Module 5.1: Option Boundaries and Parities

Learning objectives

- Briefly review standard option boundary conditions
- Briefly review the standard option parities
- Examine empirical data related to option prices on a single date

Executive summary

In this module we review option boundary conditions and put-call parties. The key to understanding these relationships and why they hold in practice is to understand the thought process of arbitrageurs.

We first review how arbitrageurs approach boundaries with the simple example of the zero boundary. Next, we explore a variety of option boundaries for both European-style and American-style options. Finally, we demonstrate why put-call parity must hold in free economic systems. All the arbitrage transaction presented in this chapter are static in nature. That is, the transaction occurs at the initial trade date and could simply be held until the option expires. Often, however, the arbitrage transaction's life is very short, even seconds or minutes but rarely days.

This module concludes with R code exploring boundaries for American-style options on SPY for one date.

Central finance concepts

The primary conceptual goal is to have you acquire a familiarity with the arbitrageur's mindset. An arbitrageur is a unique player in modern free economies who provide market liquidity and transactional efficiency. Here we simply introduce the mechanics of arbitrage as well as explore the importance of option boundary conditions as well as various parities. Option boundary conditions include upper and lower bounds on the market price. Option boundaries are expressed as inequalities. Parities can establish a precise relationship between various instruments. For example, the European-style put-call parity expresses the exact relationship between an underlying instrument, financing, calls, and puts.

All the arbitrage transaction presented in this chapter are static in nature. That is, the transaction occur at the initial trade date and could simply be held until the option expires. Often, however, the arbitrage transaction's life is very short, even seconds or minutes but rarely days. With continuous time models presented in Modules 4, 5, and 7, the arbitrage transactions are dynamic in nature theoretically occurring continuously.

First, we review the mechanics of arbitrage.

Mechanics of arbitrage

Option boundary conditions and parities are primarily determined by the activities of arbitrageurs in economically free economies. The best way to understand the activities of arbitrageurs is to understand two general principles driving arbitrage decisions. We call them the two rules of the arbitrageur.

1. Do not spend your own money.
2. Do not take any market risk.

The arbitrageur's goal is to obtain positive cash flow on the initial trade date without violating either rule.

Again, arbitrage activity can be categorized as either static or dynamic. Static arbitrage activities assume that a set of trades occurs on the trade date, the position could be held until the expiration date before being unwound. That is, there is no required intermediate trading. In practice, many static arbitrage trades are held for very short periods of time. Positions are unwound only because the anomaly ceases to exist.

Dynamic arbitrage activities assume that again a set of trades occurs on the trade date. The position, however, requires active trading while held following a set of defined rules. Finally, this dynamic portfolio could be maintained until the expiration date before being unwound. That is, there are required intermediate trading activities. Again, in practice many dynamic arbitrage trades are held for very short periods of time. Like static arbitrage, positions are unwound only because the anomaly ceases to exist.

Option boundary conditions and option parities fall within the static arbitrage category whereas the binomial-based and continuous time-based models all fall within the dynamic arbitrage category which are covered in Modules 4, 5, and 7.

Option boundary conditions
In the quantitative section below, we provide detailed documentation of the required mathematical relationship between various instruments. One obvious relationship is that option prices should be non-negative. The typical economic argument is based on liability being limited. Thus, a negative valued financial instrument that enjoys limited liability is economically irrationally.

The arbitrage perspective is different. Arbitrageurs simply seek to monetize economically irrational market prices without necessarily seeking to understand why it exists. A series of transactions are executed that results in money inflows to the arbitrageur with no chance of future money outflows. Some more complex option boundaries may be difficult to understand the economic justification. The arbitrageur simply monetizes option boundary violations. The arbitrageur's activities will force market prices back within the boundary conditions.

In general, lower boundaries are highly monitored and at times violated providing a revenue source for arbitrageurs. Upper boundaries are rarely violated; hence, they are often ignored. We provide upper boundaries below primarily for completeness.

Option parity conditions
Put-call parity establishes the relationship between put and call options with the same maturity and same exercise price. European-style put-call parity results in an exact equality. One way to conceptualize European-style put-call parity is to consider two portfolios that generate the same outcome—a floor on losses but no ceiling. The first strategy of buying stocks and buying puts is known as portfolio insurance because the put option provides a floor on your potential losses. The second strategy involves buying calls and placing the majority of funds in the bank. Thus, if the stock price falls, you just lose the call premium. If the stock price rises, the call is in-the-money and you have significant upside potential. Intuitively, these two strategies lead to similar outcomes.

For American-style put-call parity, the early exercise feature results in some complications and the relationship results in inequality expresssions. The mechanics of arbitrage, however, remain the same.

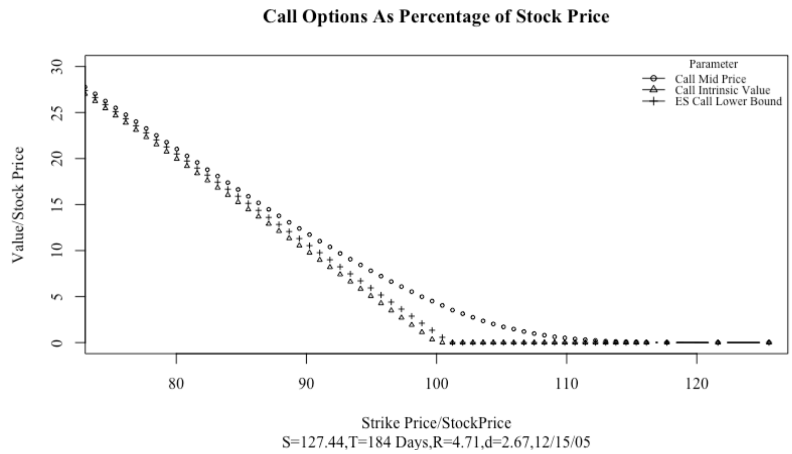
Normalization
Given the diversity of exercise prices, option maturities, and differing underlying instruments, it is often preferred to express boundaries and parities as percentages of the underlying instrument. Because the call price is lower than the underlying instrument price, we can divide the various boundaries and parities by the underlying instrument price. The result is that these conditions are expressed as a percentage of the underlying instrument. This transformed perspective aids in comparing market conditions across the different exercise prices, option maturities, and underlying instruments.

