Module 5.4: Geometric Brownian Motion-Based Option Valuation Models

Learning objectives

- Explain how to value options based on the geometric Brownian motion
- Review the standard assumptions, boundary conditions and sketch a derivation of the model
- Review digital options and explain the relationship to standard option

Executive summary

We review the assumptions underlying the standard option valuation model proposed by Black, Scholes and Merton. We identify this model with geometric Brownian motion option valuation model (GBMOVM) to distinguish from arithmetic Brownian motion option valuation model (ABMOVM). Recall that GBM results in a lognormal terminal distribution whereas ABM results in a normal terminal distribution. Next, we review the boundary conditions based on static arbitrage take was extensively covered in Module 5.1. After reviewing one representation of the GBM option valuation model, we sketch out its derivation. Finally, we review digital options and related issues.

Central finance concepts

We introduce the key financial concepts of the GBMOVM and defer the technical details later in this module.

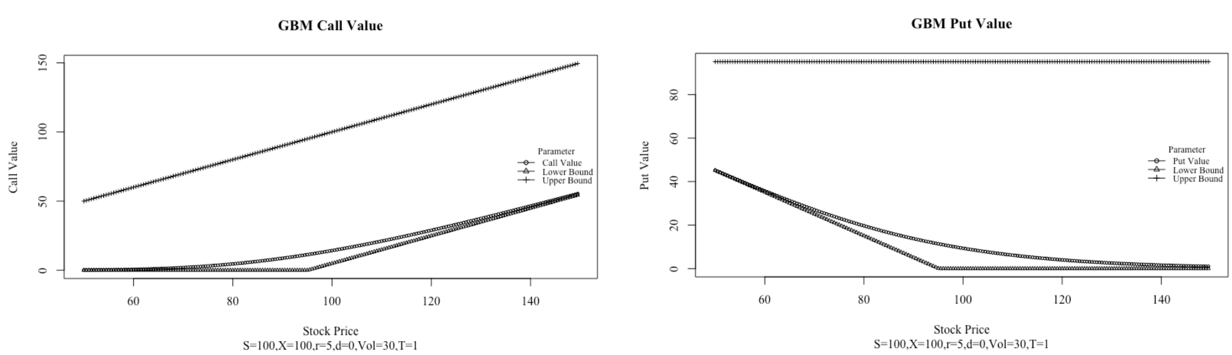
GBM option valuation model

There are several technical assumptions required for the GBMOVM to theoretically hold. The key assumptions include option are European-style, GBM, financing available at the risk-free interest rate, no market frictions, and constant volatility. Although in practice none of these assumptions are valid, still the GBMOVM is incredibly useful in providing guidance on a host of financial decisions, such as relative value (comparing one option with an alternative), future likelihoods (such as the probability of an option being in-the-money), and sensitivities (such as the Greeks like delta that measures the sensitivity of the option value to the underlying instrument price).

Because options are European-style, we assume a continuous cash flow yield such as dividend yield. Discrete dividends can be handled with the escrow method.

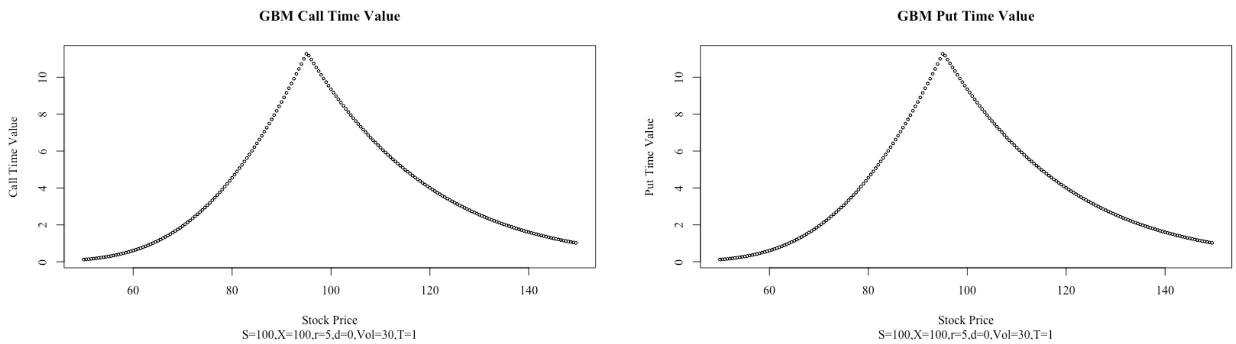
Given the importance of option boundaries within the context of valuation, we provide a concise technical review of Module 5.1 for the purpose of aiding the development and testing of computerized GBMOVM applications. Figure 5.4.1 illustrates these boundary conditions as well as the option values based on the model presented below.

