Module 5.3

Arithmetic Brownian Motion
Binomial Option Valuation Model
(ABM BOVM)

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Overview

- Explore additive binomial model
  - Arithmetic Brownian motion (ABM) in limit
- Incorporate dividends
- Digital options
- European- and American-style options
- Variety of plots generated in R

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Central Finance Concepts

- Arithmetic Brownian motion (ABM)
- Binomial option valuation model (BOVM)
- One- and two-period models
- ABM coherence conditions
- Role of dividends
- Multiperiod models
- Graphical illustrations

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ABM Binomial Framework

- Additive
- Recombining
- Incorporate dividends (discrete and continuous)
- Address early exercise with American-style options

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Binomial Lattice Frameworks

- One-period: Heuristic, intuitive, simple
- Two-period: Introduce dynamic insights
- Multiperiod: Deployable model
- European-style easier to deploy, but requires backward recursion
- American-style requires addressing potential early exercise

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ABM Coherence Conditions

- Requirements to converge to normal distribution
- Seeks to avoid arbitrage within sterile theoretical model
- Deeply useful when exploring alternative models to deploy in practice

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Dividends

- Discrete dividends: ABM approach recombines easily
- Continuous dividends: Unrealistic with stock options (pay four times per year)
- Escrow method: Bifurcate stock into two components
  - Stock without PV dividends over option life
  - PV dividends

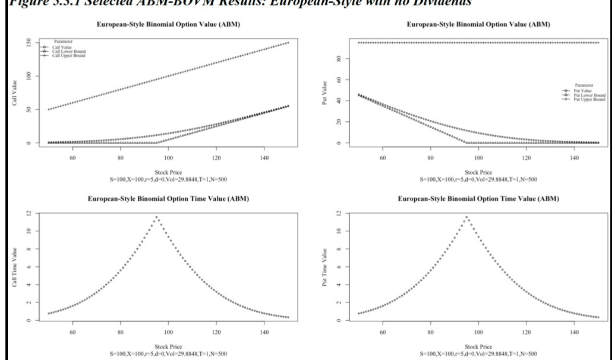
Multiperiod Models

- Challenge:
  - Probability of a single path tends to zero
  - Number of potential paths tends to infinity
- ABM requires backward recursion
- Backward recursion allows easy treatment of early exercise

Selected Illustrations

- European-style, with and without dividends
  - Sensitivity to stock price and boundaries
  - Time value sensitivity
  - Plain vanilla and digital options
  - Convergence with respect to number of steps
  - Role of dividends
- American-style contrasts

Figure 5.3.1 Selected ABM-BOVM Results: European-Style with no Dividends



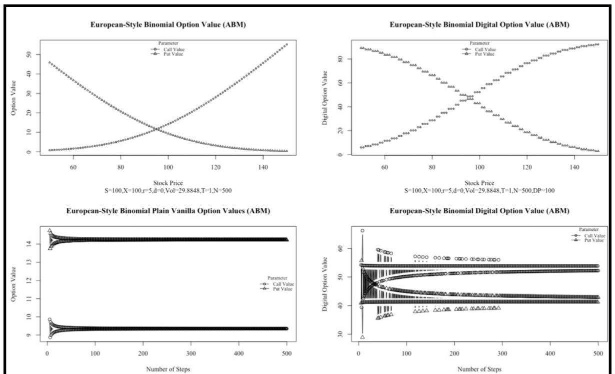
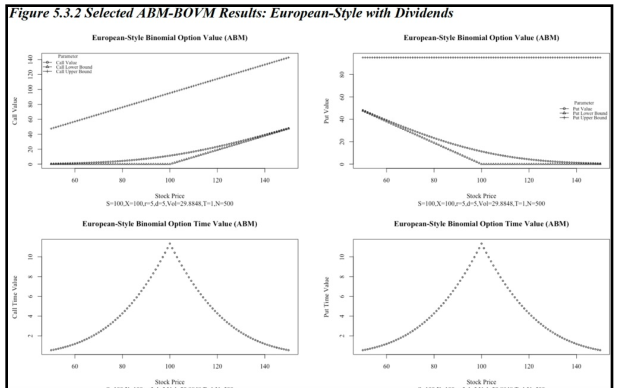