Underlying instrument cash flows
The technical materials below assume the underlying instrument is a common stock. Dividends often influence the boundaries and parities. Dividends can be modeled either as discrete cash payments or continuous cash flows. For example, the equity indexes often have hundreds of quarterly dividend-paying stocks. As each dividend goes ex-dividend on different dates, there may be a vast number of dividends paid each year. As an approximation, it is easier just to assume a dividend yield rather than discrete dividend payments.

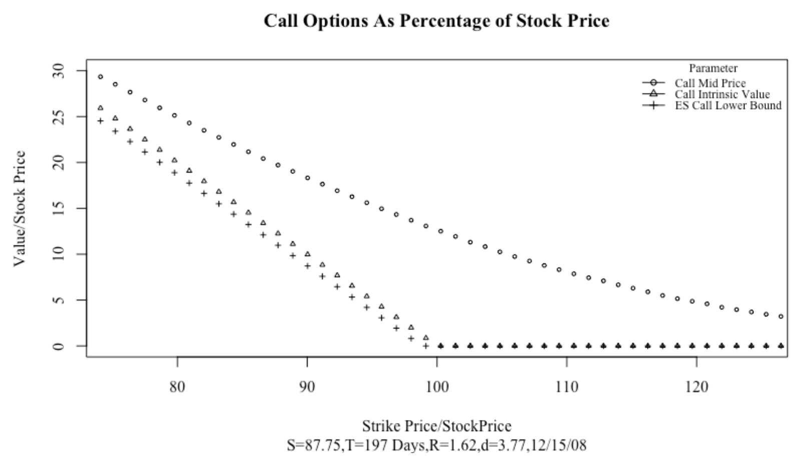
Analysis of boundaries with actual data
Based on the R code associated with this module as well as actual option data taken from three dates, December 15, 2005 (before a financial crisis), December 15, 2008 (during a financial crisis), and December 15, 2011 (after a financial crisis), we illustrate several results.

Figure 5.1.1 illustrates the call lower bound for these three selected dates, before, during and after a financial crisis. Further, the options selected were a bit over half year in time to expiration (184, 197, and 197 days, respectively). In Panel A, if you look carefully, you see the call price converges to the higher lower bound or the European-style lower bound in this case. The European-style lower bound is higher than the intrinsic value because the interest rate (R = 4.71%) exceeds the dividend yield (d = 2.67%).

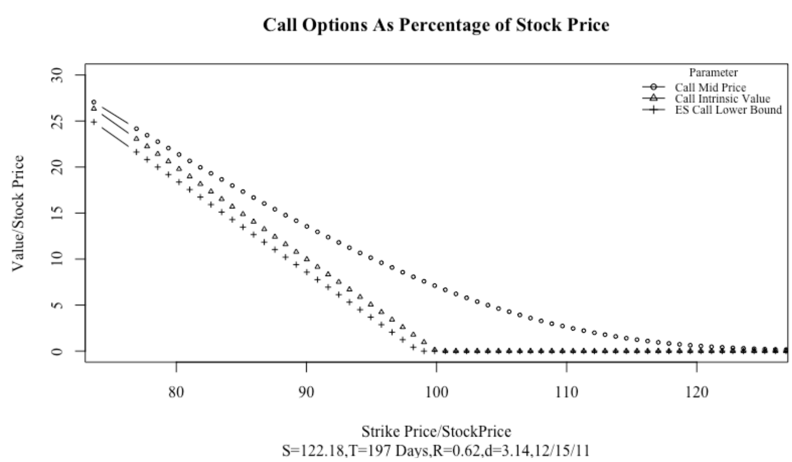
Figure 5.1.1 American-style call lower bound, call intrinsic value, and call prices for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



Panel C After a financial crisis

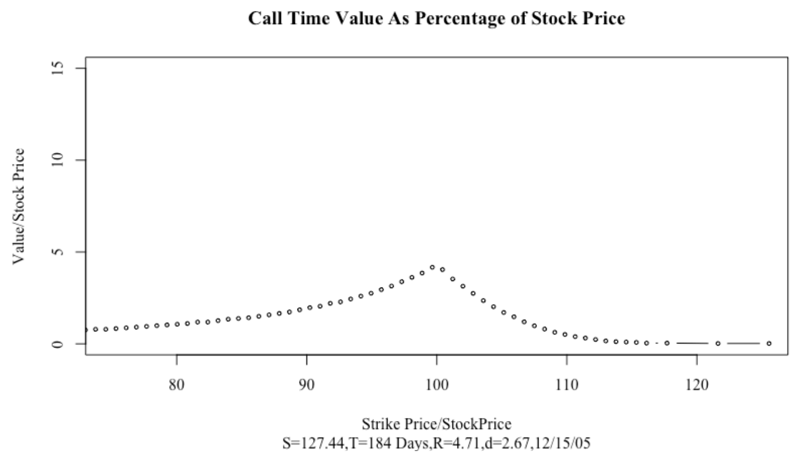


Comparing Panel B with Panel A in Figure 5.1.1, several observations can be made. First, the financial crisis resulted in significantly lower interest rates (R = 1.62%) as well as significantly higher dividend yields (d = 3.77%). Thus, the intrinsic value now exceeds the European-style lower bound. Second, normalized option values are much higher compared with boundaries. The uncertainty induced within the financial crisis resulted in higher call option values.

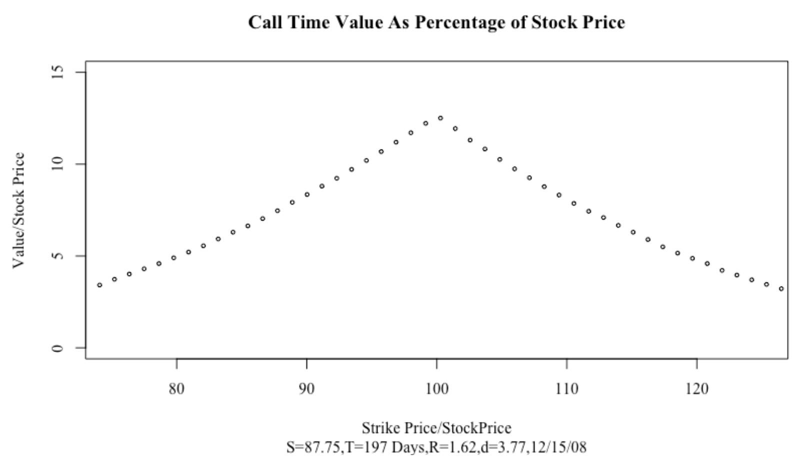
Panel C reports the post-crisis pattern. Here we clearly see the convergence to the higher lower bound or the intrinsic value. Also, the normalized option prices have once again dropped back down.

