Module 5.4

Geometric Brownian Motion
Option Valuation Model
(GBMOVM)

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Overview

- Review assumptions of GBMOVM
- Explore role of dividends
- Identify different representations of GBMOVM
- Derive the GBMOVM
- Review selected plots

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Central Finance Concepts

- Geometric Brownian motion (GBM)
- Option valuation model (GBMOVM)
- Key assumptions
- Graphical illustrations

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GBMOVM Key Assumptions

- Terminal distribution is lognormal
- Risk-free rate is constant, borrowing and lending allowed
- Volatility of the underlying instrument's continuously compounded rate of return is known, positive, and constant
- European-style options only

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Selected Plots

- GBMOVM values and boundaries
  - Quality model should not violate arbitrage boundaries
  - Should roughly correspond to observed option prices
- GBMOVM time values

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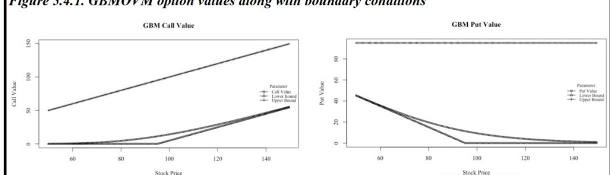
GBMOVM and Boundaries

Figure 5.4.1. GBMOVM option values along with boundary conditions



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GBM Applied to Digital Options

- Cash or Nothing (CoN): option payoff is a fixed digital payout (DP) if the option expires in-the-money
- Asset or Nothing (AoN): a fixed number of units of the underlying (assumed 1 unit) if the option expires in-the-money
- Call=long AoN call and short CoN call
- Put=long CoN put and short AoN put

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Quantitative Finance Materials

- GBMOVM assumptions
- Mathematical adjustments for dividends
- Generic boundaries and GBMOVM
- Derivation of GBMOVM
- Digital option valuation expressions

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GBMOVM Assumptions

- Standard finance presuppositions and assumptions apply (see Chapter 2)
- Underlying instrument behaves randomly and follows a lognormal distribution (or follow geometric Brownian motion)
- Risk-free interest rate exists, is constant, borrowing and lending allowed
- Volatility of the underlying instrument's continuously compounded rate of return is known, positive, and constant
- No market frictions, including no taxes, no transaction costs, unconstrained short selling allowed, and continuous trading
- Investors prefer more to less
- Option are European-style (exercise available only at maturity)
- Underlying instrument may pay a constant continuous cash flow yield (e.g., dividend yield) as well as possibly discrete cash flows (e.g., discrete dividends)

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Dividends

- Escrow method assumed
- All dividends over option life extracted

$$PV_T\left(\underline{D}\right)=S_0\left(1-e^{-\delta T}\right)+\sum_{i=1}^{N}PV_{\tau_i}\left(D_i\right)$$

- Underlying instrument sans dividends

$$S_0'=S_0-PV_T\left(\underline{D}\right)=S_0-\left[S_0\left(1-e^{-\delta T}\right)+\sum_{i=1}^{N}PV_{\tau_i}\left(D_i\right)\right]=B_\delta S_0-\sum_{i=1}^{N}PV_{\tau_i}\left(D_i\right)$$

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Generic OVM

- Generic option valuation

$$O\left(S_0',t;\iota_U,X,T\right)=PV_k\left\{E_{0,\pi}\left[O\left(\tilde{S}_T',T\right)\right]\right\}=PV_k\left(E_0\left\{\max\left[0,\iota_U\left(\tilde{S}_T'-X\right)\right]\right\}\right)$$

$$\iota_U=\begin{cases}+1 \text{ if underlying call option}\\ -1 \text{ if underlying put option}\end{cases}$$

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Generic Option Boundaries

- Upper bound

$$O_0\le\max\left(\iota_U S_0',-\iota_U B_r X\right)$$

- Lower bound

$$O_0\ge\max\left[0,\iota_U\left(S_0'-B_r X\right)\right]$$

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GBMOVM Generic Model

$$O(S_0',t;\iota_U,X,T,r,\sigma)=PV_r\left\{E_0\left[O(FV_rS_t',t)\right]\right\}$$
$$=B_r\left[\iota_U S_0' B_{-r} N(\iota_U d_1)-\iota_U XN(\iota_U d_2)\right]=\iota_U S_0' N(\iota_U d_1)-\iota_U XB_r N(\iota_U d_2)$$

$$\iota_U=\begin{cases}+1 \text{ if underlying call option}\\ -1 \text{ if underlying put option}\end{cases} \qquad B_r=e^{-r} \qquad N(d)=\int_{-\infty}^{d}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$$

$$d_1=\frac{\ln\left(\frac{S_0'}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad d_2=d_1-\sigma\sqrt{T}$$

