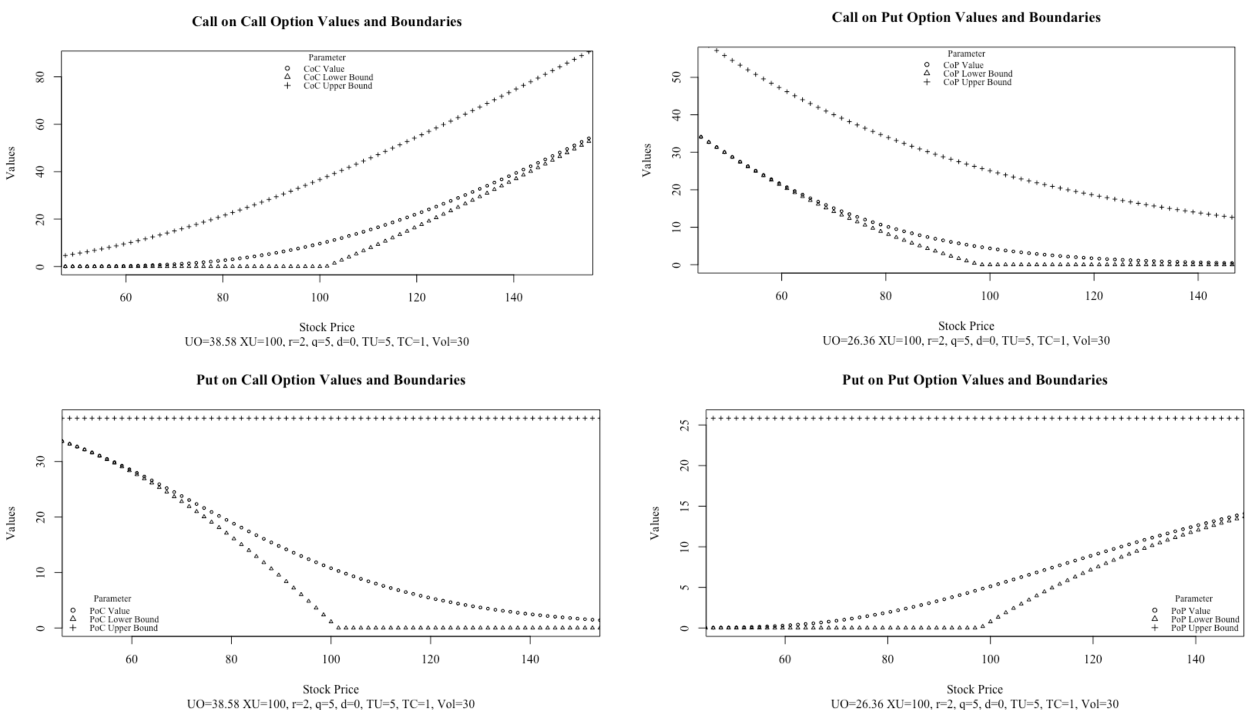
Figure 5.7.1. Three Important Dates for Compound Options with Notation



There are four different types of compound options based on whether the underlying option is a call or put as well as whether the option on the underlying option is a call or put. Thus, we have call on call (cacall), call on put (caput), put on call (pucall), and put on put (puput). We will introduce generic notation to present on single model that encapsulates all four types.

