Module 5.1 Valuation

Option Boundaries and Parities

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Overview

- Understand role of arbitrageur
- Explore numerous boundary conditions
  - American- and European-style
  - Upper and lower bounds
  - Calls and puts
- Introduce cash flow table
- Selected evidence

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Central Finance Concepts

- Role of arbitrageur
- Boundaries and parities
- Role of intermediate cash flows (dividends)
- Empirical analysis

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Role of Arbitrageur

- Arbitrage – typically a series of transactions that result in positive cash flow today, no future liability
- Guiding principles of arbitrageurs
  - Do not spend your own money
  - Do not take any market risk
- Liquidity providers
- Market makers

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Option Boundary Conditions

- Option prices should be non-negative
  - Economic argument: Liability is limited
  - Arbitrage argument: A series of transactions are executed that results in money inflows to the arbitrageur with no chance of future money outflows
- Lower boundaries are often violated
- Upper boundaries rarely violated

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Option Parity Conditions

- Establishes the relationship between put and call options with the same maturity and same exercise price
- European-style parities result in exact relationships (equalities)
- American-style parities result in ranges (inequalities)

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Normalization

- Given the diversity of exercise prices, option maturities, and differing underlying instruments, it is often preferred to express boundaries and parities as percentages of the underlying instrument
- Consider S = 100 and C = 14.23, normalized call = 14.23%

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Influence of Cash Flows

- Dividends often influence the boundaries and parities
  - Discrete cash payments (individual stocks)
  - Continuous cash flows (index ETFs)
- Equity indexes often have hundreds of quarterly dividend-paying stocks

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Actual Data

- December 15, 2005
  - Before a financial crisis
  - “Normal” markets
- December 15, 2008
  - During a financial crisis
  - “Abnormal” markets
- December 15, 2011 (after a financial crisis)

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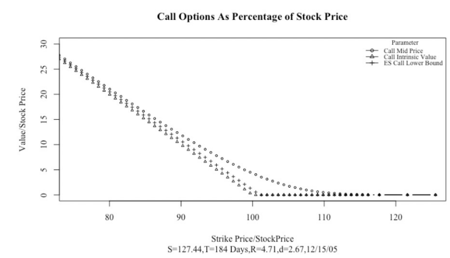
Figure 5.1.1 American-style call lower bound, call intrinsic value, and call prices for SPY
Panel A Before a financial crisis



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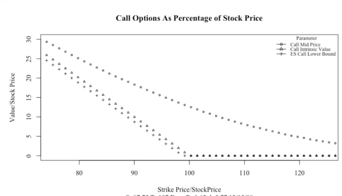
Panel B During a financial crisis



Panel C After a financial crisis

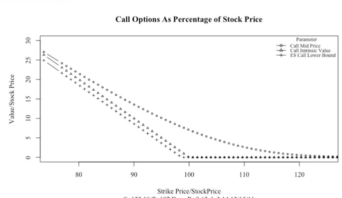


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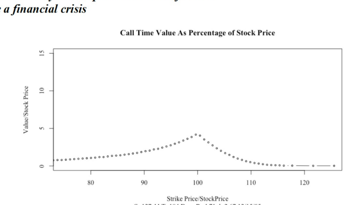
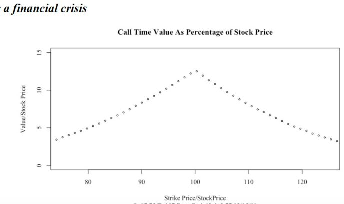
Figure 5.1.2 American-style call option time value for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



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Panel C After a financial crisis

