# Module 5.2: Geometric Brownian Motion-Based Binomial Models

**Learning objectives**

- Computing European-style and American-style call and put option values using a coherent geometric Brownian motion binomial option valuation approach
- Contrast the value of plain vanilla call and put option values with cash-or-nothing digital option values
- Using the log transformation to compute binomial probabilities
- Introduce the idea of self-financing, dynamic replication of option values

## Executive summary

A lattice approach to valuing various options consistent with a lognormal terminal distribution is presented in this module. In our context here, a lattice refers to how some underlying instrument's value may change discretely over the next time step. The valuation approach is based on dynamic arbitrage. Dynamic arbitrage is based on the capacity to continuously rebalance a custom-designed portfolio.

In this module, we present the traditional binomial valuation model we refer to as the geometric Brownian motion binomial option valuation model or GBM-BOVM. In the next module, we introduce an unorthodox binomial valuation model consistent with a normal terminal distribution that we refer to as the arithmetic Brownian motion binomial option valuation model or ABM-BOVM. Like tools in a toolbox for the quantitative analyst, the varied challenges analysts face will warrant the unique tool selected. Unorthodox tools often prove vital with particularly challenging tasks.

## Central finance concepts

We now move from option boundaries and parities covered in the last module to an option valuation model. Specifically, we move from the market option price being contained within a region established by boundaries to an exact option value based on a valuation model.

Our strategy here is to apply a lattice-based approach. For most people, lattice means strips of material in a crisscross pattern. Again, in finance, a lattice refers to how some underlying instrument's value may change discretely over the next time step. We present the GBM-BOVM in this module and illustrate valuing plain vanilla options as well as digital options. We also apply this model to both European-style and American-style calls and puts. As a contrast, we present the ABM-BOVM in the next module.

The binomial option framework presented in this module is designed to converge to a lognormal distribution in the limit to be consistent with the BSMOVM. This binomial framework has several objectives that will be further developed in this module, including

1. Multiplicative,
2. Recombining,
3. Incorporate dividends (discrete and continuous), and
4. Address early exercise with American-style options.

Multiplicative and recombining is incorporated with $u$ and $d$ parameters at each node. As we will see, without careful handling the discrete dividend payments result in a non-recombining tree—a violation of objective number 2 above.

**GBM one period binomial lattice framework**

Many of the more complex concepts can be easily understood within the context of a simple one period model. The technical details of the single period model presented below is not realistic, but it lays a solid framework for understanding the both the multiperiod model presented here as well as continuous models presented in Modules 5.4, 5.5, and 5.7.

The key inputs defined later are the total return if up occurs and total return if down occurs. When introducing coherence conditions, strict structure will be provided for these values. For the one period lattice, these values are simply assumed.

**GBM European-style option two period model**

The single period model can be extended to multiple periods and thereby accommodate options with longer lives or smaller time steps. Note that by design, described in detail below, the lattice is recombining based on a multiplicative process. Thus, in a single period model there are two potential future outcomes, whereas in the two period model there are three potential outcomes. There will be two different paths where one arrives at the middle node after two periods.

Two key features of the two period binomial model are the recombining nature of the tree, and the growth of the underlying instrument is multiplicative. The tree is recombining because the stock price is assumed to grow by multiplication. The multiplicative approach presented in this module facilitates the convergence of the stock price to the lognormal distribution.

**GBM American-style option two period model**

With American-style options, a backward recursion approach is taken within the lattice. At each node where time remains on the option, three conditions are appraised where the highest value is selected. First, the value of the option assuming the option is not early exercised is computed. Second, the cash value of immediately exercising is computed. Finally, the lower boundary of the option is computed. The maximum of these three values is placed in the lattice and the evaluation continues.

Thus, American-style options will not trade for less than their European-style counterparts. Given the vast number of different binomial frameworks possible, we explore guidelines known as coherence conditions.

**GBM coherence conditions**

The GBM-BOVM presented technically below is designed to converge to a lognormal distribution in the limit to be consistent with the Black–Scholes–Merton option valuation model (BSMOVM). The lattice will be built multiplicatively. That is, the value of the underlying at some future date is found by multiplying certain parameters. In the next module, we introduce another binomial model that converges to the normal distribution. In that case, the lattice will be built additively. That is, the value of the underlying at some future date is found by adding certain parameters.

Vital to all lattice frameworks is the need to have the lattice recombine over maturity time. In the binomial cases, the goal is to have the number of futures states grow linearly. In the binomial framework, the number of future states increase by one with each additional point in time in the lattice.

There have been numerous lattice-based option valuation models posited over the past several decades. Many of these models are not internally coherent, often admitting simple arbitrage opportunities even within the sterile theoretical environment. Seeking to thwart that potential, a set of four coherence conditions have been offered. If all four of these coherence conditions are satisfied, then the lattice model is at least internally coherent.

Although presented in detail later, we briefly sketch the coherence conditions here. First, the no arbitrage boundary condition requires that the total return from investing in the risk-free interest rate be greater than the total return on the risky instrument if the down state occurs as well as total return from investing in the risk-free interest rate be less than the total return on the risky instrument if the up state occurs. Second, there is a technical condition on the probability of an up move that it cannot be too close to either zero or one. Third, there is a mathematical relationship between the assumed probability of an up move and the values of the up and down parameters. Finally, there is a technical requirement that forces the local variance within the lattice to exactly equal to the assumed variance of the lognormal distribution in the limit.

Though highly technical, the coherence conditions are deeply useful when exploring alternative models for actual implementation.

**Dividends**

Dealing with discrete cash flows paid to the underlying instrument is a significant challenge especially for lattice models that converge to the lognormal distribution in the limit. These lattice trees often fail to recombine posing insurmountable implementation challenges.

Further, option valuation needs to be able to address known future cash flows related to the underlying instrument. In the case of stocks, both discrete and continuous dividends are discussed below.

The escrow method introduced below for handling dividend payments simply divides the current stock price into the present value of the known discrete dividend payments and the remaining stock value including any potential dividend yield component. The term escrow suggests that the present value of known discrete dividends is placed in a bankruptcy-proof trust that will be paid for sure and the remaining stock value is stochastic. Companies do not actually do this, but it is a conceptual framework for dividends.

**GBM European-style multiperiod option model**

Several different multiperiod binomial lattice frameworks are covered here and in the next module. We review both the European-style and American-style models both with and without various forms of dividends.

One key insight is known as the log transformation. With a recombining binomial lattice, the number of states grows linearly with the number of time steps. Thus, the number of different sample paths within the lattice is exponentially expanding. The likelihood of one particular sample path is quickly declining toward zero.

The valuation model basically requires multiplying two numbers together where one is exploding toward positive infinity and the other is imploding to zero, introducing significant machine error as well as the inability to even perform calculations. The log transformation resolves this tricky problem rendering the binomial lattice model highly useful in modern applications.

**GBM American-style multiperiod option model**

With American-style options, a backward recursion approach is taken within the lattice. At each node where time remains on the option, three conditions are appraised where the highest value is selected. First, the value of the option assuming the option is not early exercised is computed. Second, the cash value of immediately exercising is computed. Finally, the lower boundary of the option is computed. Again, the maximum of these three values is placed in the lattice and the evaluation continues.

Backward recursion is not necessary with the European-style GBM-BOVM although it could be used. There are numerically faster ways to solve for European-style option values. When we get to ABM-BOVM both European-style and American-style option values are found using backward recursion due to complexities related to arithmetic Brownian motion's additive discrete time framework.

We now review selected graphical results based on the quantitative models developed below along with the corresponding R code.