Figure 5.1.2 shows the time value or the option price less the binding lower bound (higher of the European-style lower bound and intrinsic value). Notice that the option prices decline more slowly for lower *X*/*S* values indicating a greater time premium for deep in-the-money call options when compared to deep out-of-the-money call options. This is consistent with investor's desire for downside protection as this reflects out-of-the-money put prices via put-call parity. As we will see later, this is inconsistent with the implications of the Black–Scholes–Merton option valuation model. Also, note that the time value is higher post-crisis compared to pre-crisis as the memory of this crisis likely still lingered.

Figure 5.1.2 American-style call option time value for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



Panel C After a financial crisis

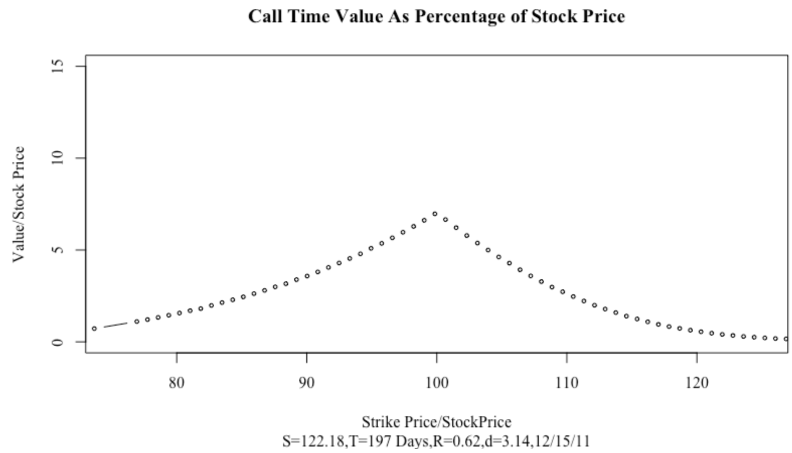
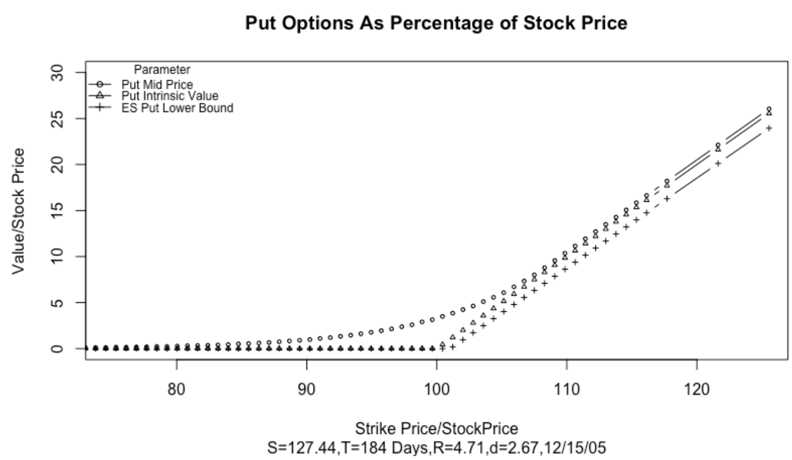
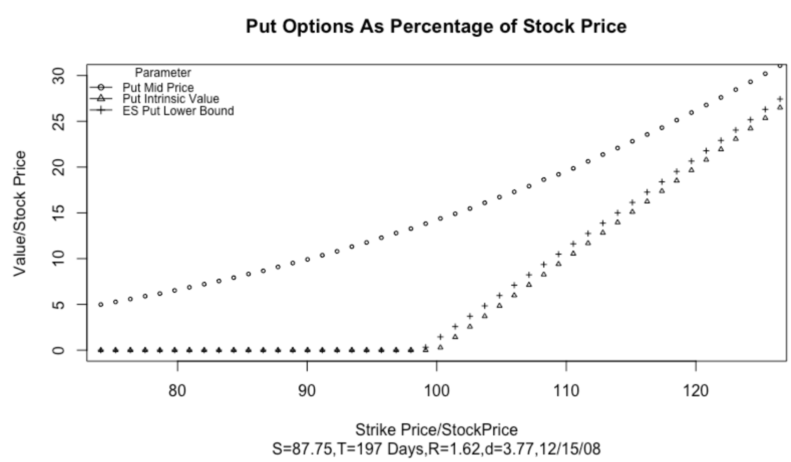


Figure 5.1.3 illustrates similar boundary convergence for the put except the prices converge much faster to the intrinsic value.

Figure 5.1.3 American-style put lower bound, put intrinsic value, and put prices for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