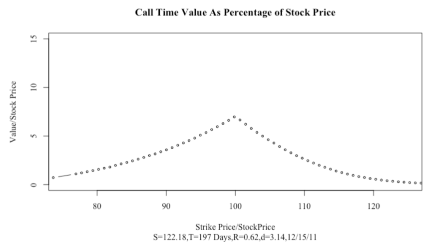

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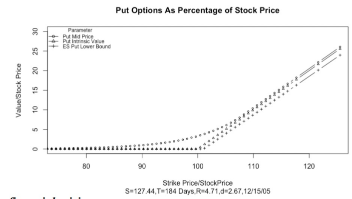
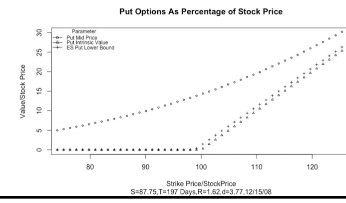
Figure 5.1.3 American-style put lower bound, put intrinsic value, and put prices for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



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Panel C After a financial crisis

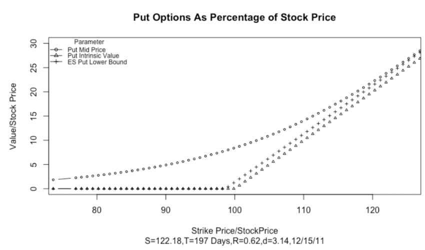

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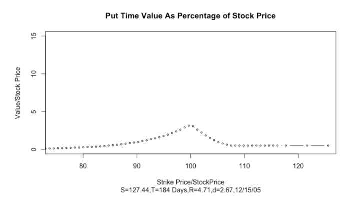
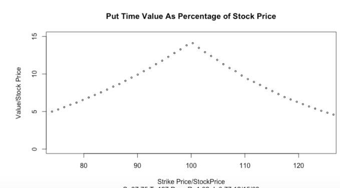
Figure 5.1.4 American-style put option time value for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



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Panel C After a financial crisis

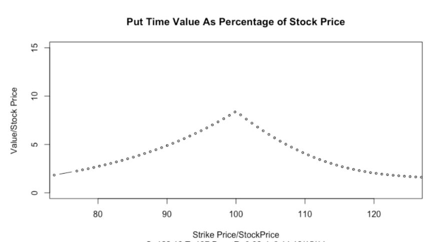

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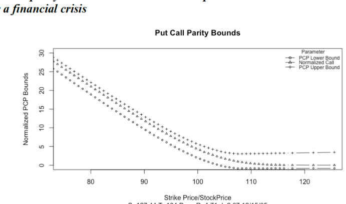
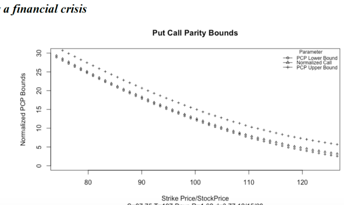
Figure 5.1.5 Put-call parity bounds and normalized call prices
Panel A Before a financial crisis



Panel B During a financial crisis



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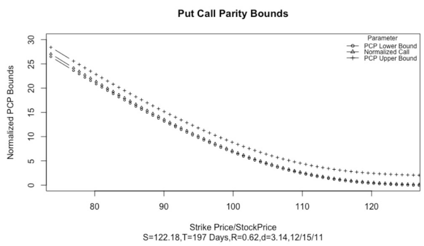
Panel C After a financial crisis



Quantitative Finance Materials

- Lower bounds
- Upper bounds
- Differences in strike prices
- Differences in maturities
- Put-call parity

Boundary Strategies

- Assume boundary condition violated
- Rearrange expression greater than zero
- Implement implied trading strategy
  - Guaranteed positive initial cash flow
  - Check to insure no future liability
- If possible, disequilibrium and arbitrageurs will aggressively pursue

Option Lower Bound

- European-style (ES) call option

$$c_t \geq Max\left(0, S_t - PVD_t - PVX\right)$$

- ES put option

$$p_t \geq Max\left(0, PVX + PVD_t - S_t\right)$$

- American-style (AS) call option

$$C_t \geq Max\left(0, S_t - PVD_t - PVX, S_t - X\right)$$

- AS put option

$$P_t \geq Max\left(0, PVX + PVD_t - S_t, X - S_t\right)$$

ES Call Lower Bound

Table 5.1.1 European-Style Call Lower Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \leq X$	$S_T > X$
Short sell stock	$+S_t$	$-D_i$	$-S_T$	$-S_T$
Lend	$-(PVD_t + PVX)$	$+D_i$	$+X$	$+X$
Buy call	$-c_t$		0	$+(S_T - X)$
Net	$+S_t - (PVD_t + PVX) - c_t$ + by assumption	0	$X - S_T$ + by column	0

Note: Cash flow tables are constructed when the stated condition is violated. The objective is to demonstrate the positive cash flow today with no chance of negative cash flow in the future.