**GBM-BOVM European-style results**

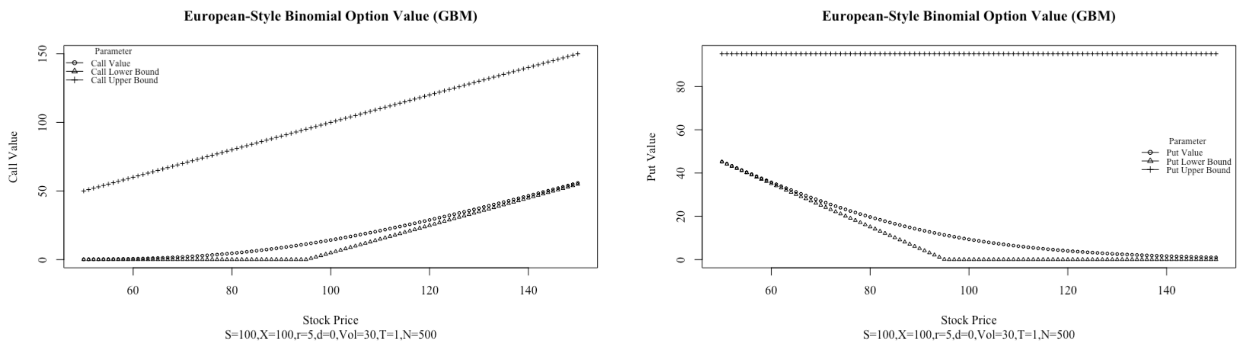
The time value plots highlight the lognormal distribution's positive skewness. Recall in Module 5.1, we introduced lower and upper boundaries. Figure 5.2.1 is based on an assumed stock price of 100, exercise price of 100, risk-free interest rate of 5% (continuously compounded), volatility of 30% (annualized, continuously compounded rates of return), dividend yield of 0%, and time to maturity of 1 year. Panel A illustrates the convergence to the lower boundary as the stock price declines (zero for call and the present value of the exercise price less the stock price for the put) as well as the convergence to the lower boundary as the stock price increases (stock price less the present value of the exercise price for the call and zero for the put).

Panel B draws attention to just the time value. Upon careful inspection, we see the time values are identical and positively skewed. The mode (peak) is technically at the present value of the exercise price. Recall based on actual option data; we documented significant observed negative skewness.

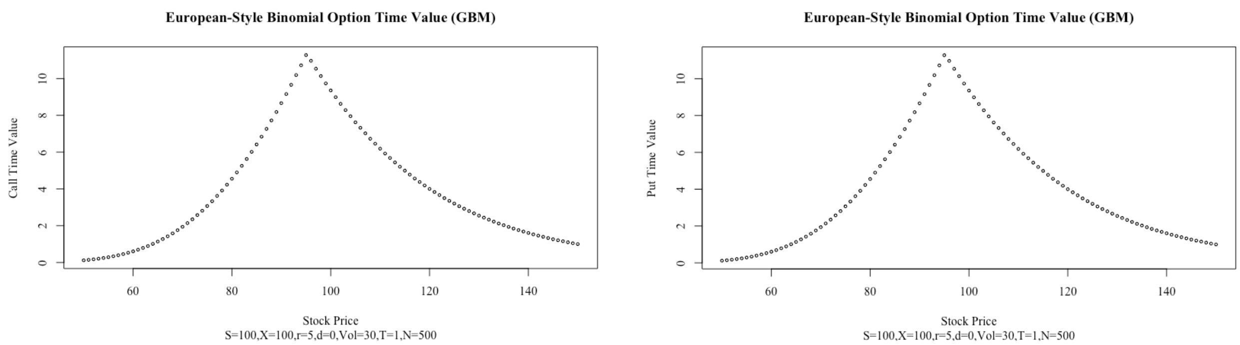
Panel C illustrates calls and put together for both plain vanilla options and digital options. The digital options are cash-or-nothing and pays the exercise price if the underlying is in-the-money at expiration. The stepped mapping with digitals reflects the use of 500 time steps introducing minor discontinuities.

Panel D shows the convergence properties for both plain vanilla and digital options as we increase the number of time steps.

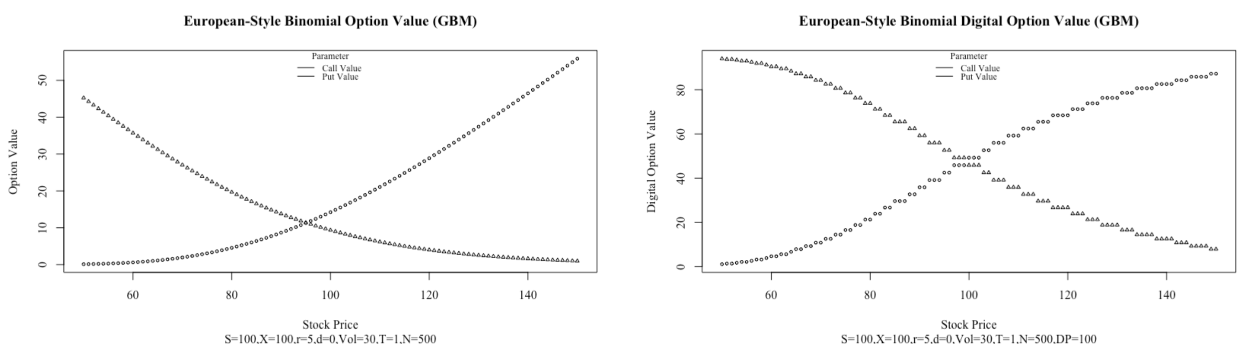
***Figure 5.2.1. Selected graphs related to European-style binomial option valuation model–No Dividends***
***Panel A. Option values with boundary conditions for different initial stock prices***



