Module 5.5: Arithmetic Brownian Motion-Based Option Valuation Models

Learning objectives

- Explain how to value options based on the arithmetic Brownian motion
- Review the unique assumption and sketch a derivation of the model

Executive summary

We make a detailed comparison of the GBMOVM and the ABMOVM addressing numerous considerations. We also review a key assumption underlying the standard option valuation model proposed by Black, Scholes and Merton and contrast it with the arithmetic Brownian motion option valuation model (ABMOVM). Recall that GBM results in a lognormal terminal distribution whereas ABM results in a normal terminal distribution. We also sketch the ABMOVM model derivation.

Central finance concepts

The technical derivation of the ABMOVM is provided later in this module. We focus here on appraising the choice of either ABMOVM or GBMOVM when faced with a particular quantitative finance application. Our position is that it is better to have model choices rather than be constrained to only one, the orthodox BSMOVM.

We now take a deep dive in exploring these two candidate models.

ABMOVM or GBMOVM[1]

We explore here the advantages and disadvantages of ABMOVM and GBMOVM. The GBMOVM is deeply embedded in the heart of modern financial analysis. It is also well known, however, that the GBMOVM is deeply flawed. The model has been extended to incorporate stochastic volatility, jumps, volatility surfaces, and so forth. While many of these extensions have been useful for specific applications, core problems remain—many of which are rarely identified, much less addressed. Now consider just two problems with the GBMOVM related to the underlying lognormal distribution assumption. First, there is zero probability of the underlying instrument being zero in the future. Second, a simple portfolio of underlying instruments follows no known statistical distribution. Hence, within the GBMOVM, internally consistent portfolio statistics cannot be computed which impairs risk management practice. Thus, the lognormal distribution suffers because it is not additive, and the underlying values are strictly positive

ABMOVM can easily address these two problems. All the extensions (such as stochastic volatility, jumps, volatility surfaces, and so forth) can also be easily applied within the ABMOVM. Ultimately, the appropriate model to deploy in practice will depend on the user's objectives, the underlying instruments involved, and the empirical support.

Historically, the shift in financial research from ABM to GBM occurred with Samuelson (1965). Samuelson (1965), based on earlier work by Osborne (1959), Spenkle (1961), Alexander (1961), Boness (1964) and many others, introduced GBM for solving financial derivatives valuation problems. GBM was introduced, in part, to "correct" prior errors by authors relying on ABM. Samuelson (1965) credits Bachelier (1900) with discovering "the mathematical theory of Brownian motion five years before Einstein's classic 1905 paper. (p. 13)" Two prior errors identified by Samuelson include stocks possessing limited liability and warrant values exceeding the underlying instrument's value. It is important to note that while stocks do possess limited liability characteristics, they still have some probability of becoming worthless. Neither of these are weaknesses of ABM, rather they result from an incomplete application of ABM to this finance problem.

Because finance is a social science, every mathematical framework has both strengths and weaknesses. As Derman (2011) notes in the context of financial theories, "Only imperfect models remain. The movements of stock prices are more like the movements of humans than molecules. It is irresponsible to

[1]See Brooks and Brooks (2017).

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pretend otherwise. (p. 187)" Because modeling human movements remains elusive, it is important to have multiple, pertinent frameworks available for the array of tasks facing the finance profession.

Although there are numerous option valuation and risk management frameworks available today, the vast majority of frameworks are encapsulated within the GBM framework. The most relevant work is Louis Bachelier's dissertation published in 1900 where he introduces what is now known as ABM and derives the first known option valuation model. Bachelier's dissertation laid the foundation for many other works including Kruizenga's dissertation (1952), which was written under Paul Samuelson. In chapter 8 of his 1952 doctoral thesis, Kruizenga provides detailed analysis of the variance of expected option gains where the very rudiments of delta hedging are apparent. Further, Kruizenga provides anecdotal evidence in support of Bachelier's model (see pages 185 to 186).

Osborne (1959) provides one of the first detailed statistical analyses of the log of the price relative for common stock prices. Osborne's work gives the early justification for the lognormal distribution (ultimately GBM) using Weber-Fechner law. It should be noted, however, that Osborne concedes, "percentage changes of less than ±15 percent, expressed as fractions from unity, are very nearly natural logarithms of the same ratio. (p. 146)" Thus, Osborne's approach could easily have provided support for either GBM or ABM models.

Sprenkle (1961), in his dissertation on warrant prices, provides both normative and positive arguments in favor of using the lognormal distribution (ultimately GBM). Sprenkle notes, "We have now developed rationales for both normally distributed expected stock prices and lognormally distributed expected stock prices. Is there any a priori reason for preferring one to the other? The answer is yes; there is a quite important reason for preferring the lognormal distribution. Normal distributions imply that the investor thinks there is a chance that future stock prices will be negative. ... The lognormal distribution restricts x_t [stock prices] to positive values, and an investor with this type of expectation does not think there is any chance of negative stock prices. (p. 195)" (Underline in original.) As any investor realizes, however, there is always a positive chance of witnessing a zero stock price, which is not possible in a lognormal distribution. For example, U.S. Airways (ticker symbol was formerly "U") stock price is presently zero after declaring bankruptcy on August 11, 2002. According to Trading Economics, there was an average 46,635 corporate bankruptcies each year in the U.S. from 1980 until 2016.[2]

Sprenkle's (1961) empirical tests led him to conclude, "that in most cases a lognormal distribution is a better assumption than a normal distribution. (p. 196)" Alexander (1961), however, notes, in response to observing fat tailed distributions, that this "sort of situation (leptokurtosis) is frequently encountered in economic statistics and would certainly overshadow any attempt to test fine points such as the difference between a logarithmic and a percentage scheme. (p. 16)"

Samuelson (1965) appears to have coined the term geometric Brownian motion. He states that this model accounts for the observed anomaly of long-term warrants increasing in value to infinity under Bachelier's model. He also notes that within this model, stocks cannot become negative in value because they are distributed lognormally. One weakness of this model, however, is that under the lognormal distribution, the probability of observing a zero stock price is zero. We take a different approach to option valuation that has the value of stock always being non-negative but with a positive probability of bankruptcy.