ES Put Lower Bound

Table 5.1.4 European-Style Put Lower Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \leq X$	$S_T > X$
Borrow	$+(PVD_t + PVX)$	$-D_i$	$-X$	$-X$
Buy stock	$-S_t$	$+D_i$	$+S_T$	$+S_T$
Buy put	$-p_t$		$+(X - S_T)$	0
Net	$+(PVD_t + PVX) - S_t - p_t$ + by assumption	0	0	$S_T - X$ + by column

Note: Again cash flow tables are constructed when the stated condition Is violated. The objective is to demonstrate the positive cash flow today with no chance of negative cash flow in the future.

Option Upper Bound

- ES call option

$$c_t \le S_t - PVD_t$$

- ES put option

$$p_t \le PVX$$

- AS call option

$$C_t \le S_t$$

- AS put option

$$P_t \le X$$

ES Call Upper Bound

Table 5.1.2 European-Style Call Upper Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	Dividend Date	$S_T \le X$	$S_T > X$
Sell call	$+c_t$		0	$-(S_T - X)$
Buy stock	$-S_t$	$+D_t$	$+S_T$	$+S_T$
Borrow	$+PVD_t$	$-D_t$		
Net	$+c_t - S_t + PVD_t$ + by assumption	0	$+S_T$ + by column	+X

AS Call Upper Bound

Table 5.1.3 American-Style Call Upper Bound Cash Flow Table

				Cash Flow at Expiration	
	Trade Date	Early Exercise	Dividend Date	$S_T \le X$	$S_T > X$
Sell call	$+C_t$	$-(S_{t'} - X)$		0	$-(S_T - X)$
Buy stock	$-S_t$	$+S_{t'}$	$+D_t$	$+S_T$	$+S_T$
Net	$+C_t - S_t$ + by assumption	+X	$+D_t$	$+S_T$ + by column	+X

ES Put Upper Bound

Table 5.1.5 European-Style Put Upper Bound Cash Flow Table

		Cash Flow at Expiration	
	Trade Date	$S_T \le X$	$S_T > X$
Sell put	$+p_t$	$-(X - S_T)$	0
Borrow	$-PVX$	$+X$	$+X$
Net	$+p_t - PVX$ + by assumption	$+S_T$ + by column	+X + by column

AS Put Upper Bound

Table 5.1.6 American-Style Put Upper Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	Early Exercise	$S_T \le X$	$S_T > X$
Sell put	$+P_t$	$-(X - S_{t'})$	$-(X - S_T)$	0
Lend	$-X$	$+FV_{t'}(X)$	$+FV_T(X)$	$+FV_T(X)$
Net	$+C_t - S_t$ + by assumption	$+S_{t'}$ + Interest + r > 0	$+S_{t'}$ + Interest + r > 0	$+ FV_T(X)$ + r > 0

Selected Boundary Relations Difference in Strike Prices

European-style call difference in strike prices

$$c_t(X_L) - c_t(X_H) \le PV(X_H - X_L).$$

American-style call difference in strike prices

$$C_t(X_L) - C_t(X_H) \le X_H - X_L.$$

European-style put difference in strike prices

$$p_t(X_H) - p_t(X_L) \le PV(X_H - X_L).$$

American-style put difference in strike prices

$$P_t(X_H) - P_t(X_L) \le X_H - X_L.$$

Selected Boundary Relations Difference in Maturity Time

Assume $T_2 > T_1$.