Panel C After a financial crisis

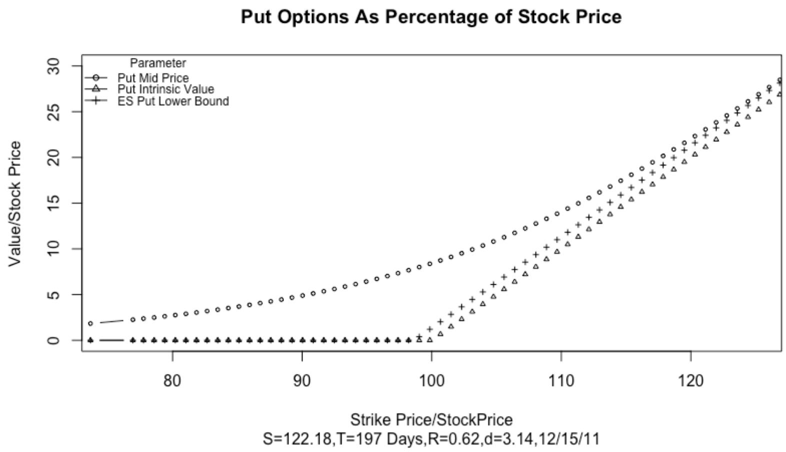
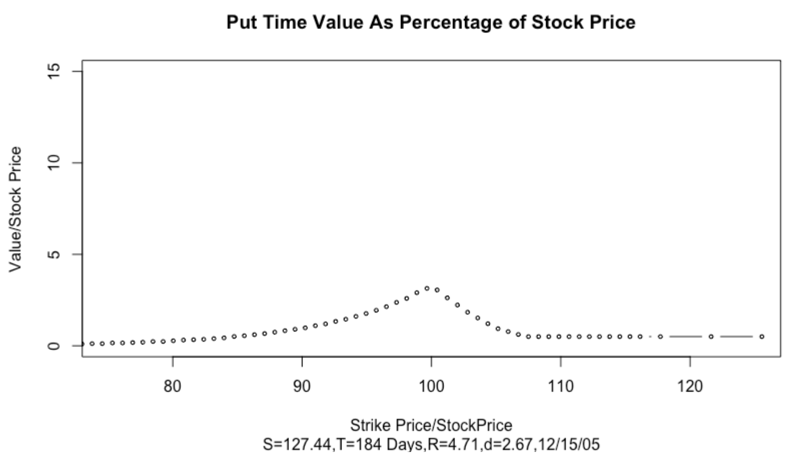
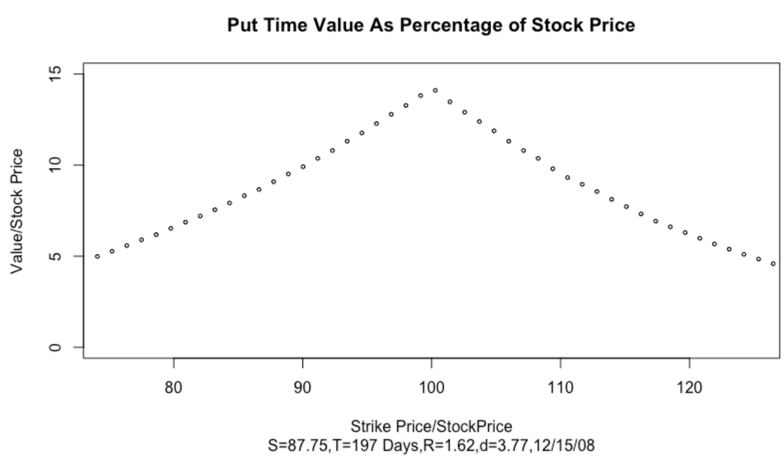


Figure 5.1.4 reports similar results for put option time values.

Figure 5.1.4 American-style put option time value for SPY
Panel A Before a financial crisis



Panel B During a financial crisis



Panel C After a financial crisis

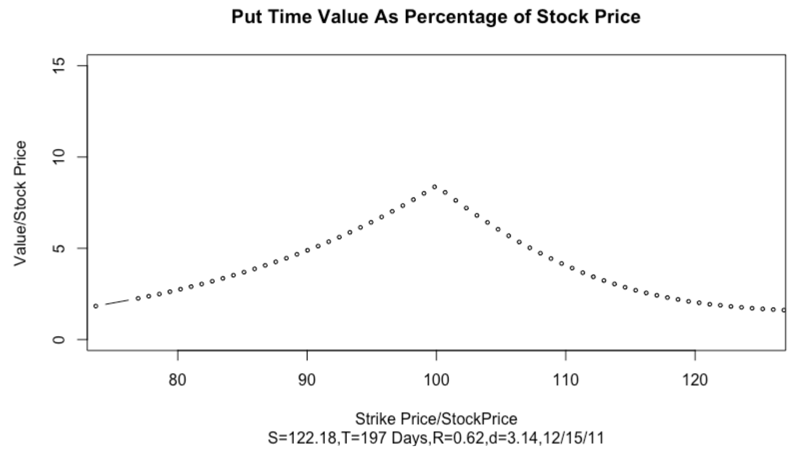
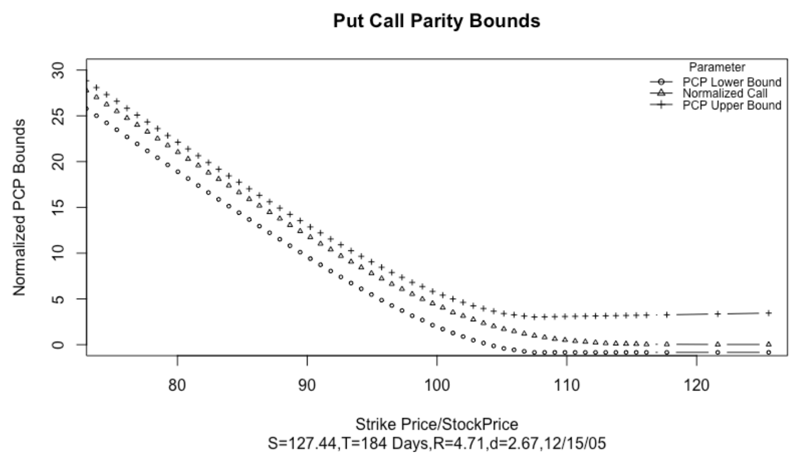
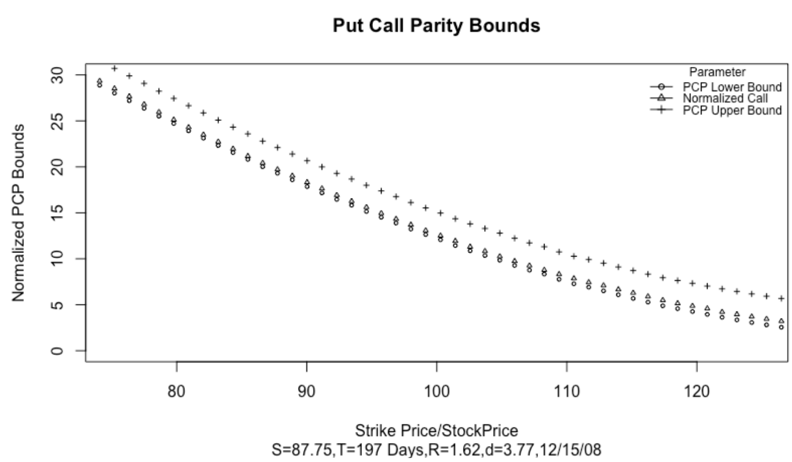


Figure 5.1.5 illustrates that the option prices stay within the American-style put-call parity boundaries for SPY on these dates.