Black and Scholes (1973) developed a risk-free portfolio under continuous rebalancing as highlighted by Merton (1973) and put forward the partial differential equation for the value of an option. Merton (1973) posits a general framework for rational option valuation that addresses a number of boundary conditions for call options and warrants. His model also relaxes several constraints in the Black–Scholes–Merton (BSM) framework. Black and Scholes (1972) use their option valuation model to empirically test their option prices against the prices of traded options. They use lagged variance as their risk term and find that their model matches fairly well to observed option prices; their model did, however, tend to overprice options on high variance stocks and underprice options on low variance stocks. Interestingly, ABMOVM results in lower option values when variance is high and higher option values when variance is low, consistent with Black and Scholes' empirical observation. Along with this theoretical model, the existence of exchange traded

[2]See http://www.tradingeconomics.com/united-states/bankruptcies (referenced on January 13, 2017).

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options led to enormous growth in option markets (Kairys and Valerio, 1997). The advent of more data led to several empirical papers on the BSM option valuation framework (e.g., MacBeth and Merville, 1979; Chiras and Manaster, 1978; Gultekin, et al, 1982; Moore, 2006; and many others), and the BSM option valuation framework gave rise to many other option valuation models.

Szpiro (2011) identifies four reasons why Samuelson preferred GBM to ABM for stocks. First, with GBM, stock prices are always positive. Second, percentage price movements are equally distributed independent of the stock price. Third, GBM is in accordance with human nature based on Osborne's work. Finally, empirical observations are consistent with price movements modeled by GBM. We empirically re-examine these four reasons by exploring the different models' strengths and weaknesses.

Underlying instrument can have zero value

Based on a brief examination of U.S. census data, roughly 0.7% of businesses file for some form of bankruptcy each year. Of the businesses filing for bankruptcy, it is reasonable to assume that a significant percentage result in equity prices being zero. Additionally, many instruments have or can potentially have negative values (for example interest rates, spreads, and measurement data). Many financial instruments also can have zero prices.

Based on the unusual price quotation conventions during Bachelier's time, a negative strike price was possible. Thus, for all we know, Bachelier may have considered GBM and discarded the GBMOVM as it would not permit negative strike prices.

Common stock prices can be viewed as an unincorporated equivalent stock price plus a zero strike put option afforded by the limited liability via incorporation. Thus, ABM can easily, and in some ways more appropriately, handle negative and zero values of the underlying. Additionally, the ABM framework allows for a closed-form solution for the probability of default. GBM has no practical way to accommodate the positive probability of zero common stock prices. It remains an interesting task to fully account for the economic value of limited liability as well as its duration.

Underlying instrument can have multiple risk factors

Enterprise risk management has now emerged as a major division within many corporations, and the C-level position of Chief Risk Officer is increasingly popular. Many enterprise risk systems identify a small set of risk factors that account for the major risks within an enterprise. Each balance sheet item is mapped to these risk factors. Once the appropriate risk variables are identified and the entity's positions are appropriately modeled, the firm-wide risk can be examined.

ABM is particularly well suited for this type of exercise. The underlying position can be modeled as a multifactor ABM. The beauty of the normal distribution is that it is additive. That is,

$$dS = \mu\left(S,t\right)dt + \sum_{j=1}^{N_F} \sigma_j\left(t\right)dw_j \text{. (normal distribution)} \tag{5.5.1}$$

With the appropriate correlation matrix, the stochastic differential equation above can immediately be written as a single factor ABM. Correspondingly, related options can be valued consistently, and an enterprise risk system can easily be deployed.

A multifactor geometric Brownian motion poses challenges as the sum of lognormal risk factors does not result in a lognormal distribution or any known distribution. That is,

$$dS = \mu\left(S,t\right)dt + \sum_{j=1}^{N_F} \sigma_j\left(t\right)S\,dw_j \text{ (lognormal distribution)} \tag{5.5.2}$$

does not easily reduce to a single factor model. Poitras (1998) develops numerical methods towards this end and emphasizes that no closed-form solutions exist.

Underlying instruments can be easily aggregated into portfolios

Similarly, aggregating different positions is straightforward with arithmetic Brownian motion. With slightly more detailed notation, we have

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$$dS_i = \mu\left(S_i,t\right)dt + \sum_{j=1}^{N_F}\sigma_{i,j}\left(t\right)dw_j \text{ . (normal distribution)} \tag{5.5.3}$$

If we build a portfolio of securities, denoted as

$$\Pi = \sum_{i=1}^{N_I} N_i S_i \text{ ,} \tag{5.5.4}$$

then it can be seen immediately that the portfolio value is normally distributed,

$$d\Pi = \sum_{i=1}^{N_I} N_i dS_i = \sum_{i=1}^{N_I} N_i \left[\mu\left(S_i,t\right)dt + \sum_{j=1}^{N_F}\sigma_{i,j}\left(t\right)dw_j\right]\text{. (normal distribution)} \tag{5.5.5}$$

As previously noted, geometric Brownian motion does not permit easy aggregation of the distribution of portfolio values.

Model Greeks based on empirical observations

After the crash of October 1987, the Black–Scholes–Merton paradigm suffered a permanent loss of market participants' confidence as an accurate representation of option values. As evidence, the implied volatility surface literature emerged. After 1987, there was an intense effort to find a replacement model. To date, unanimity of opinion has not occurred. Rather than rely on a single model, many firms deploy multiple models and monitor which model is performing better in particular markets during different seasons. The specific model used should not be based exclusively on the underlying assumptions, rather on the model's ability to aid in quality decision-making.

Relative risk or absolute risk

The appropriate way to model risk is at the heart of these two models. ABMOVM views risk from the absolute volatility perspective, and GBMOVM views risk from the relative volatility perspective. In order to illustrate these differences, suppose that we have two stocks initially valued at \$100 per share. The first stock sees a decline in value to \$50. The second stock experiences a 2-for-1 stock split. Is there a difference between these outcomes? Risk can clearly be represented in both ways. Samuelson (1965) chose to use the way that most fit his lectures on the efficient market hypothesis, the relative volatility perspective. Essentially, one can make a case for either framework depending on how risk is characterized.

