Module 5.2

Geometric Brownian Motion
Binomial Option Valuation Model
(GBM BOVM)

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Overview

- Explore multiplicative binomial model
  - Geometric Brownian motion (GBM) in limit
  - Recombining
- Incorporate dividends
- Digital options
- European- and American-style options
- Variety of plots generated in R

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Central Finance Concepts

- Geometric Brownian motion (GBM)
- Binomial option valuation model (BOVM)
- One- and two-period models
- GBM coherence conditions
- Role of dividends
- Multiperiod models
- Graphical illustrations

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GBM Binomial Framework

- Multiplicative
- Recombining
- Incorporate dividends (discrete and continuous)
- Address early exercise with American-style options

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Binomial Lattice Frameworks

- One-period: Heuristic, intuitive, simple
- Two-period: Introduce dynamic insights
- Multiperiod: Deployable model
- European-style easier to deploy
- American-style requires addressing potential early exercise

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GBM Coherence Conditions

- Requirements to converge to lognormal distribution
- Seeks to avoid arbitrage within sterile theoretical model
- Deeply useful when exploring alternative models to deploy in practice

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Dividends

- Discrete dividends: Many lattices fail to recombine
- Continuous dividends: Unrealistic with stock options (pay four times per year)
- Escrow method: Bifurcate stock into two components
  - Stock without PV dividends over option life
  - PV dividends

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Multiperiod Models

- Challenge:
  - Probability of a single path tends to zero
  - Number of potential paths tends to infinity
- Solution: Log transformation
- Backward recursion allows easy treatment of early exercise

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Selected Illustrations

- European-style, with and without dividends
  - Sensitivity to stock price and boundaries
  - Time value sensitivity
  - Plain vanilla and digital options
  - Convergence with respect to number of steps
  - Role of dividends
- American-style contrasts

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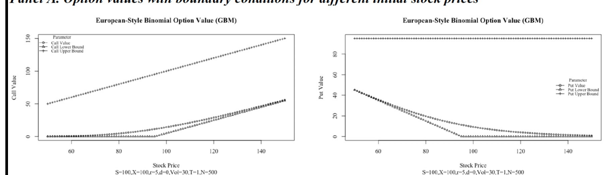
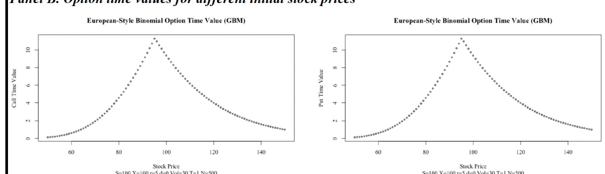
Figure 5.2.1. Selected graphs related to European-style binomial option valuation model–No Dividends

Panel A. Option values with boundary conditions for different initial stock prices



Panel B. Option time values for different initial stock prices



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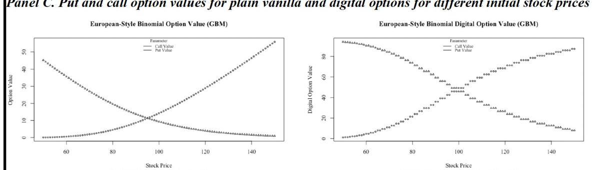
Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices



Panel D. Put and call option values for plain vanilla and digital options for different number of time steps

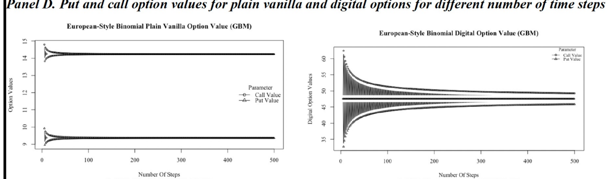


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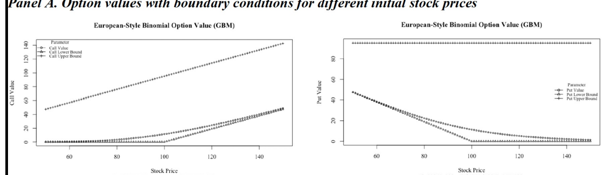
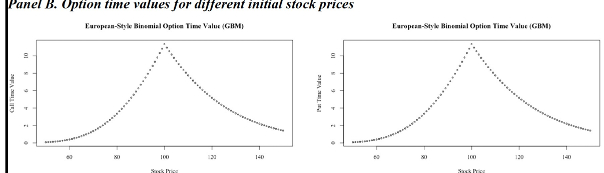
Figure 5.2.2. Selected graphs related to European-style binomial option valuation model with dividends