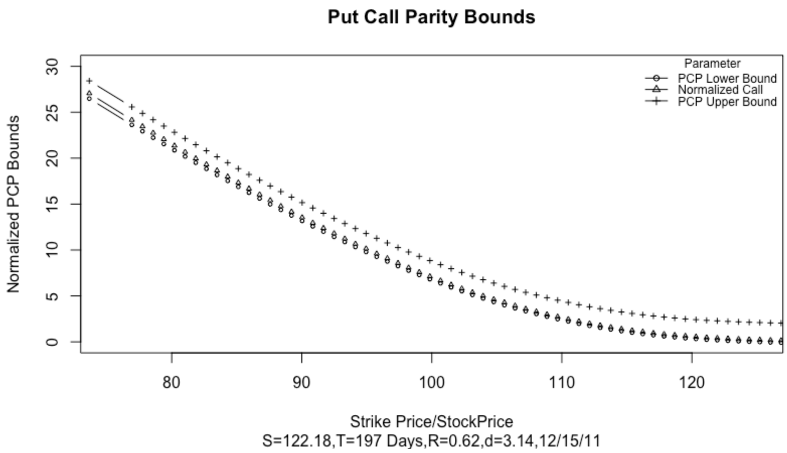
Figure 5.1.5 Put-call parity bounds and normalized call prices
Panel A Before a financial crisis



Panel B During a financial crisis



Panel C After a financial crisis



We now turn to the quantitative details that support the R code generating these results.

Quantitative finance materials

This section presents in great detail the mechanics of arbitrage with respect to option boundaries and option parities. The materials conclude with an introduction to normalization as well as addressing continuous dividend yields. We initially assume discrete dividend payments.

Option Boundary Conditions

We cover numerous option boundary conditions, including both calls and puts. The key insight is boundaries are inequalities that if violated results in potential arbitrage profit opportunities. We focus first on obvious elementary boundaries for the purpose of clarifying the arbitrage process.

Option zero boundary

Although intuitively obvious, we explore the option zero boundary condition for the purpose of understanding static arbitrage activity. The zero boundary conditions states that the minimum value of an option is zero or

$$c_t \geq 0; C_t \geq 0; p_t \geq 0; P_t \geq 0. \tag{5.1.1}$$

We use lower case symbols to denote European-style options (c_t and p_t) and upper case symbols to denote American-style options (C_t and P_t). American-style option give the buyer (a.k.a. holder or long) the ability to exercise the option at any time before or at the expiration date. European-style option give the buyer the ability to exercise the option only at the expiration date.

We focus on European-style call options, but the approach is the same for any other option. Specifically, the approach taken is known as proof by contradiction. That is, we assume the opposite condition and then show that it does not hold. Because the opposite does not hold, the original claim is validated. Specifically, if we assume the opposite condition, we have the call option is less than zero or

$$c_t < 0. \tag{5.1.2}$$

For example, the call is trading for $c_t = -\$0.05$. If we move everything to the greater than side, we obviously have

$$-c_t > 0. \tag{5.1.3}$$

The arbitrageur is interested in cash flows. Thus, to generate $-c_t$ cash flow, we would buy the call option. For example, if the call option were trading for –\$0.05, then when we buy this call and we will receive \$0.05 [= –(–0.05)]. Because these are options and not obligations, it is not possible that this option will result in a future liability.

Thus, the original claim must hold because otherwise pure static arbitrage profit is available. We receive money on the trade date *t* with no chance of future liabilities. Hence, both rules of the arbitrageur are satisfied.

With this framework, we sketch the well-known option boundary conditions. The following notation is used:

t – Initial trade date, expressed as a fraction of a year (e.g., $t = 0$).

t_{di} – Dividend payment date for ith dividend, expressed as a fraction of a year (e.g., $t_{di} = 0.15$).

T – Expiration or maturity date, expressed as a fraction of a year (e.g., $T = 0.5$).

S_t – Stock price on the initial trade date, in currency units (e.g., S_t = \$100).

X – Option exercise or strike price, in currency units (e.g., X = \$105).

D_i – Dividend amount paid on t_{di}, in currency units (e.g., D_i = \$1).

r_c – Annualized, continuously compounded risk-free interest rate (e.g., $r_c = 3\%$)

τ – Time to expiration measured in fraction of year ($\tau = T - t$, e.g., $\tau = 0.5$)

$PV()$ – Present value operator (= $e^{-r_c\tau}$, e.g., if $X = 105$, $r_c = 3\%$ and $\tau = 0.5$, then $PV(\mathrm{X}) = 105\ e^{-0.03(0.5)}$ = 103.43675.

$FV()$ – Future value operator (= $e^{r_c\tau}$, e.g., if $X = 105$, $r_c = 3\%$ and $\tau = 0.5$, then $PV(\mathrm{X}) = 105\ e^{0.03(0.5)}$ = 106.58687.

PVD_t – Present value of dividends over life of the option, in currency units (e.g., PVD_t = \$0.95).

PVX_t – Present value of the exercise price over life of the option, in currency units (e.g., PVX = \$95).

Option intrinsic value

The intrinsic value is the amount of cash generated by immediate exercise of an American-style option or zero, whichever is greater. Thus, for call options, the intrinsic value is

$$IV_{c,t} \geq \max\left(0, S_t - X\right). \tag{5.1.4}$$

For put options, the intrinsic value is

$$IV_{p,t} \geq \max\left(0, X - S_t\right). \tag{5.1.5}$$

European-style call lower bound

The lower bound for a European-style call can be expressed as

$$c_t \geq \max\left(0, S_t - PVD_t - PVX\right). \tag{5.1.6}$$

Assuming the opposite, we have

$$c_t < \max\left(0, S_t - PVD_t - PVX\right). \tag{5.1.7}$$

The zero boundary condition will be violated if $S_t - PVD_t - PVX \leq 0$. Thus, we need only consider the case where

$$c_t < S_t - PVD_t - PVX. \tag{5.1.8}$$

Moving terms to the greater than side, we have

$$0 < S_t - \left(PVD_t + PVX\right) - c_t. \tag{5.1.9}$$

Equation (5.1.9) gives us the trading strategy. Basically, we conduct transactions that yield cash flows consistent with the symbols in this equation. Note that anytime an equation involves PV, it implies financing where you will borrow if negative and lend if positive. The following cash flow table yields the results of following this implied strategy. For ease of exposition, we assume only one future dividend payment during the life of the option. Table 5.1.1 illustrates the cash flows from the implied trading strategy.