Figure 5.3.2 Selected ABM-BOVM Results: European-Style with Dividends



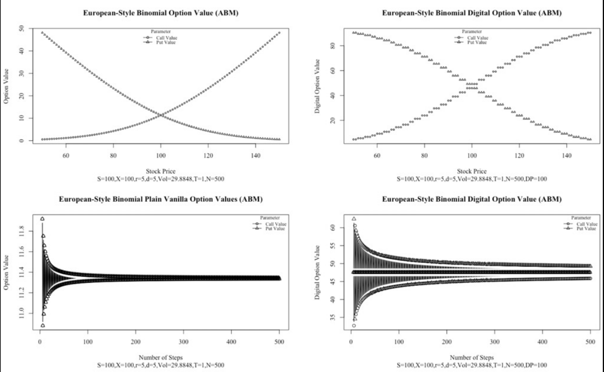


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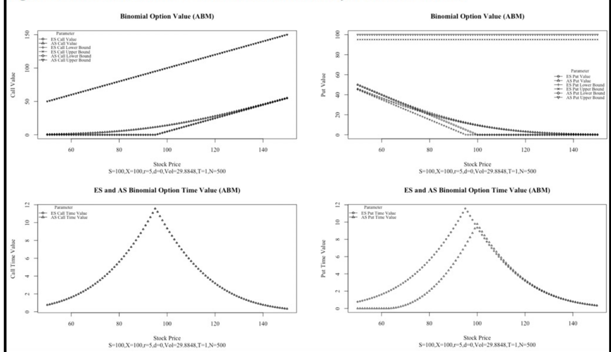
Figure 5.3.3 Selected ABM-BOVM Results: American-Style with no Dividends



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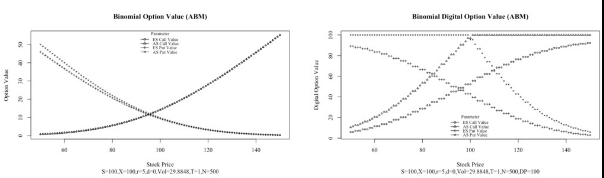

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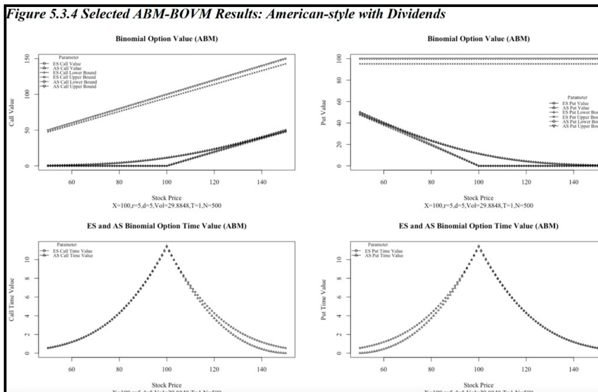
Figure 5.3.4 Selected ABM-BOVM Results: American-style with Dividends



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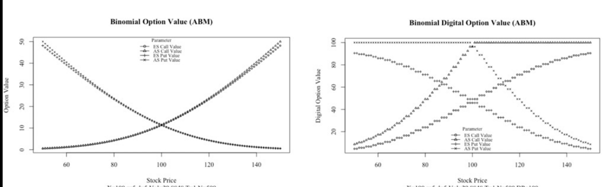

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Quantitative Finance Materials

- Notation
- One period model
  - Multiplicative
  - Dividends
- Two period model
  - Recombining
- Coherence conditions
- Multiperiod models

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Notation

$0, T, \Delta t$	initial trade date, time 0; expiration or maturity date, time T; next time step,
S_0, S_T	value of underlying instrument, e.g., stock, at time 0 and at time T,
u, d	up, dollar change in S, if up occurs ($u > 0$) and if down occurs ($d < 0$),
B_0, B_T	bond, value of risk-free investment at time 0 and at time T,
V_0, V_T	portfolio, value of some financial instrument portfolio at time 0 and at time T,
ι	indicator function, +1 for calls and –1 for puts,
O_0	option, value of options, either call or put, at time 0,
O_u, O_d	option, value of option at time T if up occurs and if down occurs,
Δ	delta, hedge ratio, units of the financial instrument to enter to hedge option position,
$FV()$	future value based on risk-free interest rate,
$PV()$	present value based on risk-free interest rate,
π	equivalent martingale probability of up move,
$E_\pi()$	expectation under equivalent martingale probability,
r	discretely compounded, periodic "risk-free" interest rate,
r_c	continuously compounded, annualized, "risk-free" interest rate,
δ	continuously compounded, annualized, dividend yield, and
D_T	known discrete dividend amount paid at time T (ex-dividend the instant *before* the next binomial point in time).