Panel A. Option values with boundary conditions for different initial stock prices



Panel B. Option time values for different initial stock prices



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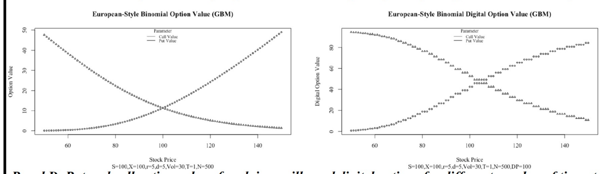
Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices



Panel D. Put and call option values for plain vanilla and digital options for different number of time step

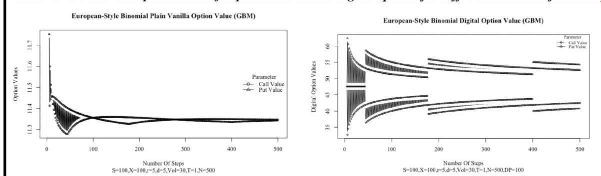
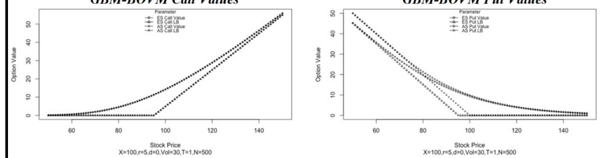
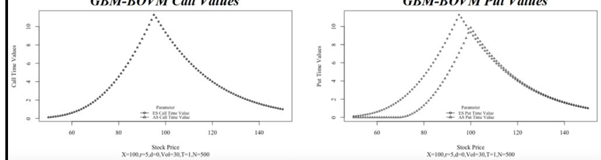


Figure 5.2.3. American-style and European-style binomial option valuation model with no dividends

Panel A. Option values with boundary conditions for different initial stock prices



Panel B. Option time values with boundary conditions for different initial stock prices



Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices

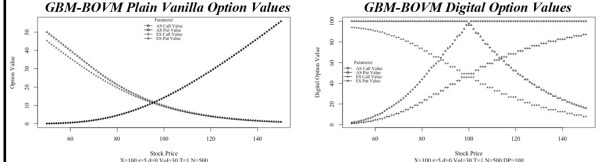
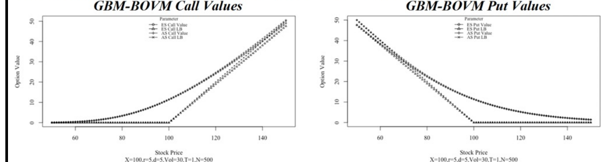
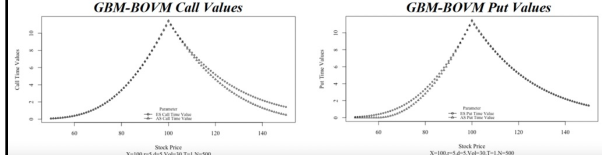


Figure 5.2.4. American-style and European-style binomial option valuation model with dividend yield

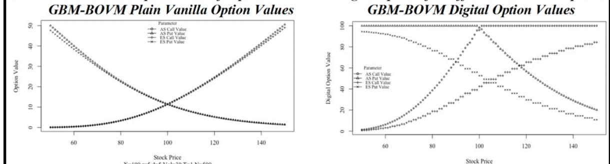
Panel A. Option values with boundary conditions for different initial stock prices



Panel B. Option time values with boundary conditions for different initial stock prices



Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices



Quantitative Finance Materials

- Notation
- One period model
  - Multiplicative
  - Dividends
- Two period model
  - Recombining
- Coherence conditions
- Multiperiod models