Some early authors made the case that stock splits should not influence risk measures. ABMOVM views risk from an absolute perspective, and hence, absolute volatility for a position would be constant with a stock split. Under this framework, our input for the price for the first stock would now be \$50 instead of the initial \$100. The absolute volatility for this position would also remain unchanged (everything else being equal), leading to the well-known leverage effect for relative volatility (see Black, 1976 and Christie, 1982, for the development of this idea; see also Ait-Sahalia, 2013, for a more recent examination of the magnitude of this effect). For the stock experiencing a stock split, options already have typical provisions to adjust the strike price for a stock split. Here, the absolute volatility could simply be cut in half of what it was initially as there is no leverage effect. Thus, with ABMOVM, one must manually adjust volatility in cases of stock splits—a task that is not necessary with GBMOVM. Conversely, GBMOVM suffers by ignoring the well-documented leverage effect. Under the GBM framework the two stocks in our example would be treated as equivalent except for the change in price input and customary change in contractual strike price. Several papers address the leverage effect by adding precisely-tuned jump processes (e.g., Carr and Wu, 2004), but this phenomenon is naturally incorporated into the ABM stochastic process. As equity prices decline, relative volatility increases because the degree of leverage has increased (e.g., Geske, 1979).

If the riskiness of firm assets is independent of the equity value, then, for leveraged firms, the equity risk will increase with declines in equity value. ABMOVM provides one way of handling the leverage effect. Specifically, ABMOVM allows for risk factors that are separate from stock values. Thus, from the ABMOVM perspective, observed relative volatility will increase when stock prices decline.

When coining the phrase geometric Brownian motion, Samuelson (1965) notes that GBM has "the property that every dollar of market value is subject to the same multiplicational or percentage fluctuations per unit

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time regardless of the absolute price of the stock" (13). When contemplating stock declines due to splits, this view seems appropriate. When contemplating stock declines due to severe economic losses to the underlying firm, this view seems erroneous.

Behavior of option models at extreme volatilities

With the popularity of the Black–Scholes–Merton paradigm, the GBMOVM has been deployed in a wide variety of settings. For example, options on electricity and options on interest rates both display unusual implied volatilities at certain times (under the GBM framework). The lognormal distribution displays some peculiar behavior when volatility is high. Note that under the equivalent martingale measure with GBM, the normal mean, μ, is reduced by the variance divided by two.[3]

Remember that the definition of median is $\hat{x}$ such that $\int_0^{\hat{x}} f(x)dx = \int_{\hat{x}}^{+\infty} f(x)dx = \frac{1}{2}$. For the lognormal distribution, the median is Median = exp(μ), whereas the median of the normal distribution is its mean, μ. Also recall that the probability distribution mode satisfies $f'(x) = 0$ and $f''(x) < 0$. For the lognormal distribution, $\text{Mode} = \exp(\mu - \sigma^2)$, whereas the mode of the normal distribution is its mean, μ. The lognormal distribution mode is an exponentially decreasing function of the normal distribution variance. This characteristic is an important property of the lognormal distribution with financial applications. Higher normal variance results in a lower peak of the distribution. The lognormal mean can be expressed as $\text{Mean} = \exp\left(\mu + \frac{\sigma^2}{2}\right)$. Thus, the mean is an exponentially increasing function of the normal distribution variance. In finance, however, the normal mean is defined in such a way as to offset this variance effect. With constant rates, yields, and volatility, we have $\text{Mean} = S_t \exp\left[(r - q)(T - t)\right]$. Thus, the lognormal distribution mean is invariant to the variance of the normal distribution. By definition, the mean, median and mode of the normal distribution are μ.

Based on analytical services like Bloomberg, it is not unusual to observe implied volatilities in excess of 100% based on the GBMOVM.[4] Implied volatilities in excess of 1,000% for commodities like electricity (electricity market prices are also negative at times) have been periodically observed. If we assume an underlying price of \$100, a one year horizon, an expected return of 12%, and a standard deviation of 130%, then the lognormal mean is \$112.75, the median is \$48.43, and the mode is \$8.94. Does it really make sense that if the underlying price is \$100, then there is a 50% chance that the underlying value will be below \$48.43 and that the most likely outcome is \$8.94 in one year when the mean return is 12%? The normal distribution does not suffer from such strange behavior with extreme volatilities as the mean, median, and mode are the same.

Binomial Convergence

Early development of the GBMOVM identified a lattice approach for valuing options (e.g., Cox, Ross, and Rubinstein, 1979). Chance (2008) provides a concise summary of binomial approaches to the GBMOVM. A similar binomial option valuation model can be constructed for the ABMOVM along with the appropriate convergence properties (see Module 5.3). The main difference between ABMOVM and GBMOVM is that the lattice is additive and recombining as opposed to multiplicative and recombining for GBMOVM. The lattice approach is important for valuing American-style options.

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[3]Specifically, $\mu(t) = \ln(S_t) + r(t) - q(t) - \frac{\sigma^2(t)}{2}$.

[4]For example, short-dated Eurodollar futures options often have implied volatilities in excess of 100. On April 4, 2017, a 9-day call on April Eurodollar futures with 99.375 strike price had an implied volatility of 235%. Further, on this same day, a 15-day call on April VIX with 16 strike price had an implied volatility of 102%.

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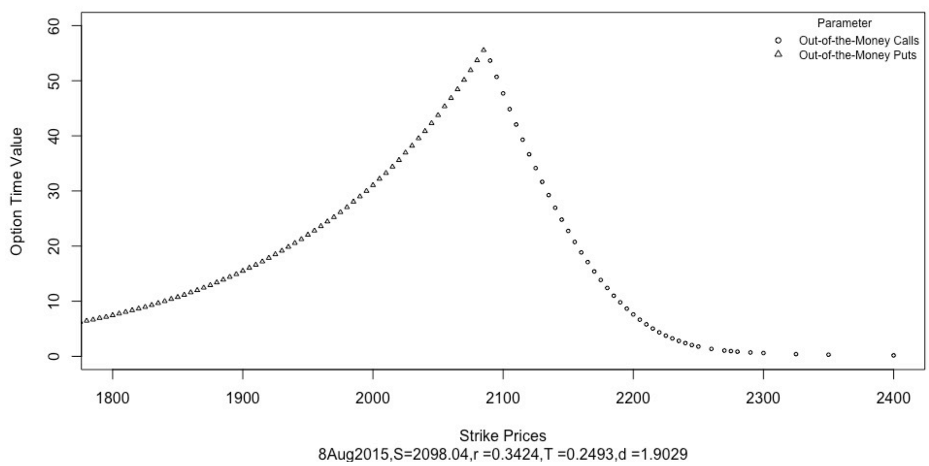
Homogeneity of Degree 1
One concern of the ABMOVM is that it is not homogeneous of degree one. That is, a stock split will result in option values that do not correspond with the degree of split. The ABMOVM can be made homogeneous of degree one by also scaling absolute volatility. Hence, if the stock price and strike prices are reduced in half, then just divide volatility in half and the resultant option values are exactly one half of the original value. The same can be done for stock dividends, reverse stock splits, and other means of changing the number of shares.
Assumed Drift
Option valuation models based on arithmetic Brownian motion often erroneously assume that the underlying grows at an arithmetic drift rate. The no-arbitrage constraint, however, negates the assumed form of drift. Both stochastic processes that we use in this paper assume a geometric drift rate based on the equivalent martingale measure. An alternative way to handle the drift term is to model forward prices on the underlying instrument, yielding a drift term of zero.[5]
Evidence from Option Prices
Recall from Module 5.4 that the time value chart reflected the positive skewness of GBMOVM. The ABMOVM is symmetric as we will see in the next section. Figure 5.5.1 illustrates option prices for SPX options with $T = 0.2493$ years. Notice the distinct negative skewness implied with actual data.