Figure 5.4.1. GBMOVM option values along with boundary conditions



A careful examination of the time value plots in Figure 5.4.2 highlights the assumed lognormal distribution with positive skewness and the fact that time value is identical for both call and put options.

Figure 5.4.2. GBMOVM time value illustration



The GBMOVM is presented below in several different ways to provide clarity in applications. Finally, the GBMOVM is derived using dynamic arbitrage that results in a solvable partial differential equation.

GBMOVM applied to digital options

Digital options are also known as binary options. The option payoff is either a fixed digital payout (DP) if the option expires in-the-money (Cash or Nothing, CoN) or a fixed number of units of the underlying (assumed 1 unit) if the option expires in-the-money (Asset or Nothing, AoN).

As with plain vanilla options, digital options have boundary conditions. These technical boundary conditions are covered in extensive detail below. Any model developed to value digital options must converge to the appropriate upper and lower boundaries.

Interestingly, a plain vanilla option can be created synthetically with static arbitrage strategies from AoN and CoN digital options. Subject to certain constraints, a plain vanilla call option is equivalent to long a particular AoN call option and short a particular CoN call option. Similarly, a plain vanilla put option is equivalent to long a particular CoN put option and short a particular AoN put option. Thus, the digital option market is intrinsically linked to the plain vanilla market.

Quantitative finance materials

We now turn to addressing the technical aspects of the GBMOVM.

GBM option valuation model

GBMOVM assumptions[1]

As with any model, the Black, Scholes and Merton option valuation model or GBMOVM is based on a set of assumptions, including

- Standard finance presuppositions and assumptions apply (see Chapter 2)
- Underlying instrument behaves randomly and follows a lognormal distribution (or follow geometric Brownian motion)
- Risk-free interest rate exists, is constant, borrowing and lending allowed
- Volatility of the underlying instrument's continuously compounded rate of return is known, positive, and constant
- No market frictions, including no taxes, no transaction costs, unconstrained short selling allowed, and continuous trading
- Investors prefer more to less
- Option are European-style (exercise available only at maturity)
- Underlying instrument may pay a constant continuous cash flow yield (e.g., dividend yield) as well as possibly discrete cash flows (e.g., discrete dividends)

[1] For more details, see Chance and Brooks (2013).

Underlying instrument cash flows

The approach taken here is known as the escrow method. The escrow method first estimates all known future cash flows that are paid to owners of the underlying instrument over the life of the option being considered. These cash flows can be enumerated as discretely paid cash flows or continuously paid cash flows.

Continuously paid cash flows yield is a useful approximation for stock index options as they may contain hundreds of quarterly discrete dividend payments.

Once the future cash flows have been identified, the present value of these cash flows is estimated. The present value of all future cash flows, assumed to be dividends here, can be expressed generically as[2]

$$PV_T\left(\underline{D}\right) = S_0\left(1 - e^{-\delta T}\right) + \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right), \tag{5.4.1}$$

where S_0 denotes the current price of the underlying instrument at time 0, D_i denotes the ith dividend paid at time τ_i, $PV_{\tau_i}\left(D_i\right)$ denotes the present value at time 0 of the ith dividend, T denotes the expiration of the option expressed in years, and δ denotes the annualized, continuously compounded cash flow yield.

We define the underlying instrument value sans (without) these cash flows as

$$S_0' = S_0 - PV_T\left(\underline{D}\right) = S_0 - \left[S_0\left(1 - e^{-\delta T}\right) + \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right)\right] = B_\delta S_0 - \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right). \tag{5.4.2}$$

Thus, the underlying instrument value is decomposed into two components. For example, the market stock price (S_0) is decomposed into the stock price sans present value of dividends (S') and the present value of dividends ($PV_T\left(\underline{D}\right)$).