Table 5.1.1 European-Style Call Lower Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \le X$	$S_T > X$
Short sell stock	$+S_t$	$-D_i$	$-S_T$	$-S_T$
Lend	$-(PVD_t + PVX)$	$+D_i$	$+X$	$+X$
Buy call	$-c_t$		0	$+(S_T - X)$
Net	$+S_t - (PVD_t + PVX) - c_t$ + by assumption	0	$X - S_T$ + by column	0

Thus, the arbitrageur receives money today and there is no chance that she will have to pay money in the future. The two rules of the arbitrageur are maintained and money is received today. Clearly, this set of transactions result in an attractive outcome that will be actively pursued in free markets. We would expect that the stock price would fall in response to the short selling and the call price will rise in response to the call buying. We really do not expect that the lending activity will have any impact in the financing market. In practice, these price movements will result in the elimination of the positive cash flows today. After the prices have reverted back within the boundary condition, arbitrageurs will unwind these positions. The net effect is positive money in their account and no position at all. All this activity may occur within minutes.

Therefore, in equilibrium, we expect Equation (5.1.6) to hold.

European-style call upper bound

The upper bound for a European-style call can be expressed as

$$c_t \le S_t - PVD_t. \tag{5.1.10}$$

Assuming the opposite, we have

$$c_t > S_t - PVD_t. \tag{5.1.11}$$

Moving terms to the greater than side, we have

$$c_t - S_t + PVD_t > 0. \tag{5.1.12}$$

Again Equation (5.1.12) gives us the trading strategy. The following cash flow table yields the results of following this implied strategy. For ease of exposition, we assume only one future dividend payment during the life of the option. Table 5.1.2 illustrates the cash flows from the implied trading strategy.

Table 5.1.2 European-Style Call Upper Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \le X$	$S_T > X$
Sell call	$+c_t$		0	$-(S_T - X)$
Buy stock	$-S_t$	$+D_t$	$+S_T$	$+S_T$
Borrow	$+PVD_t$	$-D_t$		
Net	$+ c_t - S_t + PVD_t$ + by assumption	0	$+S_T$ + by column	+X

Thus, the arbitrageur receives money today and there is no chance that he will have to pay money in the future. Note that if the stock price goes to zero, then the third column is zero. Therefore, in equilibrium, we expect Equation (5.1.10) to hold.

American-style call lower bound

The lower bound for an American-style call can be expressed as

$$C_t \ge \max\left(0, S_t - PVD_t - PVX, S_t - X\right). \tag{5.1.13}$$

Assuming the opposite, we have

$$C_t < \max\left(0, S_t - PVD_t - PVX, S_t - X\right). \tag{5.1.14}$$

The zero boundary condition will be violated if $\max\left(S_t - PVD_t - PVX, S_t - X\right) \le 0$. Thus, we need only consider the case where

$$C_t < \max\left(S_t - PVD_t - PVX, S_t - X\right). \tag{5.1.15}$$

First, consider the case where $S_t - PVD_t - PVX < S_t - X$. Thus,

$$C_t < S_t - X. \tag{5.1.16}$$

Moving terms to the greater than side, we have

$$0 < S_t - X - C_t. \tag{5.1.17}$$

In this case, the arbitrageur simply buys the call ($-C_t$), immediately exercises the right to buy the stock at X ($-X$), and sells the stock ($+S_t$). Thus, we have positive cash flow today and no future position at all!

Second, consider the case where $S_t - PVD_t - PVX > S_t - X$. Thus, we have the same condition as with the European-style call or

$$0 < S_t - \left(PVD_t + PVX\right) - C_t. \tag{5.1.18}$$

Thus, in this case, we follow the same strategy as the European-style call lower bound to capture the arbitrage profit. Notice that we are buying the call. Hence, we do not need to be concerned about early exercise. Therefore, in equilibrium, we expect Equation (5.1.13) to hold.

American-style call upper bound

The upper bound for an American-style call can be expressed as

$$C_t \le S_t. \tag{5.1.19}$$

Assuming the opposite, we have

$$C_t > S_t. \tag{5.1.20}$$

Moving terms to the greater than side, we have

$$C_t - S_t > 0. \tag{5.1.21}$$

Again Equation (5.1.21) gives us the trading strategy. The following cash flow table yields the results of following this implied strategy. Again, we assume only one future dividend payment during the life of the option. Notice we must consider the case of early exercise being force on the call writer likely before the dividend payment date. After the dividend date implies the arbitrageur also receives the dividend. Table 5.1.3 illustrates the cash flows from the implied trading strategy.

Table 5.1.3 American-Style Call Upper Bound Cash Flow Table

				Cash Flow at Expiration	
	Trade Date	**Early Exercise**	**Dividend Date**	**$S_T \le X$**	**$S_T > X$**
Sell call	$+C_t$	$-(S_{t'} - X)$		0	$-(S_T - X)$
Buy stock	$-S_t$	$+S_{t'}$	$+D_t$	$+S_T$	$+S_T$
Net	$+C_t - S_t$ + by assumption	+X	$+D_t$	$+S_T$ + by column	+X

Thus, the arbitrageur receives money today and there is no chance that he will have to pay money in the future. Note that if the stock price goes to zero, then the third column is zero. Therefore, in equilibrium, we expect Equation (5.1.19) to hold.

European-style put lower bound

The lower bound for a European-style put can be expressed as

$$p_t \geq \max\left(0, PVX + PVD_t - S_t\right). \tag{5.1.22}$$

Assuming the opposite, we have

$$p_t < \max\left(0, PVX + PVD_t - S_t\right). \tag{5.1.23}$$

The zero boundary condition will be violated if $PVX + PVD_t - S_t \leq 0$. Thus, we need only consider the case where

$$p_t < PVX + PVD_t - S_t. \tag{5.1.24}$$

Moving terms to the greater than side, we have

$$0 < \left(PVD_t + PVX\right) - S_t - p_t. \tag{5.1.25}$$

As before, we create a cash flow table based on the suggested strategy in the previous equation. Table 5.1.4 illustrates the cash flows from the implied trading strategy.

Table 5.1.4 European-Style Put Lower Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \leq X$	$S_T > X$
Borrow	$+(PVD_t + PVX)$	$-D_i$	$-X$	$-X$
Buy stock	$-S_t$	$+D_i$	$+S_T$	$+S_T$
Buy put	$-p_t$		$+(X - S_T)$	0
Net	$+(PVD_t + PVX) - S_t - p_t$ + by assumption	0	0	$S_T - X$ + by column

Once again, we observe a money machine that should not be present for long in free markets. Therefore, in equilibrium, we expect Equation (5.1.22) to hold.