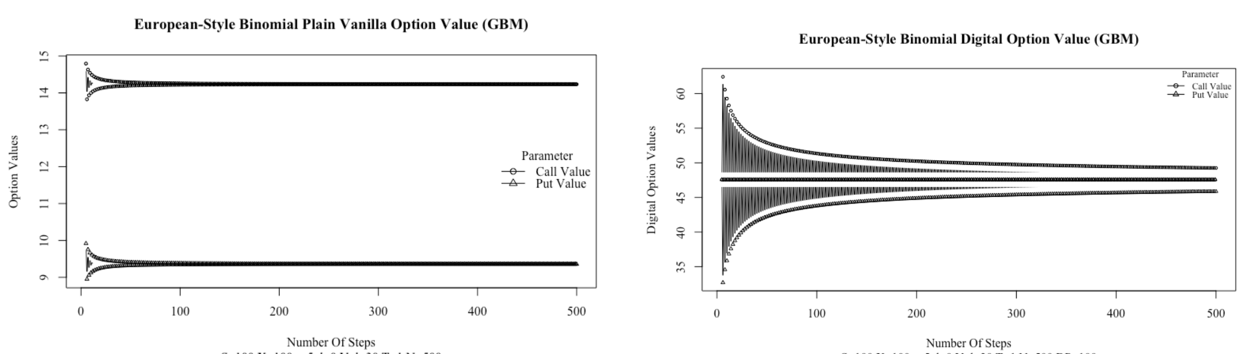

***Panel B. Option time values for different initial stock prices***




***Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices***



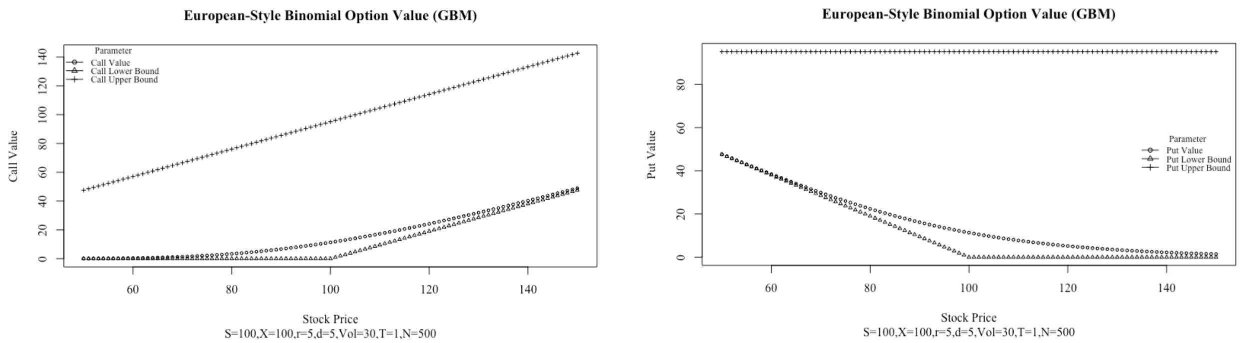

***Panel D. Put and call option values for plain vanilla and digital options for different number of time steps***



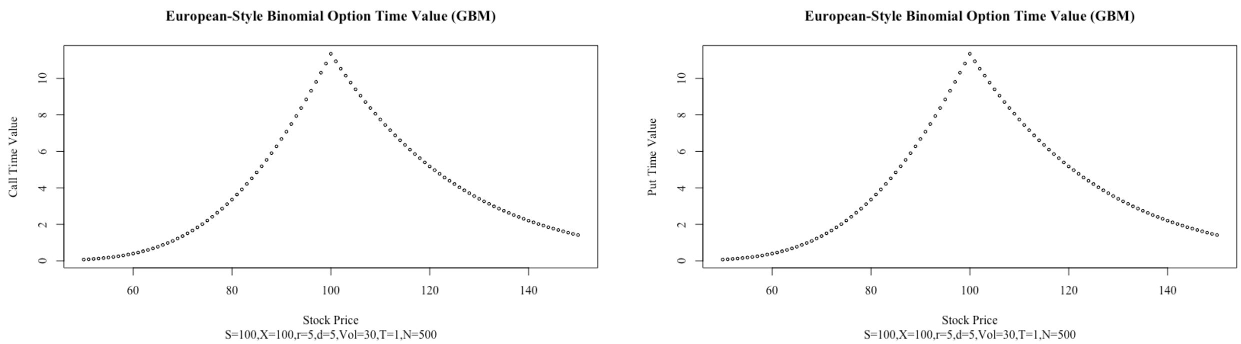

Note that the R code allows for dividends to be incorporated based on a constant dividend yield—a common approach to handling interim cash flows of an underlying instrument. Figure 5.2.2 was computed based on the same parameters above, but the dividend yield equals the interest rate of 5% (annualized, continuously compounded).

Comparing Panel A here with the prior no dividend case we see the lower boundary now changes at the current exercise price rather than at the present value of the exercise price. This effect is due to arbitrageurs being able to purchase less stocks due to dividend receipts to hedge their position. Panel B is also like the no dividend case, except the mode is at the current exercise price. Panel C is also similar, but the intersection point for both the plain vanilla and digital options is at the current exercise price. Finally, Panel D shows eventual convergence, but it is much less stable in both cases.

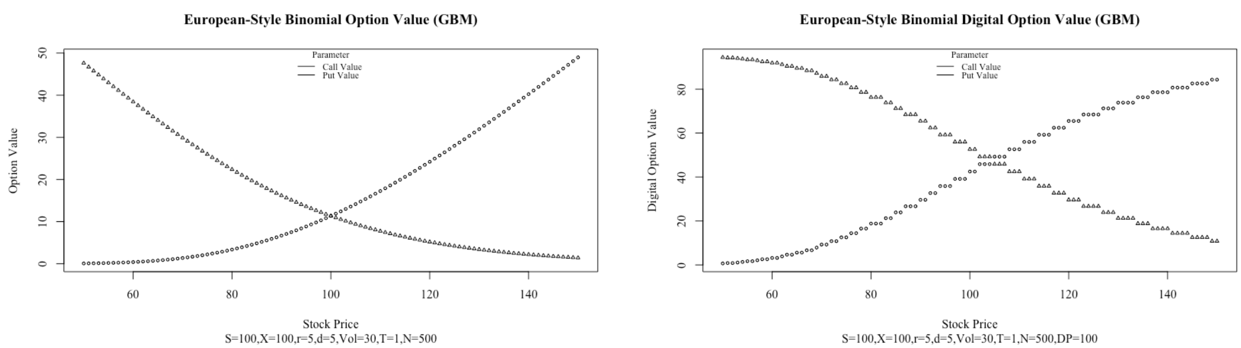
***Figure 5.2.2. Selected graphs related to European-style binomial option valuation model with dividends***
***Panel A. Option values with boundary conditions for different initial stock prices***




***Panel B. Option time values for different initial stock prices***




***Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices***




***Panel D. Put and call option values for plain vanilla and digital options for different number of time steps***

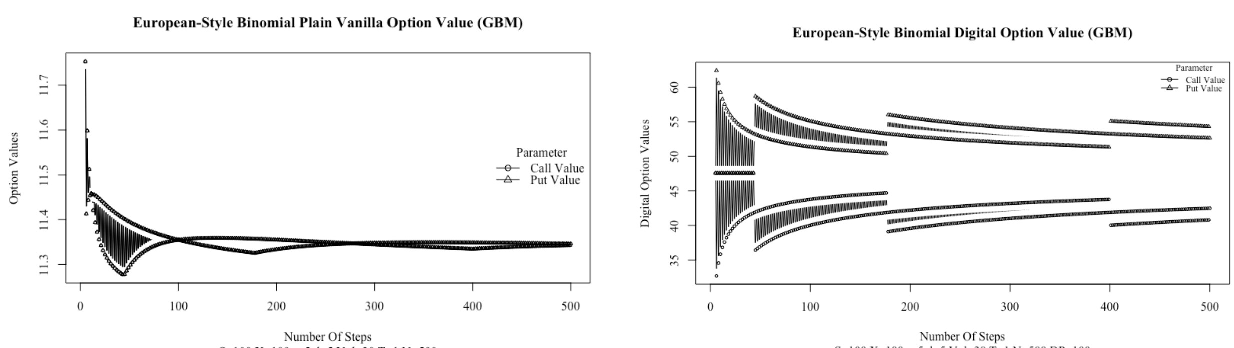



Figure 5.2.3 present values derived from both the European-style and American-style option valuation model with no dividends. Note that without dividends, we see from Panel A that call options are never exercised early; hence, the American-style (AS) call value is identical to the European-style (ES) call value.

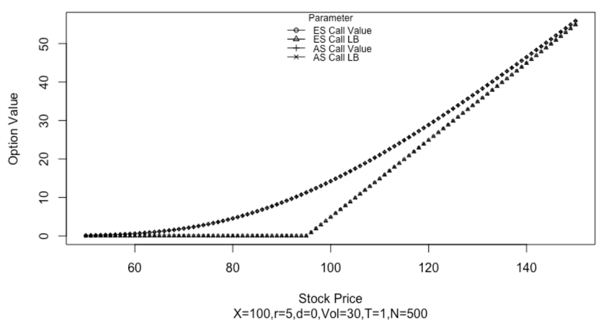
The lower boundary conditions for AS calls and ES calls are also the same. The same cannot be observed for puts. Due to arbitrage forces, AS put values are worth more than ES put values, particularly noticeable when the put options are in-the-money. Further, notice that put values converge to the the appropriate lower boundary conditions.

Panel B provides the same format, except focused solely on option time value. Again, we see there is no difference between AS and ES call option time values whereas there is significant difference between AS and ES put option time values. Although, ES put values are lower, due to the lower boundary effect, the ES put time values are higher than AS put time values. Clearly, the early exercise feature has a material effect on non-dividend paying put options based on the GBM-BOVM.