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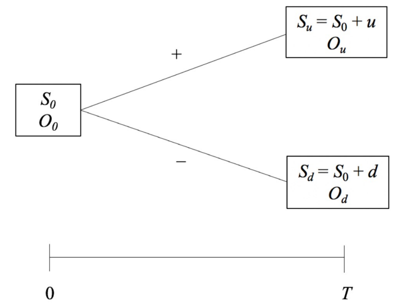
Figure 5.3.5 Additive One Period Binomial Framework



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Single Period Call Option

- Single period
- Binomial process
- Create hedged portfolio
- Derive call value
- Numerical example

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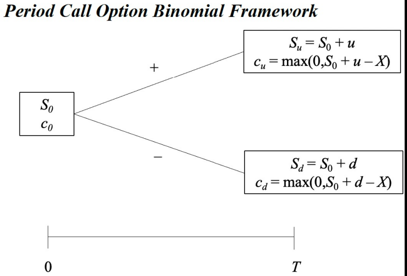
Figure 5.3.6 Additive One Period Call Option Binomial Framework



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Creating the Hedged Portfolio

- Long h shares of stock, short 1 call

$$V_0 = h_c S_0 - c_0$$

- Solve for h yielding identical future CFs

$$V_u = h_c\left(S_0 + u\right) - \max\left(0, S_0 + u - X\right) = h_c\left(S_0 + u\right) - c_u$$
$$V_d = h_c\left(S_0 + d\right) - \max\left(0, S_0 + d - X\right) = h_c\left(S_0 + d\right) - c_d$$

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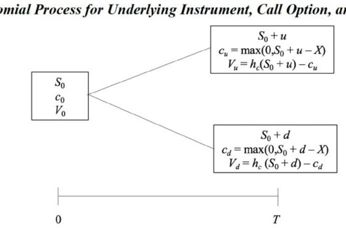
Figure 5.3.7 ABM Binomial Process for Underlying Instrument, Call Option, and Hedge Portfolio



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Hedge Ratio to Valuation

- Optimal hedge ratio ($V_u = V_d$)

$$h_c = \frac{c_u - c_d}{S_0 + u - (S_0 + d)} = \frac{c_u - c_d}{u - d}$$

- Valuation

$$c_0 = h_c S_0 - B_c \qquad c_0 = PV\left[E(c_T)\right] = \frac{\pi c_u + (1-\pi)c_d}{1+r}$$

$$B_{0,c} = \frac{h_c(S_0 + u) - c_u}{1+r} \qquad \pi_0 = \frac{S_0 r - d}{u - d}$$

Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=24.75, d=–19.8
- Intermediate calculations
  - $S_u = S + u = 99 + 24.75 = 123.75$
  - $S_d = S + d = 99 - 19.8 = 79.2$
  - $c_u = \max(0, 123.75 - 100) = 23.75$
  - $c_d = \max(0, 79.2 - 100) = 0$
  - $h_c = (c_u - c_d)/ (u - d)$
    $= (23.75 - 0)/[24.75 - (-19.8)]$
    $= 23.75/44.55 = 0.5331$

Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=24.75, d=–19.8
- No arbitrage model solution

$$\begin{aligned} c_0 &= h_c S_0 - \frac{h_c(S_0 + u) - c_u}{1+r} \\ &= 0.5331(99) - \frac{0.5331(99 + 24.75) - 23.75}{1 + 0.02} \\ &= 52.7769 - 41.3933 = 11.38 \end{aligned}$$

Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=24.75, d=–19.8
- Equivalent martingale solution

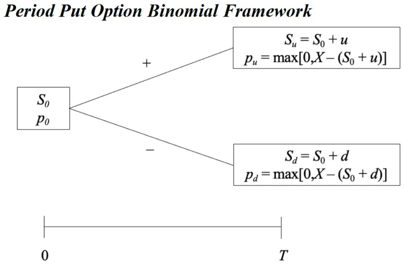
$$\begin{aligned} \pi &= \frac{1 + r - d}{u - d} \\ &= \frac{99(0.02) - (-19.8)}{24.75 - (-19.8)} = \frac{21.78}{44.55} = 0.488889 \end{aligned}$$

$$c_0 = \frac{0.4889(23.75) + (1 - 0.4889)0}{1 + 0.02} = 11.38$$

Single Period Put Option

- Single period
- Binomial process
- Create hedged portfolio
- Derive put value
- Numerical example