European-style put upper bound

The upper bound for a European-style put can be expressed as

$$p_t \leq PVX. \tag{5.1.26}$$

Assuming the opposite, we have

$$p_t > PVX. \tag{5.1.27}$$

Moving terms to the greater than side, we have

$$p_t - PVX > 0. \tag{5.1.28}$$

Again following the implied trading strategy, we have the cash flow table given in Table 5.1.5.

Table 5.1.5 European-Style Put Upper Bound Cash Flow Table

		Cash Flow at Expiration	
	Trade Date	$S_T \le X$	$S_T > X$
Sell put	$+p_t$	$-(X - S_T)$	0
Borrow	$-PVX$	$+X$	$+X$
Net	$+ p_t - PVX$ + by assumption	$+S_T$ + by column	+X + by column

Thus, the arbitrageur receives money today and there is no chance that he will have to pay money in the future. Note that if the stock price goes to zero, then the second column is zero. Therefore, in equilibrium, we expect Equation (5.1.26) to hold.

American-style put lower bound

The lower bound for an American-style put can be expressed as

$$P_t \ge \max\left(0, PVX + PVD_t - S_t, X - S_t\right). \tag{5.1.29}$$

Assuming the opposite, we have

$$P_t < \max\left(0, PVX + PVD_t - S_t, X - S_t\right). \tag{5.1.30}$$

The zero boundary condition will be violated if $\max\left(PVX + PVD_t - S_t, X - S_t\right) \le 0$. Thus, we need only consider the case where

$$P_t < \max\left(PVX + PVD_t - S_t, X - S_t\right). \tag{5.1.31}$$

First, consider the case where $PVX + PVD_t - S_t < X - S_t$. Thus,

$$P_t < X - S_t. \tag{5.1.32}$$

Moving terms to the greater than side, we have

$$0 < X - S_t - P_t. \tag{5.1.33}$$

In this case, the arbitrageur simply buys the put ($-P_t$) , immediately exercises the right to sell the stock at X ($+X$), and buys the stock ($-S_t$). Thus, we have positive cash flow today and no future position at all!

Second, consider the case where $PVX + PVD_t - S_t > X - S_t$. Thus, we have the same condition as with the European-style put or

$$PVX + PVD_t - S_t - P_t > 0. \tag{5.1.34}$$

Thus, in this case, we follow the same strategy as the European-style put lower bound to capture the arbitrage profit. Notice that we are buying the put. Hence, we do not need to be concerned about early exercise. Therefore, in equilibrium, we expect Equation (5.1.29) to hold.

American-style put upper bound

The upper bound for an American-style call can be expressed as

$$P_t \le X. \tag{5.1.35}$$

Assuming the opposite, we have

$$P_t > X. \tag{5.1.36}$$

Moving terms to the greater than side, we have

$$P_t - X > 0. \tag{5.1.37}$$

Again Equation (5.1.37) gives us the trading strategy. The following cash flow table yields the results of following this implied strategy. Table 5.1.6 illustrates the cash flows from the implied trading strategy.

Table 5.1.6 American-Style Put Upper Bound Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Early Exercise**	**$S_T \le X$**	**$S_T > X$**
Sell put	$+P_t$	$-(X - S_{t'})$	$-(X - S_T)$	0
Lend	$-X$	$+FV_{t'}(X)$	$+FV_T(X)$	$+FV_T(X)$
Net	$+C_t - S_t$ + by assumption	$+S_{t'}$ + Interest + r > 0	$+S_{t'}$ + Interest + r > 0	$+ FV_T(X)$ + r > 0

Thus, the arbitrageur receives money today and there is no chance that he will have to pay money in the future. Therefore, in equilibrium, we expect Equation (5.1.35) to hold.

For completeness, we identify a few more boundary conditions without proof. Based on the arbitrage approach applied numerous times above, you should be able to demonstrate that these conditions hold.

European-style call difference in strike prices

$$c_t(X_L) - c_t(X_H) \le PV(X_H - X_L). \quad (5.1.38)$$

American-style call difference in strike prices

$$C_t(X_L) - C_t(X_H) \le X_H - X_L. \quad (5.1.39)$$

European-style put difference in strike prices

$$p_t(X_H) - p_t(X_L) \le PV(X_H - X_L). \quad (5.1.40)$$

American-style put difference in strike prices

$$P_t(X_H) - P_t(X_L) \le X_H - X_L. \quad (5.1.41)$$

Relationship to Time to Maturity

Assume $T_2 > T_1$.

$$c_{T_2} \ge c_{T_1}, \quad (5.1.42)$$

$$C_{T_2} \ge C_{T_1}, \quad (5.1.43)$$

$$p_{T_2} \ge\le p_{T_1}, \text{ and} \quad (5.1.44)$$

$$P_{T_2} \ge P_{T_1}. \quad (5.1.45)$$

Relationship with Exercise Prices

Assume $X_L < X_M < X_H$ and $0 < \alpha < 1$ where $\alpha = (X_H - X_M)/(X_H - X_L)$. Based on properties of convexity, we have

$$\alpha c_t(X_L) + (1-\alpha)c_t(X_H) \ge c_t(X_M), \quad (5.1.46)$$

$$\alpha C_t(X_L) + (1-\alpha)C_t(X_H) \ge C_t(X_M), \quad (5.1.47)$$

$$\alpha p_t(X_L) + (1-\alpha)p_t(X_H) \ge p_t(X_M), \text{ and} \quad (5.1.48)$$

$$\alpha P_t(X_L) + (1-\alpha)P_t(X_H) \ge P_t(X_M). \quad (5.1.49)$$

We now turn to put-call parity.

Option Put-Call Parity

Put-call parity establishes the relationship between put and call options with the same maturity and same exercise price.

European-style put-call parity

European-style put-call parity can be thought of as two forms of portfolio insurance. European-style put-call parity can be expressed as

$$S_t - PVD_t + p_t = c_t + PVX \,. \tag{5.1.50}$$

The left-hand side (LHS) of buying stocks and buying puts is known as portfolio insurance because the put option provides a floor on your potential losses. The deduction of the present value of dividends is known as the escrow method. Essentially, you are buying stock without the dividends or the dividends have been placed in escrow.

The right-hand side (RHS) involves buying calls and placing the majority of funds in the bank. Thus, if the stock price falls, you just lose the call premium. If the stock price rises, the call is in-the-money and you have significant upside potential. Intuitively, the LHS and RHS lead to similar outcomes.