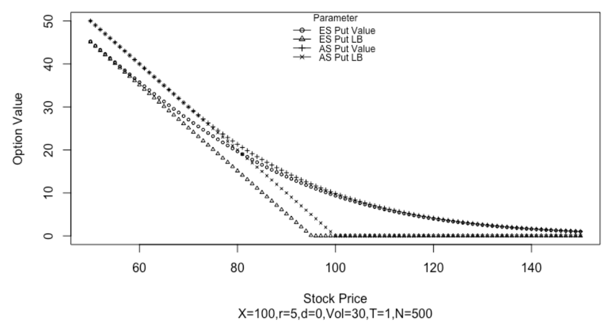
Panel C combines AS and ES as well as puts and calls. The left-hand side shows the plain vanilla options, and the right hand side shows the digital cash-or-nothing options. Obviously, the early exercise feature of AS digital options has a profound impact on option values.

***Figure 5.2.3. American-style and European-style binomial option valuation model with no dividends***
***Panel A. Option values with boundary conditions for different initial stock prices***

***GBM-BOVM Call Values***

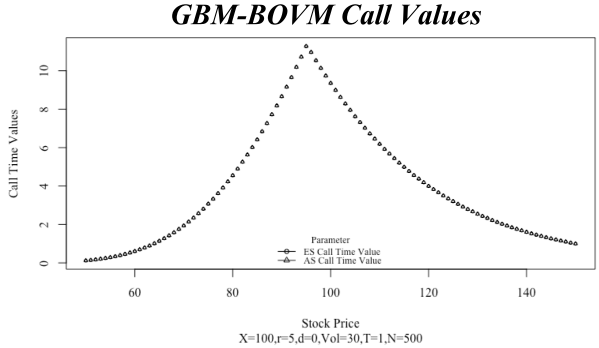



***GBM-BOVM Put Values***

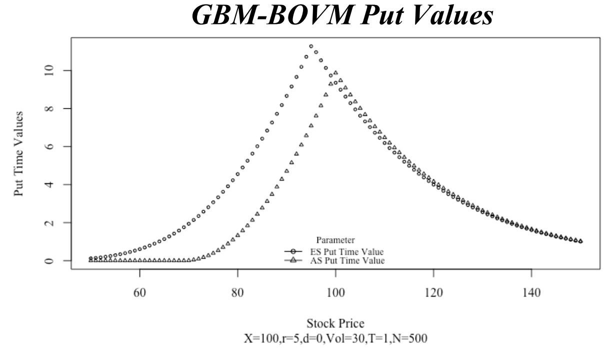



***Panel B. Option time values with boundary conditions for different initial stock prices***

***GBM-BOVM Call Values***

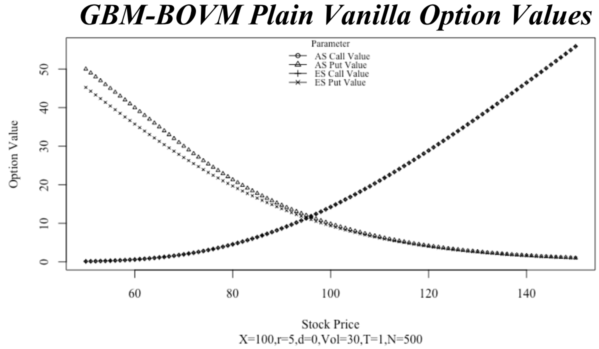



***GBM-BOVM Put Values***




***Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices***

***GBM-BOVM Plain Vanilla Option Values***




***GBM-BOVM Digital Option Values***

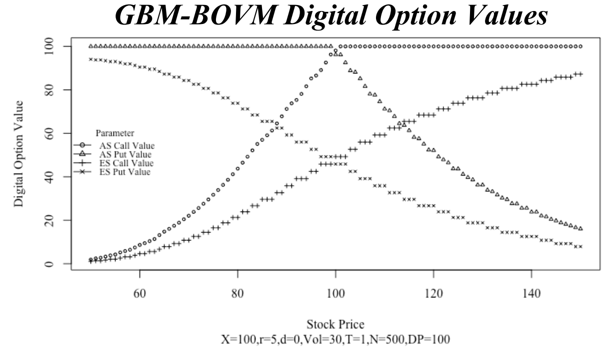



Figure 5.2.4 illustrates option values derived from both the European-style and American-style option valuation model with dividends. Here we assume the dividend yield equals the interest rate of five percent.

Note that with dividends, we see from Panel A that call options are potentially exercised early; hence, the American-style (AS) call values are no longer identical to the European-style (ES) call values when the call options are deep in-the-money. The lower boundary conditions for AS calls and ES calls are no longer the same. Due to arbitrage forces, both AS call values and AS put values are worth more than ES call values and ES put values, respectively. The is noticeable when the options are in-the-money. Also, notice that both the call and put valuation models for both AS and ES options converge to their appropriate lower boundary conditions.

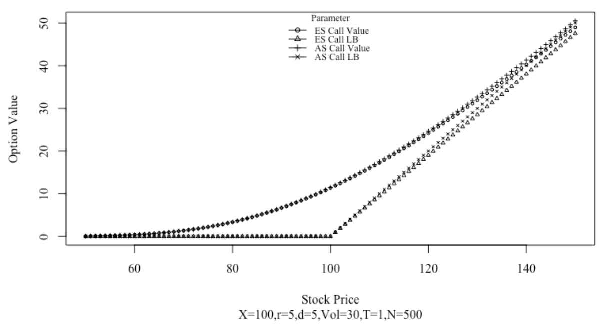
Panel B provides the same format, except focused solely on option time value. We see here that both calls and puts differ between AS and ES option time values when the options are deep in-the-money. Although, ES option values are lower, due to the lower boundary effect, the ES time values are higher than AS time values. Although the early exercise feature has a material effect on dividend paying options based on the GBM-BOVM, the impact on puts is diminished because dividends have on the lower boundary condition.

Panel C combines AS and ES as well as puts and calls. Again, the left-hand side shows the plain vanilla options, and the right hand side shows the digital cash-or-nothing options. As before, the early exercise feature of AS digital options has a profound impact on option values.

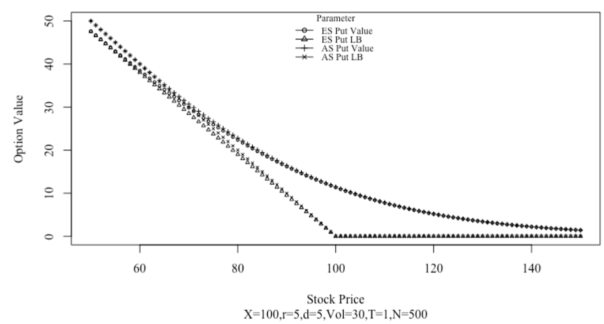
***Figure 5.2.4. American-style and European-style binomial option valuation model with dividend yield***