Figure 5.3.8 Additive One Period Put Option Binomial Framework



Creating the Hedged Portfolio

- Buy h shares of stock, buy 1 put

$$V_0 = h_p S_0 + p_0$$

- Solve for h yielding identical future CFs

$$V_u = h_p(S_0+u) + \max\left[0, X-(S_0+u)\right] = h_p(S_0+u) + p_u$$
$$V_d = h_p(S_0+d) + \max\left[0, X-(S_0+d)\right] = h_p(S_0+d) + p_d$$

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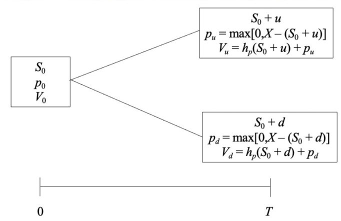
Figure 5.3.9 Binomial Process for Underlying Instrument, Put Option, and Hedge Portfolio



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Hedge Ratio to Valuation

- Optimal hedge ratio ($V_u = V_d$)

$$h_p = \frac{p_d - p_u}{u-d}$$

- Valuation

$$p_0 = B_{0,p} - h_p S_0 \qquad p_0 = PV\left[E(p_T)\right] = \frac{\pi p_u + (1-\pi)p_d}{1+r}$$

$$B_{0,p} = \frac{h_p(S_0+d)+p_d}{1+r} \qquad \pi = \frac{rS_0 - d}{u-d}$$

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Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=24.75, d=–19.8
- Intermediate calculations
  - $S_u = S + u = 99 + 24.75 = 123.75$
  - $S_d = S + d = 99 - 19.8 = 79.2$
  - $p_u = \max(0,100 - 123.75) = 0$
  - $p_d = \max(0,100 - 79.2) = 20.8$
  - $h_p = (p_d - p_u)/(u - d)$
    $= (20.8 - 0)/[24.75 - (-19.2)]$
    $= 20.8/44.55 = 0.4669$

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Single Period BOVM Example

- S=99, X=100, r=2%, T=1 , u=24.75, d=–19.8
- No arbitrage model solution

$$p_0 = \frac{h_p(S_0+d)+p_d}{1+r} - h_p S_0$$
$$= \frac{0.4669(99-19.8)+20.8}{1+0.02} - 0.4669(99)$$
$$= 56.6456 - 46.2231 = 10.42$$

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Single Period BOVM Example

- S=99, X=100, r=2%, T=1 , u=24.75, d=–19.8
- Equivalent martingale solution

$$\pi = [99(0.02) - (-19.8)]/[24.75 - (-19.8)] = 48.8889\%$$

$$p_0 = \frac{0.4889(0) + (1-0.4889)20.8}{1+0.02} = 10.42$$

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Two Period BOVM Model

- Additive
- Recombining
- Terminal values

$$c_{2u} = \max\left(0, S_0 + 2u - X\right) \qquad p_{2u} = \max\left[0, X - \left(S_0 + 2u\right)\right]$$
$$c_{ud} = \max\left(0, S_0 + u + d - X\right) \text{ and } p_{ud} = \max\left[0, X - \left(S_0 + u + d\right)\right]$$
$$c_{2d} = \max\left(0, S_0 + 2d - X\right) \qquad p_{2d} = \max\left[0, X - \left(S_0 + 2d\right)\right]$$

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Two Period BOVM Challenge

- Additive BOVM

$$c_0 = \frac{\pi_0 \pi_{1,u} c_{2u} + \left[\pi_0\left(1-\pi_{1,u}\right) + \left(1-\pi_0\right)\pi_{1,d}\right] c_{ud} + \left(1-\pi_0\right)\left(1-\pi_{1,d}\right) c_{2d}}{\left(1+r\right)^2}$$

$$p_0 = \frac{\pi_0 \pi_{1,u} p_{2u} + \left[\pi_0\left(1-\pi_{1,u}\right) + \left(1-\pi_0\right)\pi_{1,d}\right] p_{ud} + \left(1-\pi_0\right)\left(1-\pi_{1,d}\right) p_{2d}}{\left(1+r\right)^2}$$

$$\pi_0 = \frac{S_0 r - d}{u - d} \qquad \pi_{1,u} = \frac{S_u r - d}{u - d} \qquad \pi_{1,d} = \frac{S_d r - d}{u - d}$$

Interest causes up and down probabilities to be unequal.