The most efficient way to prove this relationship is to move all terms to one side and consider the consequences of trading in such a way as to generate the corresponding cash flows. That is,

$$PVX + PVD_t + c_t - S_t - p_t \overset{?}{=} 0 \,. \tag{5.1.51}$$

Table 5.1.7 illustrates the cash flow consequences.

Table 5.1.7 European-Style Put-Call Parity Cash Flow Table

			Cash Flow at Expiration	
	Trade Date	**Dividend Date**	$S_T \le X$	$S_T > X$
Borrow	$+PVX + PVD_t$	$-D_t$	$-X$	$-X$
Sell call	$+c_t$		0	$-(S_T - X)$
Buy stock	$-S_t$	$+D_t$	$+S_T$	$+S_T$
Buy put	$-p_t$		$X - S_T$	0
Net	$+\mathrm{PVX} + \mathrm{PVD}_t + c_t - S_t - p_t$	0	0	0

Clearly, a portfolio that generates zero cash flow for sure should be worth zero today. Any value today except zero will lead to arbitrage profit. For example, if $PVX + PVD_t + c_t - S_t - p_t < 0$, then move all terms to the LHS to get $0 < -(PVX + PVD_t) - c_t + S_t + p_t$. Thus, borrow, buy call, short sell stock, and sell put will generate zero cash flow in the future and positive cash flow today.

American-style put-call parity

American-style put-call parity results in boundaries. Specifically,

$$P_t + S_t - PVX \ge C_t \ge P_t + S_t - PVD_t - X \,. \tag{5.1.52}$$

To prove these boundaries, we must consider two separate cases:

$$\text{Case 1: } P_t + S_t - PVX \ge C_t \,. \tag{5.1.53}$$

$$\text{Case 2: } C_t \ge P_t + S_t - PVD_t - X \,. \tag{5.1.54}$$

Opposite of Case 1: $P_t + S_t - PVX < C_t$

Moving terms to the greater than side, we have

$$0 < C_t - P_t - S_t + PVX \,. \tag{5.1.55}$$

Table 5.1.8 illustrates the cash flow table showing arbitrage.

Table 5.1.8 American-Style Put-Call Parity Cash Flow Table (Case 1)

				Cash Flow at Expiration	
	Trade Date	**Early Exercise**	**Dividend Date**	$S_T \le X$	$S_T > X$
Sell call	$+C_t$	$-(S_{t'} - X)$		0	$-(S_T - X)$
Buy put	$-P_t$			$X - S_T$	0
Buy stock	$-S_t$	$+S_{t'}$	$+D_t$	$+S_T$	$+S_T$
Borrow	$+PVX$	$-FV_{t'}(PVX)$		$-X$	$-X$
Net	$+C_t - P_t - S_t$ +PVX	Interest	$+D_t$	0	0

Opposite Case 2: $C_t < P_t + S_t - PVD_t - X$

Moving terms to the greater than side, we have

$$0 < P_t + S_t - \left(PVD_t + X\right) - C_t. \tag{5.1.56}$$

Table 5.1.9 illustrates the cash flow table showing arbitrage.

Table 5.1.9 American-Style Put-Call Parity Cash Flow Table (Case 2)

				Cash Flow at Expiration	
	Trade Date	**Early Exercise**	**Dividend Date**	$S_T \le X$	$S_T > X$
Sell put	$+P_t$	$-(X - S_{t'})$		$-(X - S_T)$	0
Short stock	$+S_t$	$-S_{t'}$	$-D_t$	$-S_T$	$-S_T$
Lend	$-(PVD_t + X)$	$+FV_{t'}(PVX)$	$+D_t$	$+FVX$	$+FVX$
Buy call	$-C_t$			0	$-(S_T - X)$
Net	$+P_t + S_t$ $-(PVD_t + X) - C_t$	Interest	0	Interest on X	Interest on X

We now turn to normalizing put-call parity relationships.

Normalized upper and lower bounds for put-call parity

If isolate the call price and divide by the stock price, we have

$$\frac{c_t}{S_t} = 1 - \frac{\left(PVD_t + PVX_t\right)}{S_t} + \frac{p_t}{S_t}. \tag{5.1.57}$$

If we divide the American-style put-call parity by the stock price, we have

$$1 + \frac{P_t}{S_t} - \frac{PVX}{S_t} \ge \frac{C_t}{S_t} \ge 1 + \frac{P_t}{S_t} - \frac{\left(PVD_t + X\right)}{S_t}. \tag{5.1.58}$$

This relationship is important when analyzing diverse underlying instruments. We now turn to reviewing selections of the source code from this module.

Incorporating Continuous Dividend Yield

At times, it is more convenient to assume the underlying instrument pays some sort of continuous dividend yield. For example, the S&P 500 index has over 100 quarterly dividend paying stock. As each dividend goes ex-dividend on different dates, there may be over 400 dividends paid each year. As an approximation, it is easier just to assume a dividend yield rather than discrete dividend payments.

Thus, an underlying instrument trading at \$100 that is assumed to pay a annualized, continuously compounded dividend yield of say 5% would grow to $S_1 = S_0 e^{\delta T} = 100 e^{0.05(1)} =$ 105.1271. Conceptually, if you owned one share of stock priced at \$100. Then, assuming continuous dividend reinvestment, you would own 1.051271 shares after one year. If the stock was still trading for \$100, then your investment would be worth \$105.1271. Alternatively, if you invested in $e^{-\delta T} = e^{-0.05(1)} = 0.951229$ shares of stock, then after one year, you would have $e^{-\delta T} e^{\delta T} = 1$ share of stock.

In the presence of continuous dividends, all the boundary conditions above still hold when the stock price is replaced with the discounted stock price, denote $S' = S_0 e^{-\delta T}$.

Summary

In this module we reviewed option boundary conditions and put-call parties. We demonstrated the key to understanding these relationships is the behavior of arbitrageurs.

We first reviewed how arbitrageurs approach boundaries with the simple example of the zero boundary. Next, we explored a variety of option boundaries for both European-style and American-style options. Finally, we demonstrated why put-call parity must hold in free economic systems. All the arbitrage transaction presented in this chapter were static in nature. That is, the transaction occurs at the initial trade date and could simply be held until the option expires. Often, however, the arbitrage transaction's life is very short, even seconds or minutes but rarely days.

This module concluded with R code exploring boundaries for American-style options on SPY for one date.

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