Recall that stock options are typically not adjusted for routine cash dividends. Hence, the terminal stock value should be based on the initial stock value sans dividends.

Option boundary conditions review

For more details, see Module 5.1. Most valuation models are limited to hold within some upper and lower boundaries. We generically represent the value of an option, call or put, as follows:

$$O\left(S_0', t; \iota_U, X, T\right) = PV_k\left\{E_{0,\pi}\left[O\left(\tilde{S}_T', T\right)\right]\right\} = PV_k\left(E_0\left\{\max\left[0, \iota_U\left(\tilde{S}_T' - X\right)\right]\right\}\right). \tag{5.4.3}$$

Where the indicator functions is expressed as

$$\iota_U = \begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option} \end{cases}, \tag{5.4.4}$$

X denotes the strike price, and k denotes some unspecified discount rate. The value of the underlying instrument at time T equals to the value of the underlying instrument without cash flows as they have now all been paid. Hence,

$$\tilde{S}_T = \tilde{S}_T'. \tag{5.4.5}$$

Let a default-free, zero coupon, \$1 par bond be expressed as

$$B_r = e^{-r}, \tag{5.4.6}$$

For stock options, the boundaries are as follows.

[2]Typically, either continuous cash flows or discrete cash flows are modeled, not both. Modeling both poses some technical issues beyond the scope of our interest here. We represent both for simplicity of analysis later.

Option upper bound
The upper boundary for options can be expressed as

$$O_0 \leq \max\left(\iota_U S_0', -\iota_U B_r X\right). \tag{5.4.7}$$

Thus, the call upper bound is S_0' and the put upper bound is $B_r X$.
Option lower bound
The lower boundary for options can be expressed as

$$O_0 \geq \max\left[0, \iota_U\left(S_0' - B_r X\right)\right]. \tag{5.4.8}$$

Thus, the call lower bound is $\max\left(0, S_0' - B_r X\right)$ and the put lower bound is $\max\left(0, B_r X - S_0'\right)$.
Proofs for these boundary conditions can be found in most introductory financial derivatives textbooks, such as Chance and Brooks (2013), as well as Module 5.1.
GBM option valuation model (GBMOVM)
Fischer Black and Myron Scholes (1973) along with Robert Merton (1973) developed a mathematical model for valuing financial options that are European-style. European-style options can only be exercised at the expiration of the option. Based on a set of restrictive assumptions, they derive the following valuation model (the continuously compounded dividend yield version):

$$\begin{aligned} O\left(S_0', t; \iota_U, X, T, r, \sigma\right) &= PV_r\left\{E_0\left[O\left(FV_r S_t', t\right)\right]\right\} \\ &= B_r\left[\iota_U S_0' B_{-r} N\left(\iota_U d_1\right) - \iota_U X N\left(\iota_U d_2\right)\right] = \iota_U S_0' N\left(\iota_U d_1\right) - \iota_U X B_r N\left(\iota_U d_2\right) \end{aligned}, \tag{5.4.9}$$

where again the indicator functions is expressed as

$$\iota_U = \begin{cases} +1 \text{ if underlying call option} \\ -1 \text{ if underlying put option} \end{cases}, \tag{5.4.10}$$

$$B_r = e^{-r}, \tag{5.4.11}$$

$$N\left(d\right) = \int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx, \text{ (area under the standard cumulative normal distribution up to d)} \tag{5.4.12}$$

$$d_1 = \frac{\ln\left(\frac{S_0'}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \text{ and} \tag{5.4.13}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{5.4.14}$$

If there is only a cash flow yield, then the call and put option equations can be expressed as

$$C_0 = S_0 e^{-\delta T} N\left(d_1\right) - X e^{-rT} N\left(d_2\right) \text{ and} \tag{5.4.15}$$

$$P_0 = X e^{-rT} N\left(-d_2\right) - S_0 e^{-\delta T} N\left(-d_1\right). \tag{5.4.16}$$

Note here

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \text{ and} \tag{5.4.17}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{5.4.18}$$

Recall the N(d) function was covered in detail in Module 3.2.