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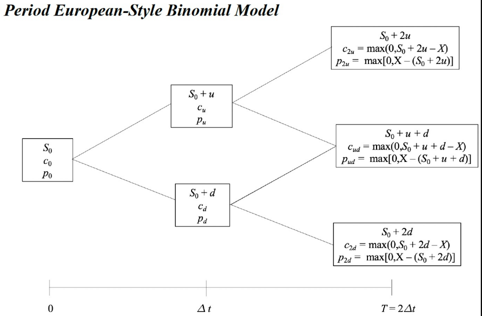
S=99, X=100, r=2%, T=1 , u=24.75, d=–19.8

Figure 5.3.9 Two Period European-Style Binomial Model



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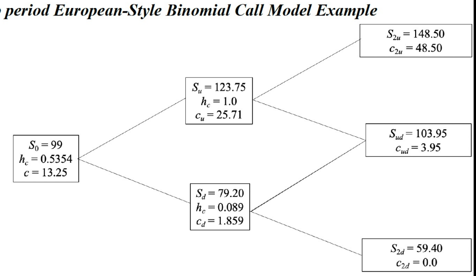
S=99, X=100, r=2%, T=1 , u=24.75, d=–19.8

Figure 5.3.10 Two period European-Style Binomial Call Model Example



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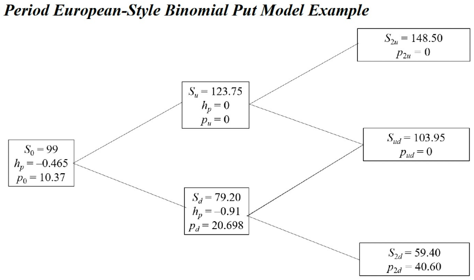
S=99, X=100, r=2%, T=1 , u=24.75, d=–19.8

Figure 5.3.11 Two Period European-Style Binomial Put Model Example



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American-Style (AS) Options

- Early exercise potential must be incorporated
- Method of backward induction
  - Start at terminal value
  - Reason backward in time
- Goal is to establish sequence of optimal actions

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AS and Dividends

- Methods for handling dividends
  - Yield method: Constant rate based on S
  - Escrow method: PVD placed in escrow
- Escrow method
  - Model stock less PVD
  - Assess early exercise decision at each node

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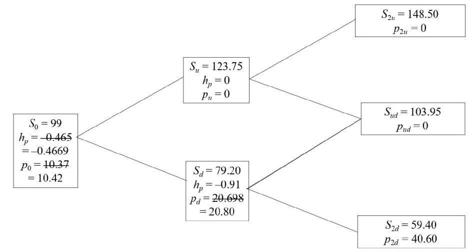
S=99, X=100, r=2%, T=1 , u=24.75, d=–19.8

Figure 5.3.12 Two period American-Style Binomial Put Model Example



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ABM Coherence Conditions

- No arbitrage boundary condition

$$d < S_0\left(e^{r\Delta t}-1\right) < u$$

- Probability condition

$$0 << \pi << 1$$

- No arbitrage condition

$$\pi = \frac{S_0\left(e^{r\Delta t}-1\right)-d}{u-d}$$

- Variance condition

$$Var_\pi\left(\Delta S_T\right) = \left(u-d\right)^2 \pi\left(1-\pi\right)$$

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Table 5.3.1. Relationship between u, d, and π

Probability	u	d	Prob Check
0	#DIV/0!	#DIV/0!	#DIV/0!
0.00000001	300000.049	0.0470125	0.00000001
0.0000001	94868.3751	0.04052567	0.0000001
0.000001	30000.035	0.02001249	0.000001
0.00001	9486.83556	-0.0448563	0.00001
0.0001	2999.90001	-0.2500025	0.0001
0.001	948.25885	-0.89914549	0.001
0.01	298.546244	-2.96510094	0.01
0.1	90.0500125	-9.9499875	0.1
0.2	60.0500125	-14.9499875	0.2
0.3	45.8757695	-19.5895976	0.3
0.4	36.7923586	-24.4448849	0.4
0.5	30.0500125	-29.9499875	0.5
0.6	24.5449099	-36.6923336	0.6
0.7	19.6896226	-45.7757444	0.7
0.8	15.0500125	-59.9499875	0.8
0.9	10.0500125	-89.9499875	0.9
0.99	3.06512595	-298.446219	0.99
0.999	0.9991705	-948.158825	0.999
0.9999	0.3500275	-2999.79998	0.9999
0.99999	0.14488131	-9486.73553	0.99999
0.999999	0.08001252	-29999.935	0.999999
0.9999999	0.05949934	-94868.275	0.9999999
0.99999999	0.0530125	-299999.948	0.99999999
1	#DIV/0!	#DIV/0!	#DIV/0!