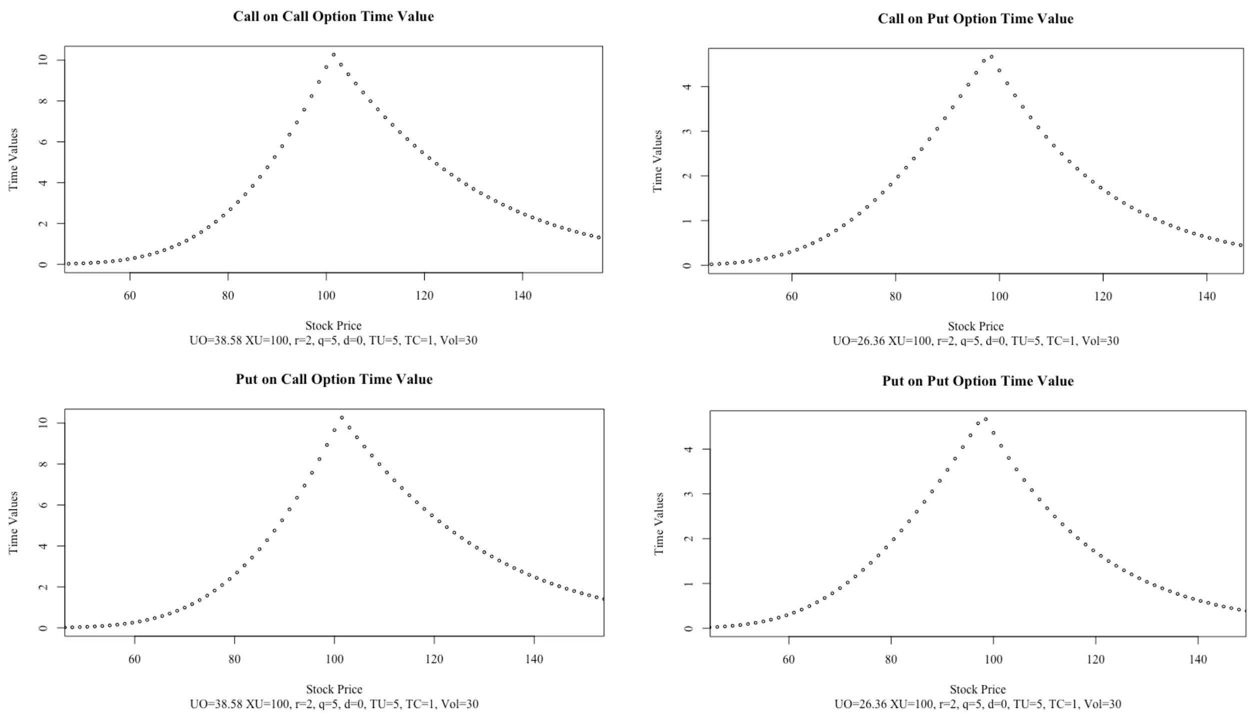
Based on the R code provided, several graphs are generated. Figure 5.7.2 illustrates the compound option values along with boundaries for a particular set of input parameters identified in the footer of each plot.

Figure 5.7.2 Compound Option Values with Boundary Conditions



We also produce time value plots in Figure 5.7.3 to illustrate the influence of the lognormal distribution assumption.

Figure 5.7.3 Compound Option Time Values



Quantitative finance materials

We now examine the technical details of compound options.

Compound option basics

A compound option is an option on an option. Thus, for European-style compound options there are two expiration dates, $t < T_1 < T_2$, where t denotes the valuation date, T_1 denotes the compound option's expiration date and T_2 denotes the underlying option's expiration date, where dates are reported in fraction of a year. There are also two strike prices, X_C and X_U, where X_C is the strike price of the compound option and X_U is the strike price of the underlying option.

We assume the strike prices are positive and $0 < X_C$ and $0 < X_U$. Therefore, the payoffs on the underlying options at maturity (T_2) are:

$$c_{T_2}\left(S, X_U, T_2\right) = \max\left(0, S_{T_2} - X_U\right) \text{ (Plain vanilla call option)} \tag{5.7.1}$$

$$p_{T_2}\left(S, X_U, T_2\right) = \max\left(0, X_U - S_{T_2}\right) \text{ (Plain vanilla put option)} \tag{5.7.2}$$

Thus, the plain vanilla call and put option values are observed at time T_2 for the underlying instrument S with strike price X_2 and the option matures at time T_2. Therefore, the payoff on the compound option at maturity (T_1) is:

$$CoC_{T_1}\left[c\left(S, X_U, T_2\right), X_C, T_1\right] = \max\left[0, c_{T_1}\left(S, X_U, T_2\right) - X_C\right] \text{ (Call on call or Cacall)} \tag{5.7.3}$$

$$CoP_{T_1}\left[p\left(S, X_U, T_2\right), X_C, T_1\right] = \max\left[0, p_{T_1}\left(S, X_U, T_2\right) - X_C\right] \text{ (Call on put or Caput)} \tag{5.7.4}$$

$$PoC_{T_1}\left[c\left(S, X_U, T_2\right), X_C, T_1\right] = \max\left[0, X_C - c_{T_1}\left(S, X_U, T_2\right)\right] \text{ (Put on call or Pucall)} \tag{5.7.5}$$

$$PoP_{T_1}\left[p\left(S, X_U, T_2\right), X_C, T_1\right] = \max\left[0, X_C - p_{T_1}\left(S, X_U, T_2\right)\right] \text{ (Put on put or Puput)} \tag{5.7.6}$$

The decision to exercise a compound option depends on whether the compound option is in the money. Specifically, whether the underlying option's value at time T_1 is greater than its strike price. The underlying option's value depend on the underlying instrument's value, S_{T_1}, and time to expiration, $T_2 - T_1$.