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Dividend Yield Only

- Call model

$$C_0=S_0e^{-\delta T}N(d_1)-Xe^{-rT}N(d_2)$$

- Put model

$$P_0=Xe^{-rT}N(-d_2)-S_0e^{-\delta T}N(-d_1)$$

$$d_1=\frac{\ln\left(\frac{S_0}{X}\right)+\left(r-\delta+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad d_2=d_1-\sigma\sqrt{T}$$

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Derivation of GBMOVM

- Geometric Brownian motion

$$dS=\mu Sdt+\sigma Sdw$$

- Itô's lemma, C(S,t),

$$dC=\left(\mu S\frac{\partial C}{\partial S}+\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}\right)dt+\frac{\partial C}{\partial S}\sigma Sdw$$

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Risk-free portfolio design

- Sell 1 call and buy delta stock

$$\Pi=-C+\frac{\partial C}{\partial S}S \qquad d\Pi=-dC+\frac{\partial C}{\partial S}dS+q\frac{\partial C}{\partial S}Sdt$$

- Hedged portfolio result (SFDR)

$$d\Pi=-\left(\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}-q\frac{\partial C}{\partial S}S\right)dt$$

- Risk free growth implied

$$d\Pi=r\Pi dt=r\left(-C+\frac{\partial C}{\partial S}S\right)dt$$

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GBM PDE

- GBM partial differential equation

$$rC=\frac{\partial C}{\partial t}+(r-q)\frac{\partial C}{\partial S}S+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}$$

- Boundary condition

$$C(S,t=T)=\max(0,S_T-X)$$

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Digital (Binary) Options

- Generic Terminal Payoffs

$$AoN(S_T,T;\iota_U,X,T,r,\sigma)=1_{\iota_U S_T>\iota_U X}S_T \text{ or}$$
$$CoN(S_T,T;\iota_U,X,T,r,\sigma,DP)=1_{\iota_U S_T>\iota_U X}DP.$$

- GBMOVM

$$AoN(S_0',t;\iota_U,X,T,r,\sigma)=S_0'N(\iota_U d_1) \text{ and}$$
$$CoN(S_0',t;\iota_U,X,T,r,\sigma,DP)=B_rDPN(\iota_U d_2).$$

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Digital Terminal Payoffs

$$C_{AoN,T} = \begin{cases} 0 & S_T \le X \\ S_T & S_T > X \end{cases}, \text{ (Digital asset-or-nothing call option)}$$

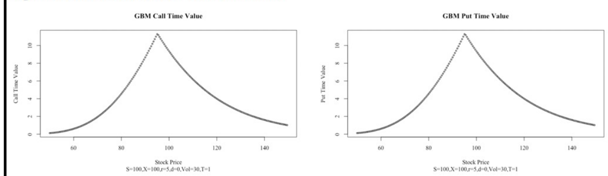
$$P_{AoN,T} = \begin{cases} S_T & S_T \le X \\ 0 & S_T > X \end{cases}, \text{ (Digital asset-or-nothing put option)}$$

$$C_{CoN,T} = \begin{cases} 0 & S_T \le X \\ DP & S_T > X \end{cases}, \text{ (Digital cash-or-nothing call option)}$$

$$P_{CoN,T} = \begin{cases} DP & S_T \le X \\ 0 & S_T > X \end{cases}. \text{ (Digital asset-or-nothing put option)}$$

GBMOVM and Time Value

Figure 5.4.2. GBMOVM time value illustration



Summary

- Review assumptions of GBMOVM
  - Lognormal distribution
  - Arbitrage (SFDR)
- Explored role of dividends
- Identified different representations of GBMOVM
- Derived the GBMOVM
- Reviewed selected plots