Notation

$0, T, \Delta t$	initial trade date, time 0; expiration or maturity date, time T; next time step,
B_0, B_T	bond, value of risk-free investment at time 0 and at time T,
Π_0, Π_T	portfolio, value of some financial instrument portfolio at time 0 and at time T,
S_0, S_T	value of underlying instrument, e.g., stock, at time 0 and at time T,
O_0	option, value of options, either call or put, at time 0,
O_u, O_d	option, value of option at time T if up occurs and if down occurs,
u, d	up, total return of S, if up occurs and if down occurs,
Δ	delta, hedge ratio, units of the financial instrument to enter to hedge option position,
$FV()$	future value based on risk-free interest rate,
$PV()$	present value based on risk-free interest rate,
π	equivalent martingale probability of up move,
r	continuously compounded, annualized, "risk-free" interest rate,
$E_\pi()$	expectation under equivalent martingale probability,
δ	continuously compounded, annualized, dividend yield, and
D_T	known discrete dividend amount paid at time T (ex-dividend the instant *before* the next binomial point in time).

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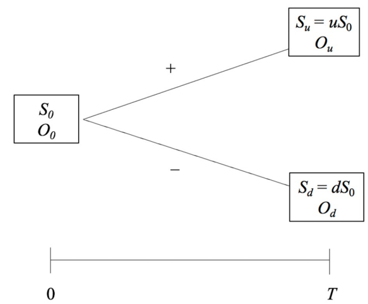
Figure 5.2.5 Multiplicative One Period Binomial Framework



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Single Period Call Option

- Single period
- Binomial process
- Create hedged portfolio
- Derive call value
- Numerical example

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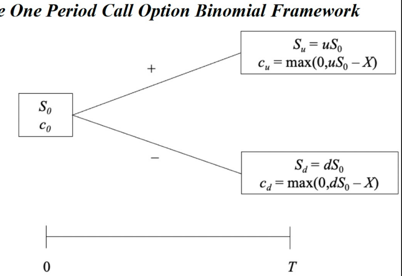
Figure 5.2.6 Multiplicative One Period Call Option Binomial Framework



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Creating the Hedged Portfolio

- Long h shares of stock, short 1 call

$$V_0 = h_c S_0 - c_0$$

- Solve for h yielding identical future CFs

$$V_u = h_c\left(uS_0\right) - \max\left(0, uS_0 - X\right) = h_c\left(uS_0\right) - c_u$$

$$V_d = h_c\left(dS_0\right) - \max\left(0, dS_0 - X\right) = h_c\left(dS_0\right) - c_d$$

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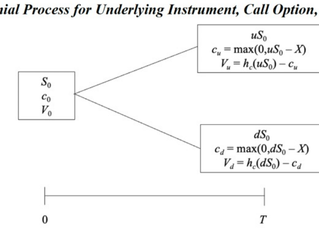
Figure 5.2.7 GBM Binomial Process for Underlying Instrument, Call Option, and Hedge Portfolio



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Hedge Ratio to Valuation

- Optimal hedge ratio ($V_u = V_d$)

$$h_c = \frac{c_u - c_d}{uS_0 - dS_0} = \frac{c_u - c_d}{S_0\left(u-d\right)}$$

- Valuation

$$c_0 = h_c S_0 - B_c \qquad c_0 = PV\left[E\left(c_T\right)\right] = \frac{\pi c_u + \left(1-\pi\right)c_d}{1+r}$$

$$B_{0,c} = \frac{h_c\left(uS_0\right) - c_u}{1+r} \qquad \pi = \frac{1+r-d}{u-d}$$

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Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=1.25, d=0.8
- Intermediate calculations
  - $S_u = uS = 1.25(99) = 123.75$
  - $S_d = dS = 0.8(99) = 79.2$
  - $c_u = \max(0, 123.75 - 100) = 23.75$
  - $c_d = \max(0, 79.2 - 100) = 0$
  - $h_c = (c_u - c_d)/[S_0(u - d)]$
    $= (23.75 - 0)/(123.75 - 79.2)$
    $= 23.75/44.55 = 0.5331$