Figure 5.5.1 Time Value of SPX Options on August 8, 2015



Interestingly, a log transformation makes this relationship more distinctly negative skewed and nearly linear as shown in Figure 5.5.2.

[5]This transformation makes our model closer to Bachelier's because he assumes a zero growth rate. See Sullivan and Weithers (1991).

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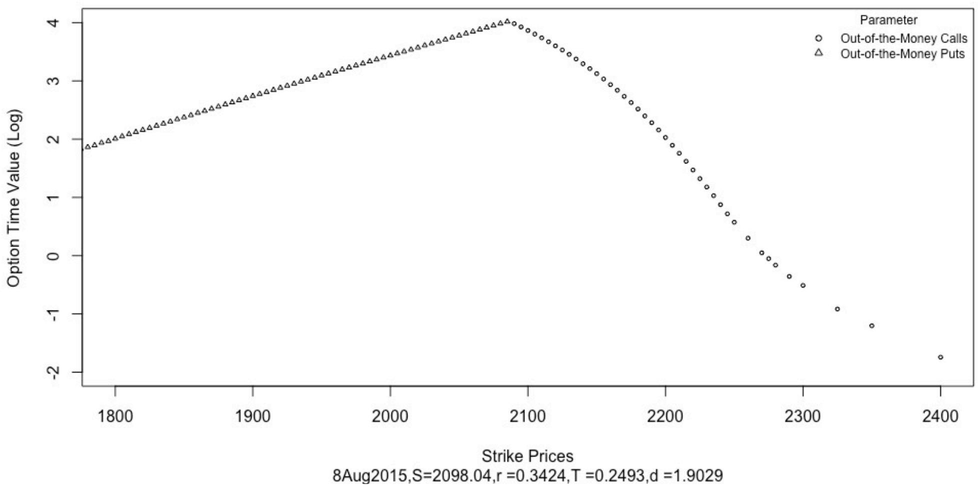
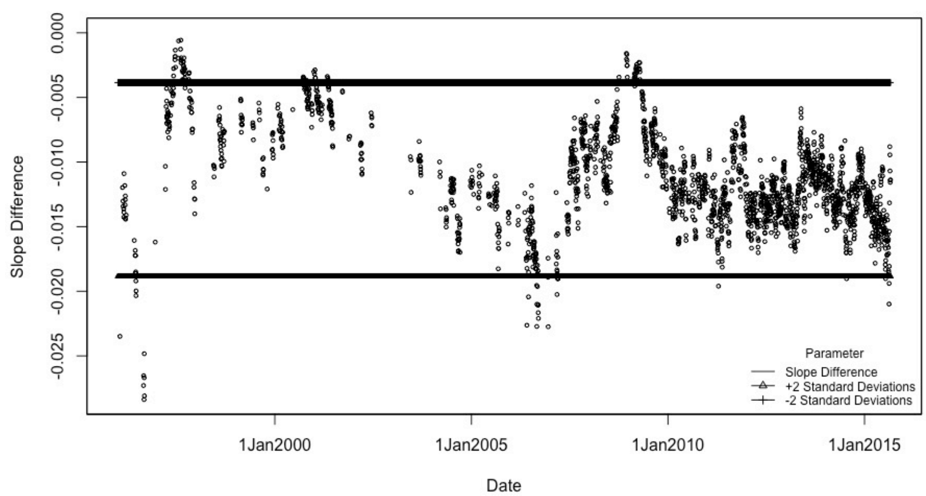
Figure 5.5.2 Log Transform of Time Value of SPX Options on August 8, 2015



If we measure the slopes of the out-of-the money puts and calls we generate Figure 5.5.3. If GBMOVM is reasonable, we would expect the slope difference to be positive. If ABMOVM is reasonable, we would expect the slope difference to be nearly zero. Notice with actual data the slope difference is always negative and significantly so.

Figure 5.5.3 Slope Difference of Log Transform of Time Value of SPX Options



We explore a simple analysis to see if ABM or GBM is better. On each day, we identify the implied volatility of at-the-money calls and puts for both the ABM and GBM models using the SPX option data. The at-the-money implied volatilities are then used to produce model values for the remaining options (±20% from the current stock price). Using the sum of the absolute values of the difference between the model and actual option price, we compute percentage improvement of the ABMOVM over the GBMOVM.[6] Over the 4,949 days evaluated, the ABMOVM was 15% closer to observed, mid-market quotes. Figure 5.5.4 shows

[6]The measure specifically is GBM Model Absolute Error Sum minus ABM Model Absolute Error Sum divided by the average of both sums.