Compound option valuation model notation

Due to the sheer volume of variables, we document the notation here:

- ι_C – denotes indicator function (= +1 if compound option is call and –1 if compound option is put),
- ι_U – denotes indicator function (= +1 if underlying option is call and –1 if underlying option is put),
- t – denotes current calendar time expressed as a fraction of a year from some 0 point, e.g., 0 years,
- T_1 – denotes calendar time when compound option expires expressed as a fraction of a year from some 0 point, e.g., 0.5 years,
- T_2 – denotes calendar time when underlying option expires expressed as a fraction of a year from some 0 point, e.g., 1.0 year,
- S_t – Underlying instrument value at time t,
- σ^2 – Annualized standard deviation of continuously compounded rates of return of the underlying instrument,
- r – Annualized continuous risk free interest rate,
- δ – Annualized continuous cash flow paid to underlying instrument holder (e.g., stock dividend yield),
- $\hat{q}$ – Annualized continuous cash flow paid to compound option holder (e.g., when stock is the underlying option on the firm, then this stock pays dividends),
- X_U – Strike price for underlying option,
- X_C – Compound option strike price,
- $O\left(S,t;\iota_U,X_U,T_2,\sigma,r,\delta,\hat{q}\right)$ – Underlying option value at time t for T_2 maturity (parameters suppressed where possible),
- $c_t = c_t\left(S,X_U,T_2\right)$ – European call option on underlying instrument, S, observed at time t, with strike price X_U, maturing at time T_2,
- $p_t = p_t\left(S,X_U,T_2\right)$ – European put option on underlying instrument, S, observed at time t, with strike price X_U, maturing at time T_2,
- $B_{t,T,x} = e^{-x(T-t)}$ – Zero coupon, discount bond with \$1 par, observed at time t, maturing at time T, with discount rate x,
- $CoC_t = CoC_t\left[c_t\left(S,X_U,T_2\right),X_C,T_1\right]$ – cacall, call on call compound option with strike price X_C, maturing at time T_1,
- $CoP_t = CoP_t\left[p_t\left(S,X_U,T_2\right),X_C,T_1\right]$ – caput, call on put compound option with strike price X_C, maturing at time T_1,
- $PoC_t = PoC_t\left[c_t\left(S,X_U,T_2\right),X_C,T_1\right]$ – pucall, put on call compound option with strike price X_C, maturing at time T_1, and
- $PoP_t = PoP_t\left[p_t\left(S,X_U,T_2\right),X_C,T_1\right]$ – puput, put on put compound option with strike price X_C, maturing at time T_1.

GBM COVM assumptions

As with any model, the GBM compound option valuation model is based on a set of assumptions, including

- Standard finance presuppositions and assumptions apply (see Chapter 2)
- Underlying instrument behaves randomly and follows a lognormal distribution (or follow geometric Brownian motion)

- Risk-free interest rate exists, is constant, borrowing and lending allowed
- Volatility of the underlying instrument's continuously compounded rate of return is known, positive and constant
- No market frictions, including no taxes, no transaction costs, unconstrained short selling allowed, and continuous trading
- Investors prefer more to less
- Option are European-style (exercise available only at maturity)
- Underlying instrument and underlying option may pay a continuous cash flow yield

Compound option boundary conditions and parities

We briefly review the appropriate compound option boundary conditions and parities followed by selected proofs. The following boundary conditions hold for the underlying options:

Underlying call lower bound: $c_t \geq \max\left(0, B_{t,T_2,\delta} S_t - B_{t,T_2,r_c} X_U\right),$ (5.7.7)

Underlying call upper bound: $c_t \leq B_{t,T_2,\delta} S_t,$ (5.7.8)