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Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=1.25, d=0.8
- No arbitrage model solution

$$c_0 = h_c S_0 - \frac{h_c\left(uS\ \right) - c_u}{1+r}$$
$$= 0.5331\left(99\right) - \frac{0.5331\left(123.75\right) - 23.75}{1+0.02}$$
$$= 52.7769 - 41.3933 = 11.38$$

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Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=1.25, d=0.8
- Equivalent martingale solution

$$\pi = \frac{1+r-d}{u-d}$$
$$= \frac{1+0.02-0.8}{1.25-0.8} = \frac{0.22}{0.45} = 0.488889$$
$$c_0 = \frac{\pi c_u + \left(1-\pi\right)c_d}{1+r}$$
$$= \frac{0.4889\left(23.75\right) + \left(1-0.4889\right)0}{1+0.02} = 11.38$$

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Single Period Put Option

- Single period
- Binomial process
- Create hedged portfolio
- Derive put value
- Numerical example

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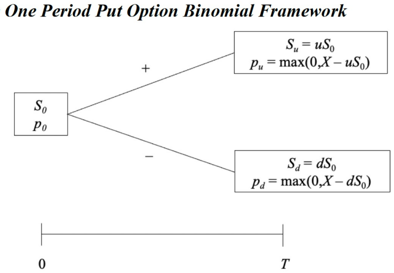
Figure 5.2.8 Multiplicative One Period Put Option Binomial Framework



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Creating the Hedged Portfolio

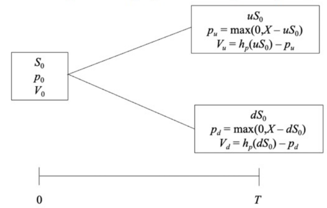
- Buy h shares of stock, buy 1 put

$$V_0 = h_p S_0 + p_0$$

- Solve for h yielding identical future CFs

$$V_u = h_p\left(uS_0\right) + \max\left(0, X - uS_0\right) = h_p\left(uS_0\right) + p_u$$
$$V_d = h_p\left(dS_0\right) + \max\left(0, X - dS_0\right) = h_p\left(dS_0\right) + p_d$$

Figure 5.2.9 GBM Binomial Process for Underlying Instrument, Put Option, and Hedge Portfolio



Hedge Ratio to Valuation

- Optimal hedge ratio ($V_u = V_d$)

$$h_p = \frac{p_d - p_u}{uS_0 - dS_0} = \frac{p_d - p_u}{S_0\left(u - d\right)}$$

- Valuation

$$p_0 = B_{0,p} - h_p S_0 \qquad p_0 = PV\left[E\left(p_T\right)\right] = \frac{\pi p_u + \left(1 - \pi\right) p_d}{1 + r}$$

$$B_{0,p} = \frac{h_p\left(dS_0\right) + p_d}{1 + r} \qquad \pi = \frac{1 + r - d}{u - d}$$

Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=1.25, d=0.8
- Intermediate calculations
  - $S_u = uS = 1.25(99) = 123.75$
  - $S_d = dS = 0.8(99) = 79.2$
  - $p_u = \max(0, 100 – 123.75) = 0$
  - $p_d = \max(0, 100 – 79.2) = 20.8$
  - $h_p = (p_d – p_u)/[S_0(u – d)]$
    $= (20.8 – 0)/(123.75 – 79.2)$
    $= 20.8/44.55 = 0.4669$

Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=1.25, d=0.8
- No arbitrage model solution

$$p_0 = \frac{h_p\left(dS_0\right) + p_d}{1 + r} - h_p S_0$$
$$= \frac{0.4669\left(79.2\right) + 20.8}{1 + 0.02} - 0.4669\left(99\right)$$
$$= 56.6460 - 46.2231 = 10.42$$

Single Period BOVM Example

- S=99, X=100, r=2%, T=1, u=1.25, d=0.8
- Equivalent martingale solution

$$\pi = \frac{1 + r - d}{u - d}$$
$$= \frac{1 + 0.02 - 0.8}{1.25 - 0.8} = \frac{0.22}{0.45} = 0.488889$$
$$p_0 = \frac{\pi p_u + \left(1 - \pi\right) p_d}{1 + r}$$
$$= \frac{0.4889\left(0\right) + \left(1 - 0.4889\right) 20.8}{1 + 0.02} = 10.42$$