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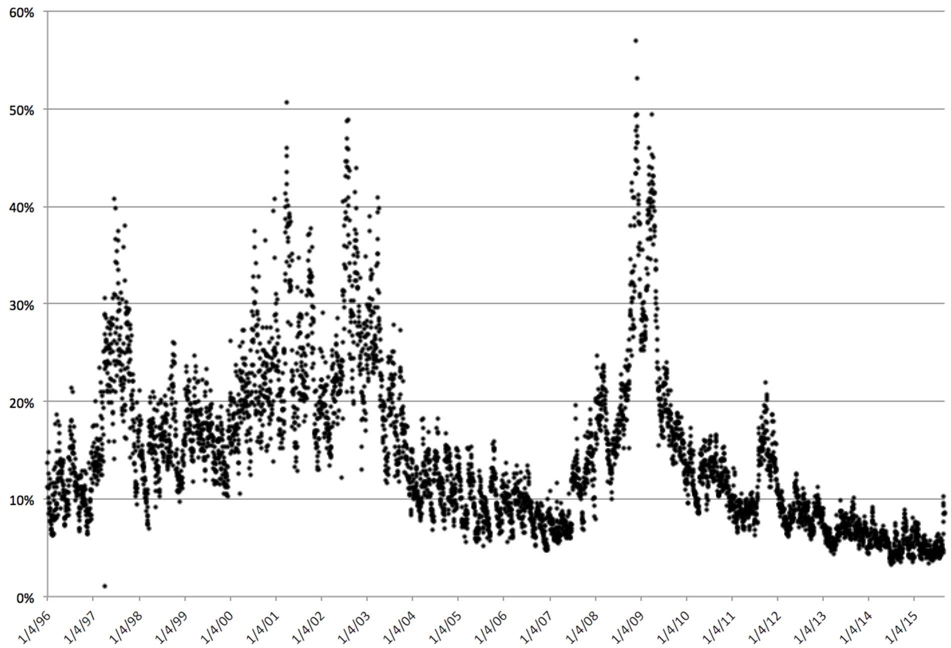
the time-series behavior of this measure. The ABM framework gives a closer match to mid-market quotes in every single day examined. Given the multi-trillion dollar size of global options markets, we conclude that ABMOVM has significant value in capturing the complexities of actual option market values.

Figure 5.5.4 Percentage Improvement of ABMOVM Over GBMOVM



Addressing limited liability

The key to developing ABMOVM appropriately is carefully defining the underlying instrument, S. We define the underlying instrument as the equivalent instrument value assuming *unlimited* liability. With GBMOVM, this definition does not result in any difference from the underlying instrument with limited liability. We denote the underlying instrument with limited liability as S_{LL} and note that, based on geometric Brownian motion, $S = S_{LL}$. With ABMOVM, the stock value with limited liability is a portfolio of the underlying instrument and a zero strike put option, thus $S \leq S_{LL} + P(S, X = 0)$.

Note that in the vast majority of cases, the market value of the limited liability put option for short maturities is extremely small. For example, for all approximately 90-day SPX options from January 4, 1996 through August 31, 2015, only two limited liability put options had values in excess of one basis point of the SPX value. Both option values were based on market data on November 20, 2008, when implied volatility was very high (implied absolute volatility was roughly 191% of the overall average during the entire period) when the SPX was 752.44. Thus, in most cases, S is a reasonable proxy for S_{LL}.[7]

Limited liability is not costless or perpetual; it requires legal fees, document filings and results in tax liabilities. If taxes are not paid, then the incorporated entity ceases to exist, and the limited liability option has expired.

Consider the following: S = \$10 (unincorporated equivalent stock value), X = \$10, σ_A = \$30, $T - t = 1.0$, r = 5% and δ = 0%. Based on the ABMOVM, the call option value is \$11.92 and the put option value is \$11.43, both apparent violations of rational boundaries. The incorporated stock value (long the unincorporated equivalent stock value plus a zero strike put option) is \$17.35 as the value of the zero strike put is \$7.35.[8] Thus, the X = \$10 put with limited liability can be interpreted as a bear spread with a short X =

[7]In analysis of this model not shown, we find that when Price/Volatility < 3.1, then the difference between S and S_{LL} is less than one basis point.

[8]The \$7.35 value is found using the ABMOVM with X = 0 and the other parameters as given. Note that with the stock price of \$17.35, the implied volatility within the GBMOVM framework is 155.50%.

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$0 put and long X = $10 put for a cost of $4.08. From this perspective, neither the call option nor the put option is in violation of rational boundaries.

Figure 5.5.5 illustrates the unincorporated equivalent stock value, the incorporated stock value and an $X =$ $10 call option. We see that the call option remains below the incorporated stock value but is above the unincorporated equivalent stock value below $14.13. Thus, ABM can be deployed in such a way to be consistent with the static boundary conditions. Note that for stocks with relatively high market values compared to their equity's absolute volatility, the role of bankruptcy diminishes, and the focus is on other distributional issues.

Figure 5.5.5 Unincorporated and Incorporated Stock With X = 10 Call

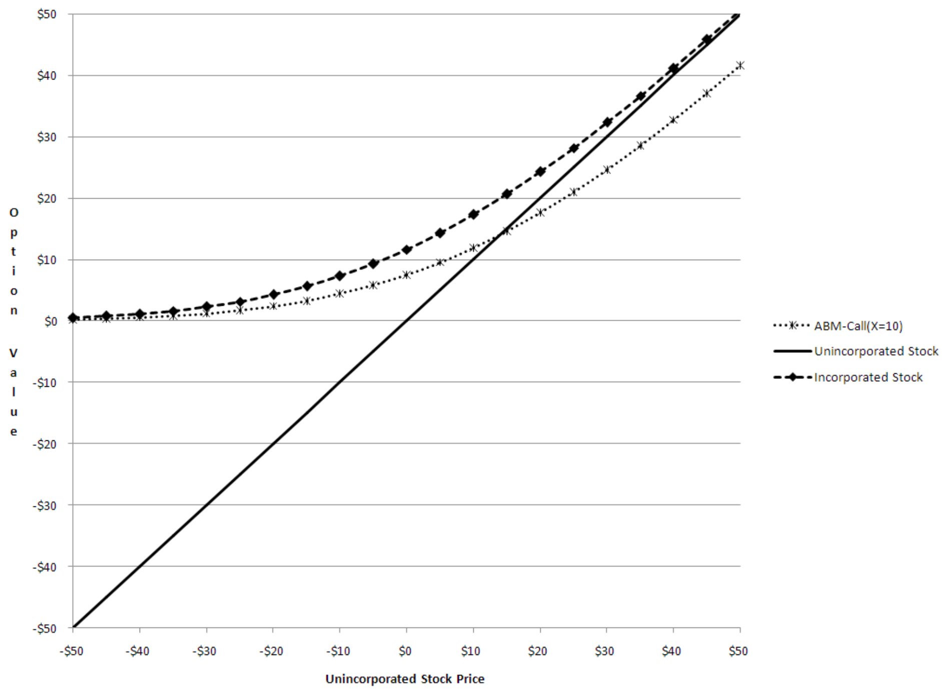

Figure 5.5.6 illustrates the value of the unincorporated equivalent stock value, the incorporated stock value and the X = $0 put option representing limited liability. We see that the limited liability put option has significant value as the unincorporated equivalent stock value declines. Again, we see the role of distress risk can easily be incorporated into the ABMOVM, whereas there is no capacity to include it in the GBMOVM.