Derivation of the call value based on the GBMOVM framework

We briefly sketch the proof of the GBMOVM model. The underlying instrument is assumed to pay a continuous yield. Consider the following three steps:

Step 1: Distribution of underlying instrument and call
Step 2: Create arbitrage cash flow table and compute hedge ratio
Step 3: Calculate option value

Step 1: Distribution of stock and call

Assume the stock price follows geometric Brownian motion,

$$dS = \mu S dt + \sigma S dw. \tag{5.4.19}$$

Further, we know that $C = C(S,t)$. Therefore, by Itô's lemma, we know the call price follows and Ito process of the form,

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} \sigma S dw. \tag{5.4.20}$$

Step 2: Create arbitrage cash flow table and compute hedge ratio

Consider selling 1 call and entering $\frac{\partial C}{\partial S}$ stock (positive number indicates purchase). Denote the portfolio as Π, the value of the portfolio is

$$\Pi = -C + \frac{\partial C}{\partial S} S. \tag{5.4.21}$$

A small change in time results in a change in the portfolio value,

$$d\Pi = -dC + \frac{\partial C}{\partial S} dS + q \frac{\partial C}{\partial S} S dt. \tag{5.4.22}$$

Note that q denotes the dividend yield. Substituting from step 1 for dC and dS, we have

$$d\Pi = -\left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt - \frac{\partial C}{\partial S} \sigma S dw + \frac{\partial C}{\partial S} \left(\mu S dt + \sigma S dw \right) + q \frac{\partial C}{\partial S} S dt \text{ or} \tag{5.4.23}$$

$$d\Pi = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - q \frac{\partial C}{\partial S} S \right) dt. \text{ (Hedged Portfolio)} \tag{5.4.24}$$

Note that for small changes in the portfolio, the portfolio is risk-free (there is no dw term). Therefore the portfolio should earn the risk-free rate, r. That is,

$$d\Pi = r\Pi dt = r\left(-C + \frac{\partial C}{\partial S} S \right) dt \text{ (Risk Free Portfolio)} \tag{5.4.25}$$

Step 3: Calculate option value

Combining the results of Hedged Portfolio equation and Risk Free Portfolio equation above, we have

$$-\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - q \frac{\partial C}{\partial S} S \right) dt = r\left(-C + \frac{\partial C}{\partial S} S \right) dt. \tag{5.4.26}$$

Cancelling dt and rearranging, we have the standard Black-Scholes-Merton partial differential equation (BSM PDE)

$$rC = \frac{\partial C}{\partial t} + \left(r - q\right)\frac{\partial C}{\partial S}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}. \text{ (BSM PDE)} \tag{5.4.27}$$

The BSM PDE is a second order, partial differential equation which when combined with the boundary condition,

$$C\left(S, t = T\right) = \max\left(0, S_T - X\right). \text{ (Call Boundary Condition)} \tag{5.4.28}$$

is the Black-Scholes-Merton differential equation. This equation was originally solved by transforming it into a representation that is isomorphic to the well-known heat transfer equation in a half space. However, solutions to problems of this nature are unique. Therefore once you have a proposed solution all you have to do is check to be sure the BSM PDE and Call Boundary Condition equations are satisfied and you are finished. The GBMOVM can be represented as

$$C = Se^{-qT}N\left(d_1\right) - Xe^{-rT}N\left(d_2\right), \text{ (GBMOVM)} \tag{5.4.29}$$

where $N\left(d\right)$ is the area under the standard cumulative normal distribution up to d (see the table nearby), or

$$N\left(d\right) = \int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx, \tag{5.4.30}$$

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{5.4.31}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{5.4.32}$$

It can be shown, based on the GBMOVM, that[3]

$$C_S = e^{-qT}N\left(d_1\right), \text{ (Delta)} \tag{5.4.33}$$

$$C_{SS} = \frac{e^{-qT}n\left(d_1\right)}{S\sigma\sqrt{T}}. \text{ (Gamma)} \tag{5.4.34}$$

$$C_t = -\frac{e^{-qT}Sn\left(d_1\right)\sigma}{2\sqrt{T}} - rXe^{-rT}N\left(d_2\right) + qSe^{-qT}N\left(d_1\right), \text{ and (Theta)} \tag{5.4.35}$$

$$n\left(d\right) = \frac{e^{-d^2/2}}{\sqrt{2\pi}}. \text{ (Probability Density Function)} \tag{5.4.36}$$

Substituting the Delta, Gamma, and Theta equations into the BSM PDE equation will result in the GBMOVM equation along with satisfying the Call Boundary Condition equation is sufficient to prove that it is the unique solution.