Two Period BOVM Model

- Multiplicative
- Recombining
- Terminal values

$$c_{u^2} = \max\left(0, u^2 S_0 - X\right) \qquad p_{u^2} = \max\left(0, X - u^2 S_0\right)$$
$$c_{ud} = \max\left(0, udS_0 - X\right) \text{ and } p_{ud} = \max\left(0, X - udS_0\right)$$
$$c_{d^2} = \max\left(0, d^2 S_0 - X\right) \qquad p_{d^2} = \max\left(0, X - d^2 S_0\right)$$

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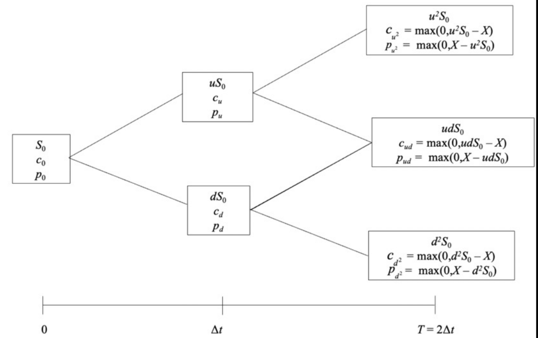
S=99, X=100, r=2%, T=1, u=1.25, d=0.8

Figure 5.2.10 Two Period European-Style Binomial Model



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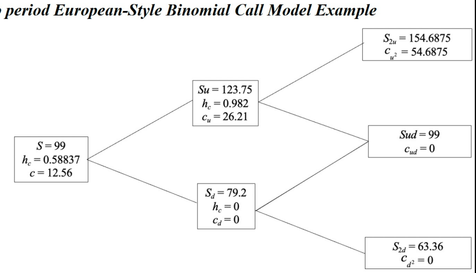
S=99, X=100, r=2%, T=1, u=1.25, d=0.8

Figure 5.2.11 Two period European-Style Binomial Call Model Example



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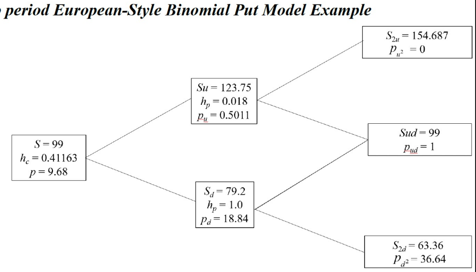
S=99, X=100, r=2%, T=1, u=1.25, d=0.8

Figure 5.2.12 Two period European-Style Binomial Put Model Example



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American-Style (AS) Options

- Early exercise potential must be incorporated
- Method of backward induction
  - Start at terminal value
  - Reason backward in time
- Goal is to establish sequence of optimal actions

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AS and Dividends

- Methods for handling dividends
  - Yield method: Constant rate based on S
  - Escrow method: PVD placed in escrow
- Escrow method
  - Model stock less PVD
  - Assess early exercise decision at each node

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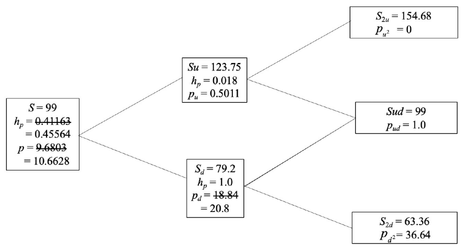
S=99, X=100, r=2%, T=1, u=1.25, d=0.8

Figure 5.2.13 Two Period American-Style Binomial Put Model Example



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GBM Coherence Conditions

- No arbitrage boundary condition

$$0 < d < e^{r\Delta t} < u$$

- Probability condition

$$0 << \pi << 1$$

- No arbitrage condition

$$\pi = \frac{e^{r\Delta t} - d}{u - d}$$

- Variance condition

$$Var_\pi\left[\ln\left(\frac{S_{\Delta t}}{S_0}\right)\right] = \left[\ln\left(\frac{u}{d}\right)\right]^2 \pi(1-\pi) = \sigma^2 \Delta t$$