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Figure 5.5.6 Unincorporated and Incorporated Stock With X = 0 Put

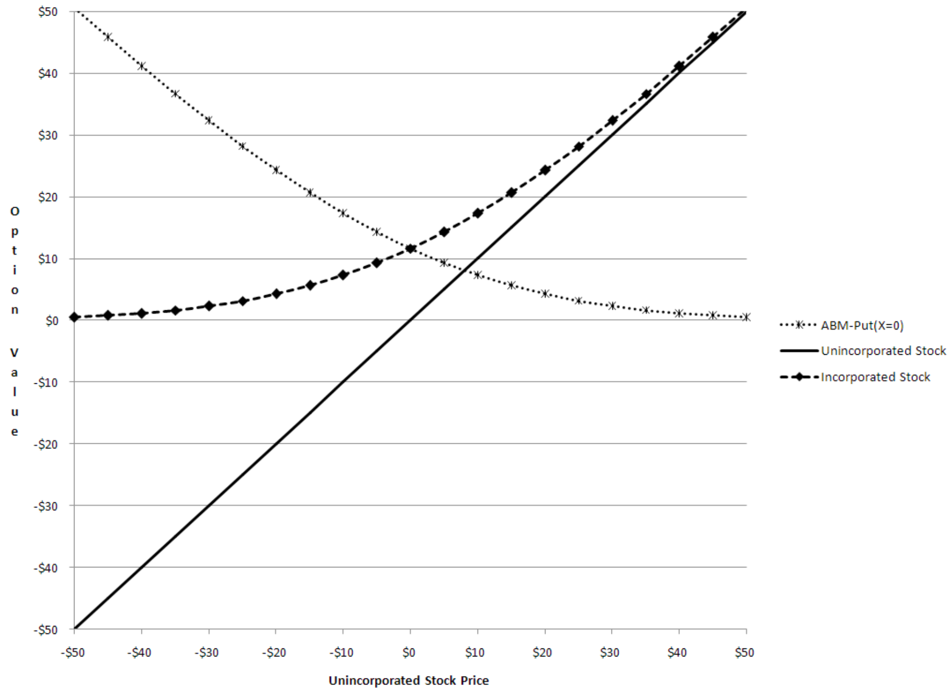

Figure 5.5.7 illustrates the value of the unincorporated equivalent stock value and two put values, one with $X = \$0$ representing the value of limited liability and the other with $X = \$10$ with limited liability (or short a $X = \$0$ put also). Again we see that the limited liability put option has significant value as the unincorporated equivalent stock value declines. The $X = \$10$ put with limited liability does not share in this value, rather it behaves more like a bear spread position. Therefore, buying a put option under ABM is equivalent to buying a bear spread because the gain available to the put holder is limited to the strike price due to the implied zero strike put option.

Figure 5.5.7 Unincorporated Stock With X = 0 Put and X = 10 Put With Limited Liability