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Generic Options

- Indicator function:

$$\iota = \iota_U = \begin{cases} +1 \text{ if call option } \left(\mathrm{O}_t = c_t\right) \\ -1 \text{ if put option } \left(\mathrm{O}_t = p_t\right) \end{cases}$$

- Terminal payoffs:
  - Calls

$$c_T \geq max\left(0, S_T - X\right)$$

  - Puts

$$p_T \geq max\left(0, X - S_T\right)$$

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u and d Conditions

- Condition for u:

$$u = S_0\left(e^{r\Delta t}-1\right) + \left(1-\pi\right)\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}$$

- Condition for d:

$$d = S_0\left(e^{r\Delta t}-1\right) - \pi\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}$$

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Multiperiod ABM BOVM

- ABM BOVM requires backward recursion
- Recall GBM BOVM

$$O_0 = PV_r\left[E_0\left(O_T\right)\right]$$
$$= PV_r\left[\sum_{j=0}^{n}\Pr\left(n,j\right)Payoff\left(\iota,n,j\right)\right]$$
$$= PV_r\left\{\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left[0,\iota\left(S_0u^jd^{n-j}-X\right)\right]\right\}$$

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Multiperiod ABM BOVM

- ABM BOVM probabilities path dependent

$$O_0 = PV_r\left[E_0\left(O_T\right)\right]$$
$$= PV_r\left[\sum_{j=0}^{n}\Pr\left(n,j\right)Payoff\left(\iota,n,j\right)\right].$$
$$= PV_r\left[\sum_{j=0}^{n}\Pr\left(n,j\right)\max\left\{0,\iota_u\left[S_0+ju+\left(n-j\right)d-X\right]\right\}\right]$$

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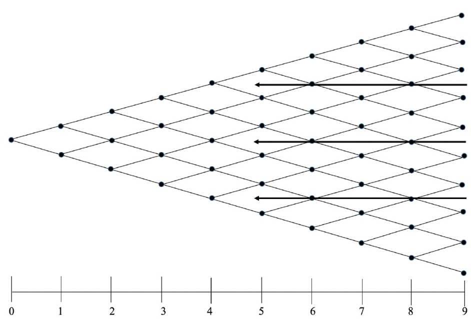
Backward Recursion

Figure 5.3.13 Nine Period Binomial Model



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American-Style Options

- Evaluate at each node

$$O_{i,j} = \max\left[O_{i,j}^B, O_{i,j}^X, O_{i,j}^L\right]$$

  - Binomial model value

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + \left(1-\pi\right)O_{i+1,j}\right]$$

  - Early exercise value

$$O_{i,j}^X = \max\left[0,\iota_U\left(S_{i,j}-X\right)\right]$$

  - Lower boundary value

$$O_{i,j}^L = \max\left\{0,\iota_U\left[S_{i,j}-PV_{r,i,n-i}\left(X\right)\right]\right\}$$

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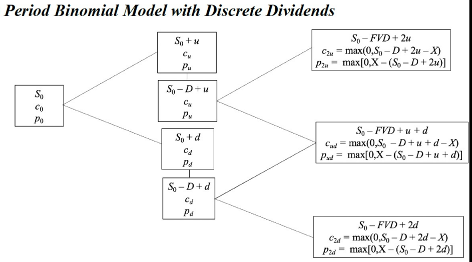
Dividends Easily Recombine

Figure 5.3.14 Two Period Binomial Model with Discrete Dividends



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Digital (Binary) Options

- Digital payout based on terminal moneyness
- Digital BOVM
  - Cash-or-nothing: Fixed cash amount
  - Asset-or-nothing: Fixed amount of asset
- Indicator function
  - I = 1 if condition is true
  - I = 0 if condition is false

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Summary

- Explored additive binomial model
- Addressed dividends
- Digital options
- European- and American-style options
- Variety of plots generated in R

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Appendices

- Appendix A: ABM BOVM single period derivation
- Appendix B: One period arbitrage examples

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