***Panel A. Option values with boundary conditions for different initial stock prices***

***GBM-BOVM Call Values***

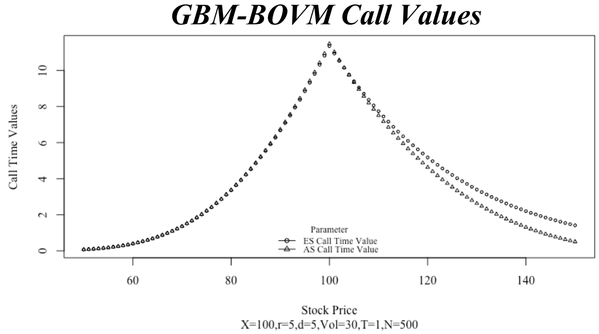



***GBM-BOVM Put Values***

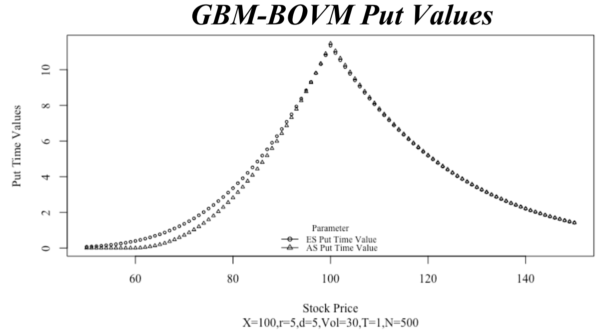



***Panel B. Option time values with boundary conditions for different initial stock prices***

***GBM-BOVM Call Values***

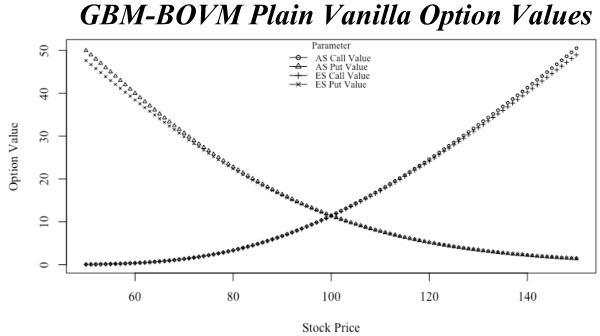



***GBM-BOVM Put Values***

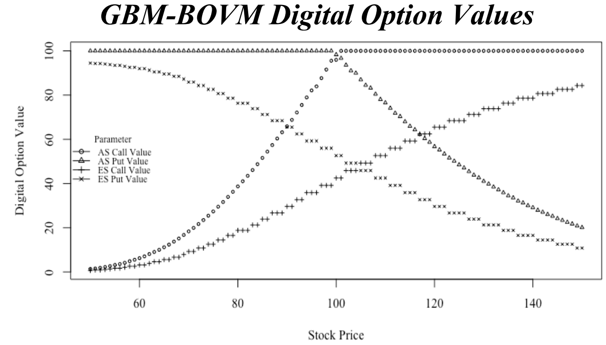



***Panel C. Put and call option values for plain vanilla and digital options for different initial stock prices***

***GBM-BOVM Plain Vanilla Option Values***




***GBM-BOVM Digital Option Values***



We now dive into the technical details related to building GBM-BOVMs.

## Quantitative finance materials

The notation used in this module is extensive, so we first explicitly define all the variables used.

### GBM notation review

| | |
|---|---|
| $0, T, \Delta t$ | initial trade date, time 0; expiration or maturity date, time $T$; next time step, |
| $S_0, S_T$ | value of underlying instrument, e.g., stock, at time 0 and at time $T$, |
| $u, d$ | up, total return of $S$, if up occurs ($u > 0$) and if down occurs ($u > d > 0$), |
| $B_0, B_T$ | bond, value of risk-free investment at time 0 and at time $T$, |
| $V_0, V_T$ | portfolio, value of some financial instrument portfolio at time 0 and at time $T$, |
| $\iota$ | indicator function, +1 for calls and –1 for puts, |
| $O_0$ | option, value of options, either call or put at time 0, |
| $O_u, O_d$ | option, value of option at time $T$ if up occurs and if down occurs, |
| $\Delta$ | delta, hedge ratio, units of the financial instrument to enter to hedge option position, |
| $FV()$ | future value based on risk-free interest rate, |
| $PV()$ | present value based on risk-free interest rate, |
| $\pi$ | equivalent martingale probability of up move, |
| $E_\pi()$ | expectation under equivalent martingale probability, |
| $r$ | discretely compounded, periodic "risk-free" interest rate, |
| $r_c$ | continuously compounded, annualized, "risk-free" interest rate, |
| $\delta$ | continuously compounded, annualized, dividend yield, and |
| $D_T$ | known discrete dividend amount paid at time $T$ (ex-dividend the instant *before* the next binomial point in time). |

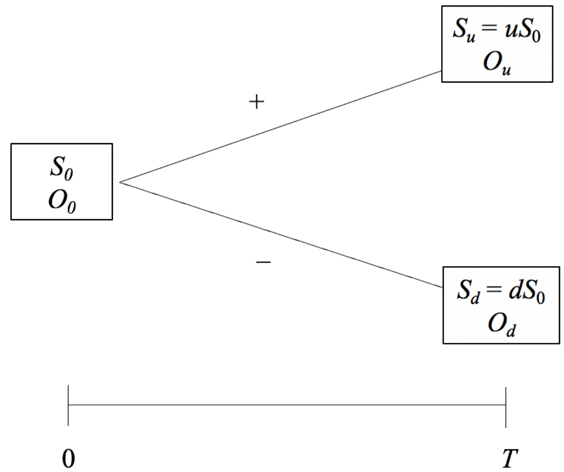
### GBM one period binomial option model

Consider a single period binomial framework: Let $S_0$ denote the underlying instrument value, such as a stock price. At each point in time, the underlying instrument will have a specific value at each of two possible states (also known as nodes). During each period, the specific value will change to one of two values (arcs). The arc will either be positive (up) or negative (down). With a multiplicative lattice, we have $S_u = uS_0$ when the state up occurs and $S_d = dS_0$ when the state down occurs.[1] For reasons detailed later, we assume $u > 1 + r > d > 0$, where $r$ denotes the discretely compounded interest rate over one period.

Figure 5.2.5 illustrates the multiplicative single period binomial framework where $O$ denotes a generic (call or put) option value. At the initial point in time, there is only one node whereas at the next point in time there are only two nodes. Also, at the initial point in time, there are two arcs emanating from the initial node, hence the name binomial. If we used three arcs, then it would be a trinomial model.

[1]A multiplicative lattice is consistent with geometric Brownian motion used in the BSMOVM. An additive lattice is consistent with arithmetic Brownian motion described in the next module. With an additive lattice, we have $S_u = S_0 + u$ and $S_d = S_0 + d$.

***Figure 5.2.5 Multiplicative One Period Binomial Framework***

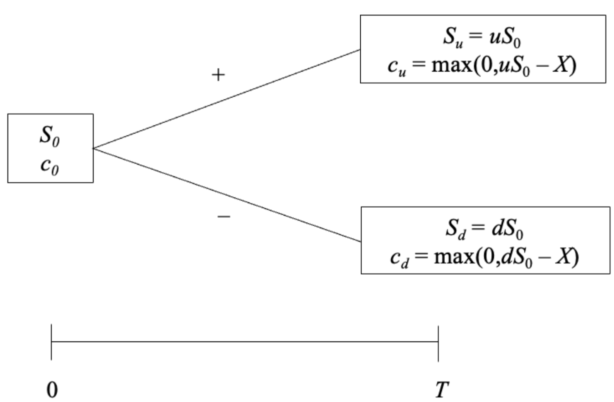



Note that the changes in $S_0$ are multiplicative; hence, this type of figure is often called a multiplicative binomial tree. Consider a generical option with exercise price $X$ that expires in one period. The two possible values for the generic option at expiration are

$$\begin{aligned} O_u &= \max\left[0, \iota\left(uS_0 - X\right)\right] \\ O_d &= \max\left[0, \iota\left(dS_0 - X\right)\right] \end{aligned}. \tag{5.2.1}$$

Of course, our objective is to determine the current option value denoted generically as $O$.
*GBM one period call option binomial model*
The basic layout with the corresponding call option prices inserted at each node is in Figure 5.2.6.