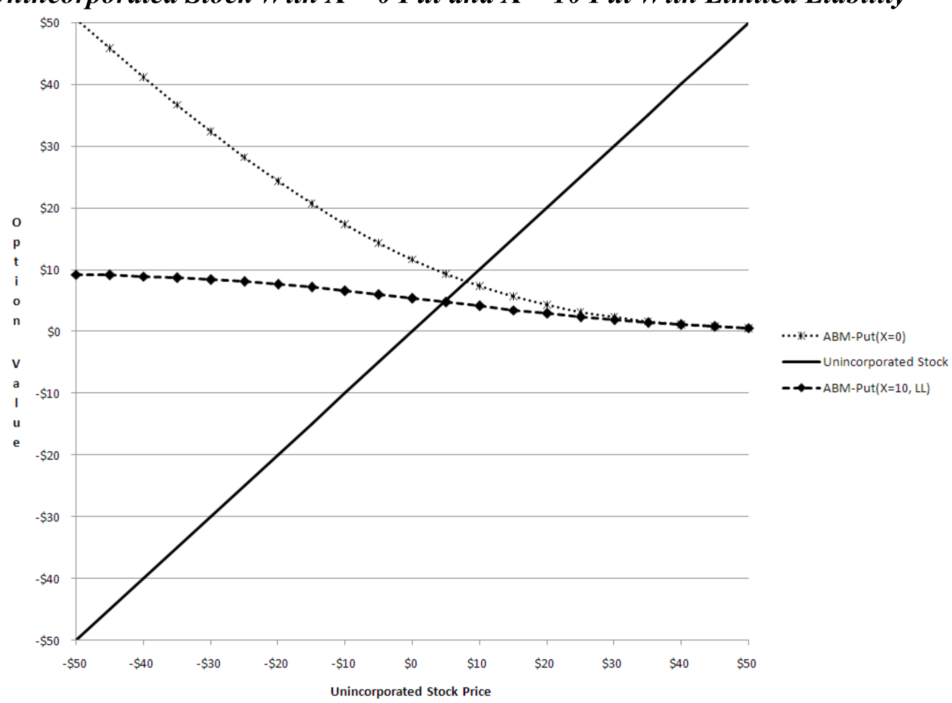

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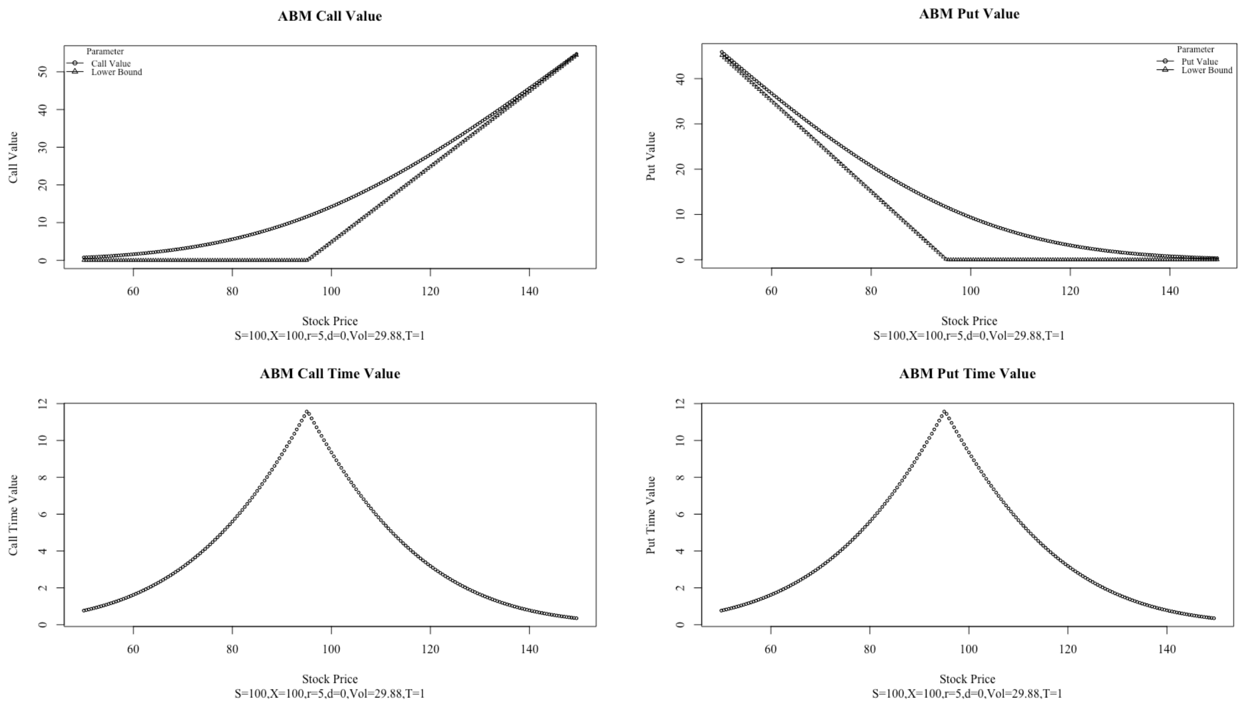
Figure 5.5.8 provides the familiar plots of ABMOVM option values and time values. A careful examination of the time value plots highlights the assumed normal distribution that results in no skewness.

Figure 5.5.8 ABMOVM option values and time values for call and put options



We now turn to the technical details of the ABMOVM.

Quantitative finance materials

The focus here is the technical derivation of the ABMOVM.

ABM option valuation model (ABMOVM)

ABMOVM assumptions[9]

As with any model, the ABMOVM is based on a similar set of assumptions, including (italics used to identify unique ABMOVM assumption)

- Standard finance presuppositions and assumptions apply (see Chapter 2)
- *Underlying instrument behaves randomly and follows a normal distribution (or follow arithmetic Brownian motion)*
- Risk-free interest rate exists, is constant, borrowing and lending allowed
- Volatility of the underlying instrument's continuously compounded rate of return is known, positive and constant
- No market frictions, including no taxes, no transaction costs, unconstrained short selling allowed, and continuous trading
- Investors prefer more to less
- Option are European-style (exercise available only at maturity)
- Underlying instrument pays a constant continuous cash flow yield (e.g., dividend yield) as well as possibly discrete cash flows (e.g., discrete dividends)

[9]For more details, see Chance and Brooks (2013).

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Underlying instrument cash flows
We follow the GBMOVM approach to handling dividends known as the escrow method. The present value of all future cash flows, assumed to be dividends here, can be expressed generically as

$$PV_T\left(\underline{D}\right)=S_0\left(1-e^{-\delta T}\right)+\sum_{i=1}^{N}PV_{\tau_i}\left(D_i\right), \tag{5.5.6}$$

where S_0 denotes the current price of the underlying instrument at time 0, D_i denotes the ith dividend paid at time τ_i, $PV_{\tau_i}\left(D_i\right)$ denotes the present value at time 0 of the ith dividend, T denotes the expiration of the option expressed in years, and δ denotes the annualized, continuously compounded cash flow yield. Again, we define the underlying instrument value sans (without) these cash flows as

$$S_0'=S_0-PV_T\left(\underline{D}\right)=S_0-\left[S_0\left(1-e^{-\delta T}\right)+\sum_{i=1}^{N}PV_{\tau_i}\left(D_i\right)\right]=B_\delta S_0-\sum_{i=1}^{N}PV_{\tau_i}\left(D_i\right). \tag{5.5.7}$$

ABM option valuation model (ABMOVM)
ABMOVM is a mathematical model for valuing financial options that are European-style. European-style options can only be exercised at the expiration of the option and can be expressed as:

$$\begin{aligned}&O\left(S_0',t;\iota_U,X,T,r,\sigma\right)=PV_r\left\{E_0\left[O\left(FV_rS_t',T\right)\right]\right\}\\&=B_r\left[\iota_U\left(S_0'B_{-r}-X\right)N\left(\iota_Ud_n\right)+\sigma_An\left(d_n\right)\right]=\iota_U\left(S_0'-B_rX\right)N\left(\iota_Ud_n\right)+B_r\sigma_An\left(d_n\right)\end{aligned}, \tag{5.5.8}$$

where again the indicator function is expressed as

$$\iota_U=\begin{cases}+1 \text{ if underlying call option}\\-1 \text{ if underlying put option}\end{cases}, \tag{5.5.9}$$

$$B_r=e^{-r(T-t)}, \tag{5.5.10}$$

$$N\left(d\right)=\int_{-\infty}^{d}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx, \text{ (area under the standard normal cumulative distribution up to d)} \tag{5.5.11}$$

$$n\left(d\right)=\frac{e^{-d^2/2}}{\sqrt{2\pi}}, \text{ (standard normal probability density function)} \tag{5.5.12}$$

$$\sigma_A^2=\sigma^2\frac{B_{-2r}-1}{2r}, \text{ (periodic adjusted volatility)} \tag{5.5.13}$$

$$d_n=\frac{S_t'B_{-r}-X}{\sigma_A}. \text{ (quasi “z” score)} \tag{5.5.14}$$

This presentation of the ABMOVM assumes the escrow method for handling dividends. Thus, the periodic adjusted volatility does not have any adjustment for dividends. The variable d_n is the distance in periodic adjusted volatility units that the expected terminal stock price is from the strike price. We now sketch the derivation for this model.