GBMOVM applied to digital options

Digital options are also known as binary options. The option payoff is either a fixed digital payout (DP) if the option expires in-the-money (Cash or Nothing, CoN) or a fixed number of units of the underlying (assumed 1 unit) if the option expires in-the-money (Asset or Nothing, AoN). Using the indicator function, we note the payoff at expiration is

$$AoN\left(S_T, T; \iota_U, X, T, r, \sigma\right) = 1_{\iota_U S_T > \iota_U X} S_T \text{ or} \tag{5.4.37}$$

[3]See Module 9.3 for the technical derivations of these derivatives (also known as Greeks).

$$CoN\left(S_T,T;\iota_U,X,T,r,\sigma,DP\right)=1_{\iota_U S_T>\iota_U X}DP. \tag{5.4.38}$$

The resulting digital option payoffs can alternatively be expressed as

$$C_{AoN,T}=\begin{cases}0 & S_T\le X\\ S_T & S_T>X\end{cases}, \text{ (Digital asset-or-nothing call option)} \tag{5.4.39}$$

$$P_{AoN,T}=\begin{cases}S_T & S_T\le X\\ 0 & S_T>X\end{cases}, \text{ (Digital asset-or-nothing put option)} \tag{5.4.40}$$

$$C_{CoN,T}=\begin{cases}0 & S_T\le X\\ DP & S_T>X\end{cases}, \text{ (Digital cash-or-nothing call option)} \tag{5.4.41}$$

$$P_{CoN,T}=\begin{cases}DP & S_T\le X\\ 0 & S_T>X\end{cases}. \text{ (Digital asset-or-nothing put option)} \tag{5.4.42}$$

Digital option upper bound
The digital option upper bounds can be expressed as

$$AoN\left(S_0,0;\iota_U,X,T,r,\sigma\right)\le S_0' \text{ and} \tag{5.4.43}$$

$$CoN\left(S_0,0;\iota_U,X,T,r,\sigma,DP\right)\le B_r DP. \tag{5.4.44}$$

The upper boundary for plain vanilla options when X = DP can be expressed as

$$O_0\le \max\left(\iota_U S_0',-\iota_U B_r DP\right). \tag{5.4.45}$$

Thus, the AoN upper bound is S_0' and the CoN upper bound is $B_r DP$.
Digital option lower bound
The digital option lower bounds is a bit more difficult to verify. Due to limited liability, we know

$$AoN\left(S_0,0;\iota_U,X,T,r,\sigma\right)\ge 0 \text{ and} \tag{5.4.46}$$

$$CoN\left(S_0,0;\iota_U,X,T,r,\sigma,DP\right)\ge 0. \tag{5.4.47}$$

We seek, however, the highest lower bound. The lower boundary for plain vanilla options when X = DP can again be expressed as

$$O_0\ge \max\left[0,\iota_U\left(S_0'-B_r DP\right)\right]. \tag{5.4.48}$$

Thus, the plain vanilla call lower bound is $\max\left(0,S_0'-B_r X\right)$ and the plain vanilla put lower bound is $\max\left(0,B_r X-S_0'\right)$ as demonstrated next.
European-style plain vanilla option lower boundary condition
The lower boundary condition for a European-style option: (r > 0)

$$O_0\ge \max\left[0,\iota_U\left(S_0'-B_r X\right)\right]. \tag{5.4.49}$$

The technique consistent with arbitrage is, in mathematical terms, proof by contradiction. Assuming the opposite,

$$O_0< \max\left[0,\iota_U\left(S_0'-B_r X\right)\right]. \tag{5.4.50}$$

Note if $\iota_U\left(S_0' - B_r X\right) \le 0$, then C < 0 and thus buy option and receive money—a case of disequilibrium. If $\iota_U\left(S_0' - B_r X\right) > 0$, then

$$O_0 < \iota_U\left(S_0' - B_r X\right). \tag{5.4.51}$$

Rearranging by moving every term to the greater than side,

$$0 < \iota_U S_0' - \iota_U B_r X - O_0. \tag{5.4.52}$$

Substituting for S_0' [recall $S_0' = B_\delta S_0 - \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right)$],

$$0 < \iota_U\left[B_\delta S_0 - \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right)\right] - \iota_U B_r X - O_0, \tag{5.4.53}$$

or

$$0 < \iota_U B_\delta S_0 - \iota_U \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right) - \iota_U B_r X - O_0. \tag{5.4.54}$$

Table 5.4.1 presents the cash flow table below illustrates arbitrage.