***Figure 5.2.6 Multiplicative One Period Call Option Binomial Framework***




A portfolio consisting of the call option and the underlying instrument is created in such a way that it is hedged. That is, the future value is known for certain and therefore should earn the risk-free rate. We can then solve for the price of the call option that is consistent with a risk-free return. Let us buy $h_c$ units of the underlying instrument and sell one call. The value of this portfolio today ($V_0$) is

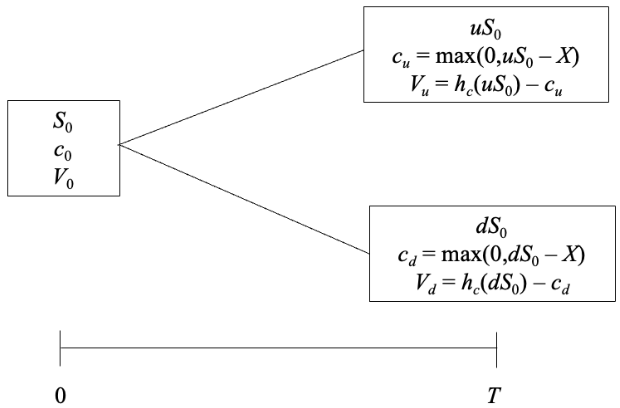
$$V_0 = h_c S_0 - c_0. \tag{5.2.2}$$

The value at expiration of this portfolio in the two future states are

$$V_u = h_c\left(uS_0\right) - \max\left(0, uS_0 - X\right) = h_c\left(uS_0\right) - c_u$$
$$V_d = h_c\left(dS_0\right) - \max\left(0, dS_0 - X\right) = h_c\left(dS_0\right) - c_d \quad (5.2.3)$$

Figure 5.2.7 illustrates the process thus far. The top row is the underlying instrument's price process. The second row illustrates the call option's price process. Finally, the third row illustrates the portfolio value's process, where the portfolio is comprised of $h_c$ units of the underlying instrument and short one call option.

***Figure 5.2.7 GBM Binomial Process for Underlying Instrument, Call Option, and Hedge Portfolio***




Up to this point, we have two instruments and have entered opposite exposures. Specifically, we are long the underlying instrument and short the call. We now introduce a third instrument, the risk-free instrument. If the portfolio represented by $V$ can exactly replicate a risk-free instrument, it must produce a risk-free return, meaning that these two outcomes are the same, as specified by the terminal value condition,

$$V_u = V_d. \quad (5.2.4)$$

If we set the terminal portfolio values equal to each other, we have one equation with only one unknown, $h_c(uS_0) - c_u = h_c(dS_0) - c_d$ whose solution can be expressed as

$$h_c = \frac{c_u - c_d}{uS_0 - dS_0} = \frac{c_u - c_d}{S_0\left(u - d\right)}. \quad (5.2.5)$$

This result is known as the optimal hedge ratio. Specifically, it tells us how many underlying instruments to buy for every call written. The sign of $h_c$ will be positive as $c_u > c_d$ and $u > d$. Recall we assume $u > 1 + r > d > 0$. Thus, if the number of units of the underlying instrument that we hold is set to $h_c$, the two future values of the underlying instrument will be identical. Hence, the portfolio is risk-free. To avoid arbitrage, the portfolio must be priced to earn the risk-free rate. Again, the discretely compounded periodic risk-free rate is denoted $r$. Thus, the following condition must hold:

$$V_0 = \frac{V_u}{1+r} = \frac{V_d}{1+r}. \quad (5.2.6)$$

Consequently, we can substitute into Equation (5.2.6) using either $V_u$ or $V_d$. We choose $V_u$, thus

$$\frac{h_c\left(uS_0\right) - c_u}{1+r} = h_c S_0 - c_0. \quad (5.2.7)$$

Therefore, the initial call price can be represented based on the *no arbitrage model* as

$$c_0 = h_c S_0 - B_{0,c}, \quad (5.2.8)$$

where

$$B_{0,c} = \frac{h_c\left(uS_0\right) - c_u}{1+r}. \tag{5.2.9}$$

Thus, a call option can be replicated by purchasing $h_c$ units of the underlying instrument partially financed through borrowing of $B_{0,c}$. From this analysis, a call option is simply a leveraged position in the underlying instrument.

To solve for the *equivalent martingale measure model*, the next step is to insert the solution for $h_c$, Equation (5.2.8), and solve for $c$:

$$c_0 = PV\left[E\left(c_T\right)\right] = \frac{\pi c_u + \left(1-\pi\right)c_d}{1+r}, \tag{5.2.10}$$

where the equivalent martingale measure probability is time and state independent[2]

$$\pi = \frac{1+r-d}{u-d}. \tag{5.2.11}$$

The derivation of Equation (5.2.10) is provided in Appendix 5.2A. Thus, another view is that the call price is simply the present value of the expected future call payoffs discounted at the risk free rate. The probabilities used in forming the expectations, however, are not the investor's subjective probabilities. They are based on the equivalent martingale measure or the risk neutral probabilities.

*GBM one period call option binomial model example*

For example, suppose the current stock price is \$99, the strike price is \$100, the annual, discretely compounded, risk free rate is 2%, the time to expiration is one year, and $u = 1.25$, and $d = 0.8$. We can compute the call price in two ways. First, note:

$$c_u = \max(0, 123.75 - 100) = 23.75$$
$$c_d = \max(0, 79.2 - 100) = 0.$$

For the no arbitrage model, we first find the hedge ratio:

$$h_c = (c_u - c_d)/[S_0(u-d)] = (23.75 - 0)/(123.75 - 79.2) = 23.75/44.55 = 0.5331.$$

Therefore, based on Equation (5.2.8), we have

$$\begin{aligned} c_0 &= h_c S_0 - \frac{h_c\left(uS\ \right) - c_u}{1+r} \\ &= 0.5331\left(99\right) - \frac{0.5331\left(123.75\right) - 23.75}{1+0.02}. \\ &= 52.7769 - 41.3933 = 11.38 \end{aligned}$$

Alternatively, we can use apply the risk neutral model. The binomial probability of an up move is

$$\begin{aligned} \pi &= \frac{1+r-d}{u-d} \\ &= \frac{1+0.02-0.8}{1.25-0.8} = \frac{0.22}{0.45} = 0.488889 \end{aligned}.$$

Therefore, based on Equation (5.2.10), we find the same results or

---

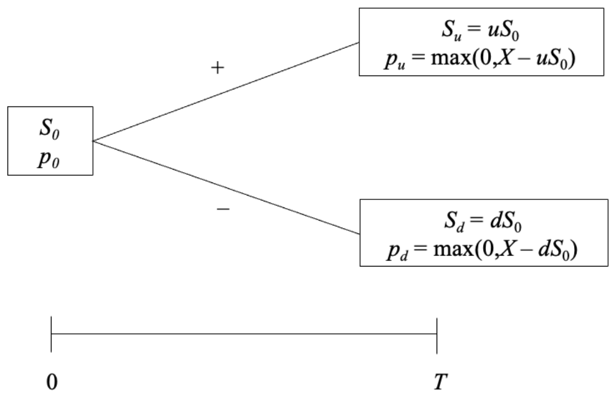
[2]This independence is an important feature for optimizing calculations of European-style option values. As we will see in the next module, arithmetic Brownian motion-based binomial valuation models will have dependent equivalent martingale measure probabilities requiring a bit more effort to build binomial models.

$$
\begin{aligned}
c_0 &= \frac{\pi c_u + (1-\pi) c_d}{1+r} \\
&= \frac{0.4889(23.75) + (1-0.4889)0}{1+0.02} = 11.38
\end{aligned}.
$$

*GBM one period put option binomial model*

Following the structure from the previous sections on calls, the basic layout with the corresponding put option prices inserted at each node is in Figure 5.2.8.