Underlying put lower bound: $p_t \geq \max\left(0, B_{t,T_2,r_c} X_U - B_{t,T_2,\delta} S_t\right)$, and (5.7.9)

Underlying put upper bound: $p_t \leq B_{t,T_2,r_c} X_U.$ (5.7.10)

The following boundary conditions hold for compound options:

Call on call lower bound: $CoC_t \geq \max\left(0, B_{t,T_1,\hat{q}} c_t - B_{t,T_1,r_c} X_C\right),$ (5.7.11)

Call on call upper bound: $CoC_t \leq B_{t,T_1,\hat{q}} c_t,$ (5.7.12)

Call on put lower bound: $CoP_t \geq \max\left(0, B_{t,T_1,\hat{q}} p_t - B_{t,T_1,r_c} X_C\right),$ (5.7.13)

Call on put upper bound: $CoP_t \leq B_{t,T_1,\hat{q}} p_t,$

Put on call lower bound: $PoC_t \geq \max\left(0, B_{t,T_1,r_c} X_C - B_{t,T_1,\hat{q}} c_t\right),$ (5.7.14)

Put on call upper bound: $PoC_t \leq B_{t,T_1,r_c} X_C,$ (5.7.15)

Put on put lower bound: $PoP_t \geq \max\left(0, B_{t,T_1,r_c} X_C - B_{t,T_1,\hat{q}} p_t\right)$, and (5.7.16)

Put on put upper bound: $PoP_t \leq B_{t,T_1,r_c} X_C.$ (5.7.17)

Important parities related to the univariate and bivariate normal cumulative distribution function are as follows:

$$N_2\left(d_1, d_2; \rho\right) = N_1\left(d_1\right) - N_2\left(d_1, -d_2; -\rho\right), \quad (5.7.18)$$

$$N_2\left(d_1, d_2; \rho\right) = N_1\left(d_2\right) - N_2\left(-d_1, d_2; -\rho\right), \quad (5.7.19)$$

$$N_2\left(d_1, d_2; \rho\right) = N_1\left(d_1\right) + N_1\left(d_2\right) - 1 + N_2\left(-d_1, -d_2; \rho\right), \text{ and} \quad (5.7.20)$$

$$N_1\left(-d\right) = 1 - N\left(d\right). \quad (5.7.21)$$

In our notation, the underlying option put-call parity is:[3]

[3]Note that option yields would be expected to be different between calls and puts within the context of common stocks as equity, although puts in this context are difficult to interpret. Within the context of the generic compound options model, one could easily have yield paying calls and puts.

$$B_{t,T_2,\hat{q}_c} c_t - B_{t,T_2,\hat{q}_p} p_t = B_{t,T_2,\delta} S_t - B_{t,T_2,r_c} X_C. \quad (5.7.22)$$

In our notation, the compound option put-call parity relations include:

$$CoC_t - PoC_t = B_{t,T_1,\hat{q}} c_t - B_{t,T_1,r_c} X_C, \quad (5.7.23)$$

$$CoP_t - PoP_t = B_{t,T_1,\hat{q}} p_t - B_{t,T_1,r_c} X_C, \text{ and} \quad (5.7.24)$$

$$CoC_t - PoC_t - \left(CoP_t - PoP_t\right) = B_{t,T_2,\delta} S_t - B_{t,T_2,r_c} X_U. \quad (5.7.25)$$

The proofs for selected boundary conditions, one parity, and the valuation model can be found in Brooks (2019). Next, we present the compound option valuation model.

Compound option valuation model

Compound option pricing model (CO) observed at time t under geometric Brownian motion based on an underlying instrument (S_t) with the compound option exercise price (X_C) expiring at time 2 (T_1) and the underlying option exercise price (X_U) expiring at time 1 ($T_2 > T_1$) can be expressed as

$$\begin{aligned} &CO\left(S,t,T_1,T_2,\iota_C,\iota_U\right) = \iota_C \iota_U S_t B_{t,T_2,\delta} B_{T_1,T_2,-\hat{q}} N_2\left(\iota_C \iota_U d_{11}, \iota_U d_{12}; \iota_C \rho\right) \\ &-\iota_C \iota_U X_U B_{t,T_2,r} B_{T_1,T_2,-\hat{q}} N_2\left(\iota_C \iota_U d_{21}, \iota_U d_{22}; \iota_C \rho\right) - \iota_C X_C B_{t,T_1,r} N\left(\iota_C \iota_U d_{21}\right) \end{aligned}, \quad (5.7.26)$$

where indicator functions denote

$$\iota_C = \begin{cases} +1 \text{ if compound call option} \\ -1 \text{ if compound put option} \end{cases} \text{ and} \quad (5.7.27)$$

$$\iota_U = \begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option} \end{cases}. \quad (5.7.28)$$