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Table 5.2.14. Relationship between u, d, and π

Probability	u	d	Prob Check
0	#DIV/0!	#DIV/0!	#DIV/0!
0.00000001	#NUM!	#NUM!	#NUM!
0.0000001	#NUM!	#NUM!	#NUM!
0.000001	1051271.096	5.411E-125	0.000001
0.00001	105127.1096	6.6178E-37	0.00001
0.0001	10512.71095	9.8227E-10	0.0001
0.001	977.554438	0.07379045	0.001
0.01	17.95514001	0.88052495	0.01
0.1	2.438626379	0.89712051	0.1
0.2	1.819144125	0.85930284	0.2
0.3	1.58387078	0.82301409	0.3
0.4	1.449553247	0.78574966	0.4
0.5	1.357519626	0.74502257	0.5
0.6	1.287020681	0.69764672	0.6
0.7	1.228283162	0.63824294	0.7
0.8	1.175296215	0.55517062	0.8
0.9	1.122208183	0.41283732	0.9
0.99	1.061364243	0.05204959	0.99
0.999	1.05232334	7.9434E-05	0.999
0.9999	1.051376234	9.8236E-14	0.9999
0.99999	1.051281609	6.6179E-42	0.99999
0.999999	1.051272148	5.411E-131	0.999999
0.9999999	#NUM!	#NUM!	#NUM!
0.99999999	#NUM!	#NUM!	#NUM!
1	#DIV/0!	#DIV/0!	#DIV/0!

Note: #DIV/0! denotes division by zero and #NUM! denotes here a number that is too small to be represented in this spreadsheet.

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Generic Options

- Indicator function:

$$\iota = \iota_U = \begin{cases} +1 \text{ if call option } (\mathrm{O}_t = c_t) \\ -1 \text{ if put option } (\mathrm{O}_t = p_t) \end{cases}$$

- Terminal payoffs:
  - Calls

$$c_T \geq max(0, S_T - X)$$

  - Puts

$$p_T \geq max(0, X - S_T)$$

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u and d Conditions

- Condition for u:

$$u = \frac{e^{r\Delta t + \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}$$

- Condition for d:

$$d = \frac{e^{r\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}$$

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Multiperiod BOVM Definitions

$$A \equiv \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}} \qquad Den \equiv \pi e^{A} + (1-\pi)$$

$$\inf\left\{\text{int } j : u^j d^{n-j} S_0 > X\right\} > a = \frac{-\ln\left(\frac{S}{X}\right) - rT + n\ln(Den)}{A}$$

$$\pi_1 = \frac{\pi e^{A}}{Den} \qquad \pi_2 = \pi = \frac{e^{r\Delta t} - d}{u - d}$$

$$O_0 = PV_r\left[\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j(1-\pi)^{n-j}\max\left(0, \iota_U u^j d^{n-j} S_0 - \iota_U X\right)\right]$$

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Multiperiod BOVM

- Generic version:

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U S_0 Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U}$$

- Binomial summations:

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j},$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j},$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j},$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j},$$

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Log Transformation

- Binomial probabilities
  - Number of paths explodes
  - Probability of single path tends to zero

$$\Pr(j) = \left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}$$

- Log transformation

$$\ln\left[\Pr(j)\right] = \sum_{k=j+1}^{n}\ln(k) - \sum_{k=1}^{n-j}\ln(k) + j\ln(\pi) + (n-j)\ln(1-\pi)$$

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American-Style Options

- Evaluate at each node

$$O_{i,j} = \max\left[O_{i,j}^B, O_{i,j}^X, O_{i,j}^L\right]$$

  - Binomial model value

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + (1-\pi)O_{i+1,j}\right]$$

  - Early exercise value

$$O_{i,j}^X = \max\left[0, \iota_U\left(S_{i,j} - X\right)\right]$$

  - Lower boundary value

$$O_{i,j}^L = \max\left\{0, \iota_U\left[S_{i,j} - PV_{r,i,n-i}(X)\right]\right\}$$