***Figure 5.2.8 Multiplicative One Period Put Option Binomial Framework***




As with calls, a portfolio consisting of the put option and the underlying instrument is created in such a way that it is hedged. That is, the future value is known for certain and therefore should earn the risk-free rate. We can then solve for the price of the put option that is consistent with a risk-free return. Let us buy $h_p$ units of the underlying instrument and buy one put. Note that to hedge, we need to be on the same side of the market. Here, we show buying both the underlying instrument and buying the put. The value of this portfolio today ($V_0$) is

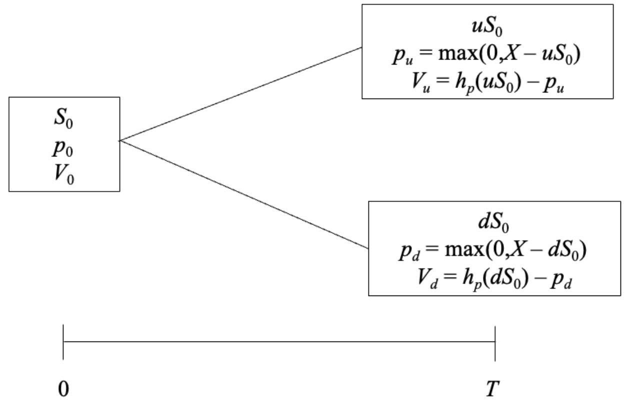
$$V_0 = h_p S_0 + p_0. \tag{5.2.12}$$

The value at expiration of this portfolio in the two future states are

$$
\begin{aligned}
V_u &= h_p(uS_0) + \max(0, X - uS_0) = h_p(uS_0) + p_u \\
V_d &= h_p(dS_0) + \max(0, X - dS_0) = h_p(dS_0) + p_d
\end{aligned}. \tag{5.2.13}
$$

Figure 5.2.9 illustrates the process thus far. The top row is the underlying instrument's price process. The second row illustrates the put option's price process. Finally, the third row illustrates the portfolio value's process, where the portfolio is comprised of $h_p$ units of the underlying instrument, and one put option.

***Figure 5.2.9 GBM Binomial Process for Underlying Instrument, Put Option, and Hedge Portfolio***




Up to this point, we have two instruments and have entered similar (long) exposures. Specifically, we are long the underlying instrument and long the put. We now introduce a third instrument, the risk-free instrument. If the portfolio represented by $V$ can exactly replicate a risk-free instrum(5.2.14)ent, it must produce a risk-free return, meaning that these two outcomes are the same, as specified by the terminal value condition,

$$V_u = V_d. \tag{5.2.15}$$

If we set the terminal portfolio values equal to each other, we have one equation with only one unknown,

$$h_p\left(uS_0\right) + p_u = h_p\left(dS_0\right) + p_d, \tag{5.2.16}$$

whose solution can be expressed as

$$h_p = \frac{p_d - p_u}{uS_0 - dS_0} = \frac{p_d - p_u}{S_0\left(u-d\right)}. \tag{5.2.17}$$

This result is known as the optimal hedge ratio. Specifically, it tells us how many underlying instruments to buy for every put purchased. The sign of $h_p$ will be positive as $p_d > p_u$ and $u > d$. Recall we assume $u > 1 + r > d > 0$. Thus, if the number of units of the underlying instrument that we hold is set to $h_p$, the two future values of the underlying instrument will be identical. Hence, the portfolio is risk-free. To avoid arbitrage, the portfolio must be priced to earn the risk-free rate. Again, the discretely compounded periodic risk-free rate is denoted $r$. Thus, the following condition must hold:

$$V_0 = \frac{V_u}{1+r} = \frac{V_d}{1+r}. \tag{5.2.18}$$

Consequently, we can substitute into Equation (5.2.18) using either $V_u$ or $V_d$. We choose $V_d$, thus

$$\frac{h_p\left(dS_0\right) + p_d}{1+r} = h_p S_0 + p_0. \tag{5.2.19}$$

Therefore, the initial put price can be represented based on the *no arbitrage model* as

$$p_0 = B_{0,p} - h_p S_0, \tag{5.2.20}$$

where

$$B_{0,p} = \frac{h_p\left(dS_0\right) + p_d}{1+r}. \tag{5.2.21}$$

Thus, a put option can be replicated by short selling $h_p$ units of the underlying instrument and lending of $B_{0,p}$. From this analysis, a put option is simply shorting a stock with lending.

To solve for the *equivalent martingale measure model*, the next step is to insert the solution for $h_p$ into Equation (5.2.20), and solve for $p_0$:

$$p_0 = PV\left[E\left(p_T\right)\right] = \frac{\pi p_u + \left(1-\pi\right)p_d}{1+r}, \tag{5.2.22}$$

where the equivalent martingale measure probability is time and state independent[3]

$$\pi = \frac{1+r-d}{u-d}. \tag{5.2.23}$$

The derivation of Equation (5.2.22) is provided in Appendix 5.2A. Thus, another view is that the put price is simply the present value of the expected future put payoffs discounted at the risk free rate. The probabilities used in forming the expectations, however, are not the investor's subjective probabilities. They are based on the equivalent martingale measure or the risk neutral probabilities.

*GBM one period put option binomial model example*

Again for example, suppose the current stock price is \$99, the strike price is \$100, the annual, discretely compounded, risk free rate is 2%, the time to expiration is one year, and $u$ = 1.25, and $d$ = 0.8. We can compute the put price in two ways. First, note:

$$p_u = \max(0, 100 - 123.75) = 0$$
$$p_d = \max(0, 100 - 79.2) = 20.8.$$

For the no arbitrage model, we first find the hedge ratio:

$$h_p = (p_d - p_u)/[S_0(u - d)] = (20.8 - 0)/(123.75 - 79.2) = 20.8/44.55 = 0.4669.$$

Therefore, based on Equation (5.2.20), we have

$$\begin{aligned} p_0 &= \frac{h_p\left(dS_0\right) + p_d}{1+r} - h_p S_0 \\ &= \frac{0.4669\left(79.2\right) + 20.8}{1+0.02} - 0.4669\left(99\right). \\ &= 56.6460 - 46.2231 = 10.42 \end{aligned}$$

Alternatively, we can use apply the risk neutral model. The binomial probability of an up move is

$$\begin{aligned} \pi &= \frac{1+r-d}{u-d} \\ &= \frac{1+0.02-0.8}{1.25-0.8} = \frac{0.22}{0.45} = 0.488889 \end{aligned}.$$

Therefore, based on Equation (5.2.22), we find the same results or

$$\begin{aligned} p_0 &= \frac{\pi p_u + \left(1-\pi\right)p_d}{1+r} \\ &= \frac{0.4889\left(0\right) + \left(1-0.4889\right)20.8}{1+0.02} = 10.42 \end{aligned}.$$

[3]This independence is an important feature for optimizing calculations of European-style option values. As we will see in the next module, arithmetic Brownian motion-based binomial valuation models will have dependent equivalent martingale measure probabilities requiring a bit more effort to build binomial models.

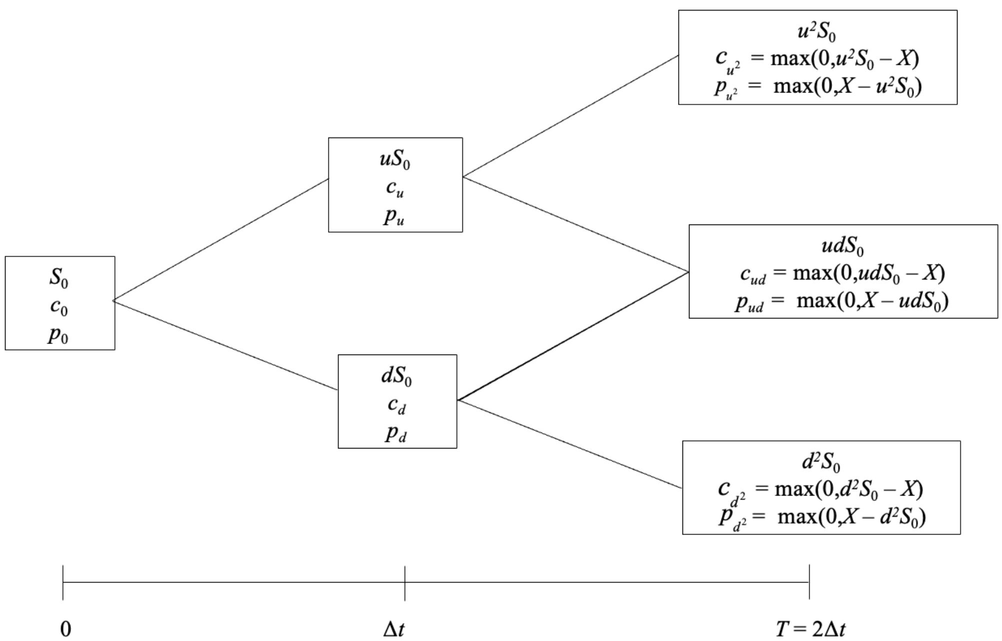
**GBM European-style option two period model**