Recall a default-free, zero coupon, \$1 par bond be expressed as

$$B_{t,T,r} = e^{-r(T-t)}, \quad (5.7.29)$$

and the bivariate cumulative standard normal distribution

$$N_2\left(a,b;\rho\right) \equiv \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{\exp\left\{-\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2\left(1-\rho^2\right)}\right\}}{2\pi\sqrt{1-\rho^2}} dz_1 \, dz_2. \quad (5.7.30)$$

Using a generic time to maturity, *T*, the periodic standard deviation are

$$\sigma_{t,T} = \sigma\sqrt{T-t}. \quad (5.7.31)$$

The correlation coefficient used in the bivariate distribution is

$$\rho = \frac{\sqrt{T_1 - t}}{\sqrt{T_2 - t}}, \quad (5.7.32)$$

and thus

$$\sqrt{1-\rho^2} = \frac{\sqrt{T_2 - T_1}}{\sqrt{T_2 - t}}. \quad (5.7.33)$$

Let $S_{T_1}^*$ be defined such that underlying option is at-the-money or

$$\iota_U S_{T_1}^* B_{T_1,T_2,\delta-\hat{q}} N_1\left(\iota_U d_{1,T_1,T_2}^*\right) - \iota_U X_U B_{T_1,T_2,r-\hat{q}} N_1\left(\iota_U d_{2,T_1,T_2}^*\right) - X_C = 0, \quad (5.7.34)$$

where

$$d_{2,T_1,T_2}^* = \frac{\ln\left(\frac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) - \frac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}}, \quad (5.7.35)$$

$$d_{1,T_1,T_2}^* = \frac{\ln\left(\frac{S_{T_1}^* B_{T_1,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma_{T_1,T_2}^2}{2}}{\sigma_{T_1,T_2}} = d_{2,T_1,T_2}^* + \sigma_{T_1,T_2}, \text{ and} \quad (5.7.36)$$

$$N_1(d) = \int_{-\infty}^{d} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx. \quad (5.7.37)$$

Let d_{ij} denote the upper bound of the bivariate normal cumulative distribution function where $i = 1, 2$ denotes whether the volatility term is added ($i = 1$) or subtracted ($i = 2$) and $j = 1, 2$ denotes whether the evaluation is S^* at T_1 ($j = 1$) or X_U at T_2 ($j = 2$). We define

$$d_{21} \equiv \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right) - \frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}}, \quad (5.7.38)$$

$$d_{11} \equiv \frac{\ln\left(\frac{S_t B_{t,T_1,-(r-\delta)}}{S_{T_1}^*}\right) + \frac{\sigma_{t,T_1}^2}{2}}{\sigma_{t,T_1}} = d_{21} + \sigma_{t,T_1}, \quad (5.7.39)$$

$$d_{22} \equiv \frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right) - \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}}, \text{ and} \quad (5.7.40)$$

$$d_{12} \equiv \frac{\ln\left(\frac{S_t B_{t,T_2,-(r-\delta)}}{X_U}\right) + \frac{\sigma_{t,T_2}^2}{2}}{\sigma_{t,T_2}} = d_{22} + \sigma_{t,T_2}. \quad (5.7.41)$$

We illustrate the proof of this model with the partial differential equation and the appropriate partial derivatives in Module 9.5.

Summary

We reviewed the assumptions underlying the compound option valuation model proposed by Geske (1979) and extended by Brooks (2019). This model is based on the underlying instrument following GBM. Details

on the derivation and extension can be found in Brooks (2019). Next, we review the compound option boundary conditions based on static arbitrage. We reviewed one representation of the GBM compound option valuation model. Finally, selected excerpts from the R code was discussed.

References

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