$$c_{T_2} \geq c_{T_1},$$
$$C_{T_2} \geq C_{T_1},$$
$$p_{T_2} \geq\leq p_{T_1}, \text{ and}$$
$$P_{T_2} \geq P_{T_1}.$$

Selected Boundary Relations Difference in Exercise Prices

Assume $X_L < X_M < X_H$ and $0 < \alpha < 1$, $\alpha = (X_H - X_M)/(X_H - X_L)$

$$\alpha c_t(X_L) + (1-\alpha)c_t(X_H) \geq c_t(X_M),$$
$$\alpha C_t(X_L) + (1-\alpha)C_t(X_H) \geq C_t(X_M),$$
$$\alpha p_t(X_L) + (1-\alpha)p_t(X_H) \geq p_t(X_M), \text{ and}$$
$$\alpha P_t(X_L) + (1-\alpha)P_t(X_H) \geq P_t(X_M).$$

Option Put-Call Parity

- European-style

$$S_t - PVD_t + p_t = c_t + PVX$$

- American-style

$$P_t + S_t - PVX \geq C_t \geq P_t + S_t - PVD_t - X$$

ES Put-Call Parity CF Table

Table 5.1.7 European-Style Put-Call Parity Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \leq X$	$S_T > X$
Lend	$PVX + PVD_t$	$-D_t$	$-X$	$-X$
Sell call	$+c_t$		0	$-(S_T - X)$
Buy stock	$-S_t$	$+D_t$	$+S_T$	$+S_T$
Buy put	$-p_t$		$X - S_T$	0
Net	$+PVX + PVD_t + c_t - S_t - p_t$	0	0	0

AS Put-Call Parity CF Tables

Table 5.1.8 American-Style Put-Call Parity Cash Flow Table (Case 1)

				Cash Flow at Expiration	
	Trade Date	**Early Exercise**	**Dividend Date**	$S_T \leq X$	$S_T > X$
Sell call	$+C_t$	$-(S_{t'} - X)$		0	$-(S_T - X)$
Buy put	$-P_t$			$X - S_T$	0
Buy stock	$-S_t$	$+S_{t'}$	$+D_t$	$+S_T$	$+S_T$
Borrow	$+PVX$	$-FV_{t'}(PVX)$		$-X$	$-X$
Net	$+C_t - P_t - S_t$ $+PVX$	Interest	$+D_t$	0	0

Table 5.1.9 American-Style Put-Call Parity Cash Flow Table (Case 2)

				Cash Flow at Expiration	
	Trade Date	**Early Exercise**	**Dividend Date**	$S_T \leq X$	$S_T > X$
Sell put	$+P_t$	$-(X - S_{t'})$		$-(X - S_T)$	0
Short stock	$+S_t$	$-S_{t'}$	$-D_t$	$-S_T$	$-S_T$
Lend	$-(PVD_t + X)$	$+FV_{t'}(PVX)$	$+D_t$	$+FVX$	$+FVX$
Buy call	$-C_t$			0	$-(S_T - X)$
Net	$+P_t + S_t$ $-(PVD_t + X) - C_t$	Interest	0	Interest on X	Interest on X

Selected Evidence

- AS call converges to ES call lower bound
- AS put converges to ES put lower bound due to high dividends
- Time value is more pronounced for lower strike prices
- Call and put prices within AS put-call parity bounds

Normalized Option Values

- Stock prices and underlying options are positive
- 0 < C/S < 1 (call less than stock)
- 0 < P/S (X >> S => P > (possible) S
- Normalized put-call parity

$$1+\frac{P_t}{S_t}-\frac{PVX}{S_t}\geq\frac{C_t}{S_t}\geq 1+\frac{P_t}{S_t}-\frac{\left(PVD_t+X\right)}{S_t}$$

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Summary

- Reviewed role of arbitrageur
- Explored numerous boundary conditions
  - American- and European-style
  - Upper and lower
  - Calls and puts
- Introduced cash flow table
- Examined selected evidence

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