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Digital (Binary) Options

- Digital payout based on terminal moneyness
- Digital BOVM
  - Cash-or-nothing: Fixed cash amount
  - Asset-or-nothing: Fixed amount of asset
- Indicator function
  - I = 1 if condition is true
  - I = 0 if condition is false

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Digital Options

- Cash-or-nothing

$$c_{CoN} = e^{-rT} DP \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} I_{u^j d^{n-j} S_0 > X}$$

$$p_{CoN} = e^{-rT} DP \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} I_{u^j d^{n-j} S_0 < X}$$

- Asset-or-nothing

$$c_{AoN} = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} u^j d^{n-j} S_0 I_{u^j d^{n-j} S_0 > X} = c + c_{CoN}\left(DP = X\right)$$

$$p_{AoN} = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} u^j d^{n-j} S_0 I_{u^j d^{n-j} S_0 < X} = p + p_{CoN}\left(DP = X\right)$$

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Summary

- Explored multiplicative binomial model
- Addressed dividends
- Digital options
- European- and American-style options
- Variety of plots generated in R

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Appendices

- Appendix A: GBM BOVM with and without dividends derivation
- Appendix B: One period arbitrage examples
- Appendix C: Dividends

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Dividends (Appendix C)

- Continuously compounded dividend yield, δ
  - $V_u = ue^{\delta\Delta t}S_0$ and $V_d = de^{\delta\Delta t}S_0$
  - Future value of the dividend payment is $D_u = u(e^{\delta\Delta t} - 1)S_0$ and $D_d = d(e^{\delta\Delta t} - 1)S_0$
  - Note that $D_u \neq D_d$
- Discrete dollar terms, D
  - $V_u = uS_0 + D_u$ and $V_d = dS_0 + D_d$
  - Note that $D_u = D_d$ *is possible*

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Dividend Adjusted Parameters

- Conditions: $0 < d < e^{(r-\delta)\Delta t} < u$

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u - d}$$

$$u = \frac{e^{(r-\delta)\Delta t + \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)} \qquad d = \frac{e^{(r-\delta)\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}$$

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Adjusted Single Period BOVM

- General condition:

$$O_0 = PV\left[E_\pi\left(O_{\Delta t}\right)\right]$$

- No arbitrage model:

$$O_0 = \Delta S_0 - PV\left(\Delta e^{\delta\Delta t} uS_0 - O_u\right)$$

- Expectations model

$$O_0 = PV\left[\frac{e^{(r-\delta)\Delta t} - d}{u - d}O_u + \frac{u - e^{(r-\delta)\Delta t}}{u - d}O_d\right] = PV\left[\pi O_u + (1-\pi)O_d\right]$$

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Multiperiod BOVM

- Generic version:

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U Se^{-\delta T} Bin_{1,\iota_U} - \iota_U Xe^{-rT} Bin_{2,\iota_U}$$

- Alternate version:

$$O_0 = PV_r\left[\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j(1-\pi)^{n-j}\max\left(0, \iota_U u^j d^{n-j} S_0 - \iota_U X\right)\right]$$

$$\pi = \frac{e^{(r-\delta)T} - d}{u - d}$$

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Figure 5.2C.1. Multiplicative Two Period Binomial Framework Without Dividends

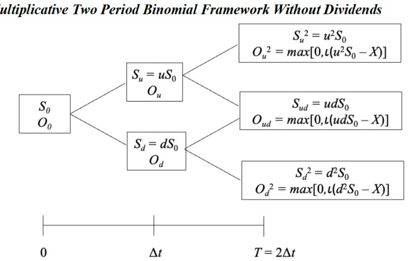

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Figure 5.2C.2. Multiplicative Two Period Binomial Framework with Discrete Dividend

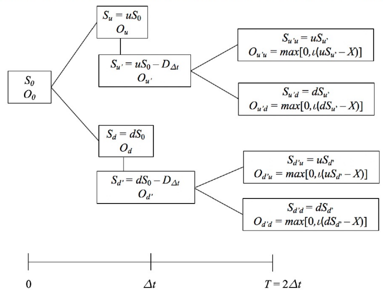

Figure 5.2C.3. Multiplicative Two Period Binomial Framework With Discrete Dividend (Escrow Method)