Table 5.4.1. Cash flow table illustrating influence of dividends

Strategy	**Today**	**D_1**	**D_2**	**…**	**D_N**	**At Expiration**
Stock	$\iota_U B_\delta S_0$	$-\iota_U D_1$	$-\iota_U D_2$		$-\iota_U D_N$	$-\iota_U S_T$
X Financing	$-\iota_U B_r X$					$+\iota_U X$
CF Financing	$-\iota_U \sum_{i=1}^{N} PV_{\tau_i}\left(D_i\right)$	$+\iota_U D_1$	$+\iota_U D_2$		$+\iota_U D_N$	
Buy option	$-O_0$					$+O_T$
Net	$Net_0 > 0$	0	0		0	$Net_T \ge 0$

Recall

$$O_T = \max\left[0, \iota_U\left(S_T - X\right)\right]. \tag{5.4.55}$$

Thus,

$$Net_T = O_T - \iota_U\left(S_T - X\right) = \max\left[0, \iota_U\left(S_T - X\right)\right] - \iota_U\left(S_T - X\right). \tag{5.4.56}$$

For a call option,

$$Net_T = \max\left(0, S_T - X\right) - \left(S_T - X\right). \tag{5.4.57}$$

Thus, when $S_T < X$, then $Net_T = X - S_T > 0$ and when $S_T \ge X$, then $Net_T = 0$.
For a put option,

$$Net_T = \max\left(0, X - S_T\right) - \left(X - S_T\right). \tag{5.4.58}$$

Thus, when $S_T \le X$, then $Net_T = S_T - X > 0$ and when $S_T < X$, then $Net_T = 0$.
Therefore, the cash flow table above evidences an arbitrage opportunity. Arbitrage opportunities should not exist in equilibrium; hence, the contradictory assumption is false, validating the original claim.

Selected musing on digital option boundaries
Assuming X = DP,

$$O_0 = AoN_{\iota_U,0} - CoN_{\iota_U,0} \geq \max\left[0, \iota_U\left(S_0' - B_r DP\right)\right]. \tag{5.4.59}$$

Thus,

$$AoN_{\iota_U,0} \geq \max\left[0, \iota_U\left(S_0' - B_r DP\right)\right] + CoN_{\iota_U,0} \tag{5.4.60}$$

and

$$CoN_{\iota_U,0} \leq AoN_{\iota_U,0} - \max\left[0, \iota_U\left(S_0' - B_r DP\right)\right]. \tag{5.4.61}$$

Recall from Equation (5.4.9),

$$O\left(S_0', t; \iota_U, X, T, r, \sigma\right) = \iota_U S_0' N\left(\iota_U d_1\right) - \iota_U X B_r N\left(\iota_U d_2\right). \tag{5.4.62}$$

Thus,

$$AoN\left(S_0', t; \iota_U, X, T, r, \sigma\right) = S_0' N\left(\iota_U d_1\right) \text{ and} \tag{5.4.63}$$

$$CoN\left(S_0', t; \iota_U, X, T, r, \sigma, DP\right) = B_r DPN\left(\iota_U d_2\right). \tag{5.4.64}$$

We turn now to review selected R code.

Summary

We reviewed the assumptions underlying the standard option valuation model proposed by Black, Scholes and Merton (or GBMOVM) to distinguish from ABMOVM. GBM results in a lognormal terminal distribution whereas ABM results in a normal terminal distribution. Next, we reviewed the boundary conditions based on static arbitrage take was extensively covered in Module 5.1. After reviewing one representation of the GBM option valuation model, we sketch out its derivation. Finally, we reviewed digital options and related issues.

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