Derivation of the call value based on the ABMOVM framework
We briefly sketch the proof of the ABMOVM model. The underlying instrument is assumed to pay a continuous yield. Consider the following three steps:

Step 1: Distribution of underlying instrument and call
Step 2: Create arbitrage cash flow table and compute hedge ratio
Step 3: Calculate option value

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Step 1: Distribution of stock and call
Assume the stock price follows arithmetic Brownian motion,

$$dS = \mu S dt + \sigma dw. \tag{5.5.15}$$

Further, we know that *C = C(S,t)*. Therefore, by Itô's lemma, we know the call price follows and Ito process of the form,

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} \sigma dw. \tag{5.5.16}$$

Step 2: Create arbitrage cash flow table and compute hedge ratio
Consider selling 1 call and entering $\frac{\partial C}{\partial S}$ stock (positive number indicates purchase). Denote the portfolio as Π, the value of the portfolio is

$$\Pi = -C + \frac{\partial C}{\partial S} S. \tag{5.5.17}$$

A small change in time results in a change in the portfolio value,

$$d\Pi = -dC + \frac{\partial C}{\partial S} dS + q \frac{\partial C}{\partial S} S dt. \tag{5.5.18}$$

Note that q denotes the dividend yield. Substituting from step 1 for dC and dS, we have

$$d\Pi = -\left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} \right) dt - \frac{\partial C}{\partial S} \sigma dw + \frac{\partial C}{\partial S} \left(\mu S dt + \sigma dw \right) + q \frac{\partial C}{\partial S} S dt \text{ or} \tag{5.5.19}$$

$$d\Pi = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} - q \frac{\partial C}{\partial S} S \right) dt. \text{ (Hedged Portfolio)} \tag{5.5.20}$$

Note that for small changes in the portfolio, the portfolio is risk-free (there is no dw term). Therefore the portfolio should earn the risk-free rate, r. That is,

$$d\Pi = r\Pi dt = r\left(-C + \frac{\partial C}{\partial S} S \right) dt \text{ (Risk Free Portfolio)} \tag{5.5.21}$$

Step 3: Calculate option value
Combining the results of Hedged Portfolio equation and Risk Free Portfolio equation above, we have

$$-\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} - q \frac{\partial C}{\partial S} S \right) dt = r\left(-C + \frac{\partial C}{\partial S} S \right) dt. \tag{5.5.22}$$

Cancelling dt and rearranging, we have the ABM partial differential equation (ABM PDE)

$$rC = \frac{\partial C}{\partial t} + \left(r - \delta \right) \frac{\partial C}{\partial S} S + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2}. \text{ (ABM PDE)} \tag{5.5.23}$$

The ABM PDE is a second order, partial differential equation which when combined with the boundary condition,

$$C\left(S, t = T \right) = \max\left(0, S_T - X \right). \text{ (Call Boundary Condition)} \tag{5.5.24}$$

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is the ABM differential equation. Recall solutions to problems of this nature are unique. Therefore once you have a proposed solution all you have to do is check to be sure the ABM PDE and Call Boundary Condition equations are satisfied and you are finished. The ABMOVM can be represented as

$$C=\left(S_0 B_\delta - B_r X\right)N\left(d_n\right)+B_r\sigma_A n\left(d_n\right),\ \text{(ABMOVM)} \tag{5.5.25}$$

where

$$\begin{gathered}\sigma_A^2=\sigma^2\frac{B_{-2(r-\delta)}-1}{2\left(r-\delta\right)};r\neq\delta\ \text{ and}\\ \sigma_A^2=\sigma^2\left(T-t\right);r=\delta\end{gathered} \tag{5.5.26}$$

$$d_n=\frac{S_t B_{-(r-\delta)}-X}{\sigma_A}\quad d_n=\frac{S_t' B_{-(r-\delta)}-X}{\sigma_A}. \tag{5.5.27}$$

It can be shown, based on the ABMOVM, that

$$C_S\equiv\frac{\partial C}{\partial S}=B_\delta N\left(d_n\right),\ \text{(Delta)} \tag{5.5.28}$$

$$C_{SS}\equiv\frac{\partial^2 C}{\partial S^2}=n\left(d_n\right)\frac{B_{-(r-2\delta)}}{\sigma_A}.\ \text{(Gamma)} \tag{5.5.29}$$

$$C_t\equiv\frac{\partial C}{\partial t}=\left(S_t\delta B_\delta - XrB_r\right)N\left(d_n\right)+r\sigma_A B_r n\left(d_n\right)-\frac{\sigma^2 B_{-(r-2\delta)}}{2\sigma_A}n\left(d_n\right).\ \text{(Theta)} \tag{5.5.30}$$

Substituting the Delta, Gamma, and Theta equations into the BSM PDE equation will result in the GBMOVM equation along with satisfying the Call Boundary Condition equation is sufficient to prove that it is the unique solution.

Proof: Substituting into the ABM PDE, we have

$$\begin{aligned}rC&=\frac{\partial C}{\partial t}+\left(r-q\right)\frac{\partial C}{\partial S}S+\frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2}\\ &=\left(S_t\delta B_q - XrB_r\right)N\left(d_n\right)+r\sigma_A B_r n\left(d_n\right)-\frac{\sigma^2 B_{-(r-2\delta)}}{2\sigma_A}n\left(d_n\right).\\ &+\left(r-\delta\right)SB_\delta N\left(d_n\right)+\frac{1}{2}\sigma^2 n\left(d_n\right)\frac{B_{-(r-2\delta)}}{\sigma_A}\end{aligned} \tag{5.5.31}$$

Cancelling terms,

$$\begin{aligned}rC&=\left(S_t\delta B_\delta - XrB_r\right)N\left(d_n\right)+r\sigma_A B_r n\left(d_n\right)-\frac{\sigma^2 B_{-(r-2\delta)}}{2\sigma_A}n\left(d_n\right)\\ &+\left(r-\delta\right)SB_\delta N\left(d_n\right)+\frac{1}{2}\sigma^2 n\left(d_n\right)\frac{B_{-(r-2\delta)}}{\sigma_A}\\ &=rSB_\delta N\left(d_n\right)-rXB_r N\left(d_n\right)+r\sigma_A B_r n\left(d_n\right)\end{aligned}. \tag{5.5.32}$$

Before reviewing the R code, we provide justification for an alternative framework.

We turn now to review selected R code.

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Summary

We made a detailed comparison of the GBMOVM and the ABMOVM addressing numerous considerations. Further, we reviewed a key assumption underlying the standard option valuation model proposed by Black, Scholes and Merton and contrast it with the arithmetic Brownian motion option valuation model (ABMOVM). Recall that GBM results in a lognormal terminal distribution whereas ABM results in a normal terminal distribution. We also sketched the ABMOVM model derivation.

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