The model can be extended to multiple periods and thereby accommodate options with longer lives or smaller time steps. For example, we can let the underlying instrument move from $uS_0$ to $u^2S_0$ or $udS_0$. From $dS_0$, the underlying instrument can move to $udS_0$ or $d^2S_0$. Note that $udS_0 = duS_0$, so over two periods, there are only three possible outcomes. The underlying instrument can go up twice to $u^2S_0$, up and then down or down and then up to $udS_0$, or down twice to $d^2S_0$. The call and put option payoffs in those states are

$$\begin{array}{lll} c_{u^2} = \max\left(0, u^2S_0 - X\right) & & p_{u^2} = \max\left(0, X - u^2S_0\right) \\ c_{ud} = \max\left(0, udS_0 - X\right) & \text{and} & p_{ud} = \max\left(0, X - udS_0\right). \\ c_{d^2} = \max\left(0, d^2S_0 - X\right) & & p_{d^2} = \max\left(0, X - d^2S_0\right) \end{array} \tag{5.2.24}$$

The layout is illustrated in Figure 5.2.10. The illustration is looking more like a branching tree or lattice. Two key features of the binomial model here is the recombining nature of the tree, and the growth of the underlying instrument is multiplicative. The tree is recombining because the stock price is assumed to grow by multiplication such that $udS_0 = duS_0$. Clearly, the order of multiplication does not matter. The multiplicative approach presented here facilitates the convergence of the stock price to the lognormal distribution.

***Figure 5.2.10 Two Period European-Style Binomial Model***




Let us position ourselves in the time 1 up-state, where the underlying instrument price is $uS_0$. At this point, we are now back in a one-period world. There are two outcomes possible outcomes in the next period, which is the expiration. It should be easy to see that the value of the call and put at this point would be:

$$c_u = \frac{\pi c_{u^2} + (1-\pi)c_{ud}}{1+r} \text{ and } p_u = \frac{\pi p_{u^2} + (1-\pi)p_{ud}}{1+r}, \tag{5.2.25}$$

where recall

$$\pi = \frac{1+r-d}{u-d}. \tag{5.2.26}$$

Likewise, in the time 1 down-state, the option value would be

$$c_d = \frac{\pi c_{ud} + (1-\pi) c_{d^2}}{1+r} \text{ and } p_d = \frac{\pi p_{ud} + (1-\pi) p_{d^2}}{1+r}. \tag{5.2.27}$$

Stepping back to time 0, the value of the call and put options are again found with Equation (5.2.10), where the values of $c_u$ and $p_u$ are given in Equation (5.2.25) and $c_d$ and $p_d$ are given in Equation (5.2.27). Thus, one way to price options in the binomial framework in any multiperiod model, we start at the end—the exercise date—and work backwards to the present.

Because the equivalent martingale measure is constant, the special case for two-period options does lend itself to a simple formula that relates the initial option value to the value two periods later, essentially skipping over the first period.

$$c_0 = \frac{\pi^2 c_{u^2} + 2\pi(1-\pi) c_{ud} + (1-\pi)^2 c_{d^2}}{(1+r)^2} \tag{5.2.28}$$

and

$$p_0 = \frac{\pi^2 p_{u^2} + 2\pi(1-\pi) p_{ud} + (1-\pi)^2 p_{d^2}}{(1+r)^2}. \tag{5.2.29}$$

Note that the three option payoffs two periods later are each weighted by the equivalent martingale measure probabilities, $\pi^2$, $2\pi(1-\pi)$, and $(1-\pi)^2$. These are the binomial probabilities for two trials, and they add up to 1.

*GBM two period call and put option binomial model example*

Suppose the current stock price is \$99, the strike price is \$100, the annual, discretely compounded, risk free rate is 2%, the time to expiration is two years, $u$ = 1.25, and $d$ = 0.8. Now assume a two-period binomial model. Based on Equations (5.2.28) and (5.2.29), we can compute the call and put prices. First, we compute the terminal payoffs for both calls and puts as

$$\begin{aligned} c_{u^2} &= \max(0, u^2 S_0 - X) = \max\left[0, (1.25)^2 99 - 100\right] = 54.6875 \\ c_{ud} &= \max(0, udS_0 - X) = \max\left[0, 1.25(0.8)99 - 100\right] = 0 \quad \text{and} \\ c_{d^2} &= \max(0, d^2 S_0 - X) = \max\left[0, (0.8)^2 99 - 100\right] = 0 \end{aligned} \tag{5.2.30}$$

$$\begin{aligned} p_{u^2} &= \max(0, X - u^2 S_0) = \max\left[0, 100 - (1.25)^2 99\right] = 0 \\ p_{ud} &= \max(0, X - udS_0) = \max\left[0, 100 - 1.25(0.8)99\right] = 1 \quad . \\ p_{d^2} &= \max(0, X - d^2 S_0) = \max\left[0, 100 - (0.8)^2 99\right] = 36.64 \end{aligned} \tag{5.2.31}$$

The binomial probability of an up move is $\pi$ = (1.02 – 0.8)/(1.25 – 0.8) = 48.89%. Therefore, based on Equation (5.2.28), we find[4]

[4]All calculations are done with software, so if you work out the numbers by hand you will often observe slight rounding differences.

$$c_0 = \frac{\pi^2 c_{u^2} + 2\pi(1-\pi)c_{ud} + (1-\pi)^2 c_{d^2}}{(1+r)^2}$$
$$= \frac{0.4889^2 54.6875 + 2(0.4889)(1-0.4889)0 + (1-0.4889)^2 0}{(1+0.02)^2} \quad (5.2.32)$$
$$= 12.56$$

and, based on Equation (5.2.29), we have

$$p_0 = \frac{\pi^2 p_{u^2} + 2\pi(1-\pi)p_{ud} + (1-\pi)^2 p_{d^2}}{(1+r)^2}$$
$$= \frac{0.4889^2 0 + 2(0.4889)(1-0.4889)1.0 + (1-0.4889)^2 36.64}{(1+0.02)^2}. \quad (5.2.33)$$
$$= 9.68$$

Alternatively, the two period binomial model can be viewed as three one period binomial models and the no arbitrage model applied. The call results are illustrated in Figure 5.2.11. Note that at node (1,0) both the call value and hedge ratio are zero because it is not possible that this option will end up in-the-money at time 2. Node (2,0) is out-of-the-money and node (2,1) is at-the-money. At node (1,1), the call value is

$$c_u = \frac{\pi c_{u^2} + (1-\pi)c_{ud}}{1+r}$$
$$= \frac{0.4889(54.6875) + (1-0.4889)0}{1+0.02} = 26.21. \quad (5.2.34)$$

The hedge ratio at node (1,1) is equal to one because both subsequent nodes are not out-of-the-money or

$$h_{c,1,1} = \frac{c_{u^2} - c_{ud}}{S_u(u-d)}$$
$$= \frac{54.6875 - 0}{123.75(1.25-0.8)} = 0.98204. \quad (5.2.35)$$

At time 0, the call hedge ratio is

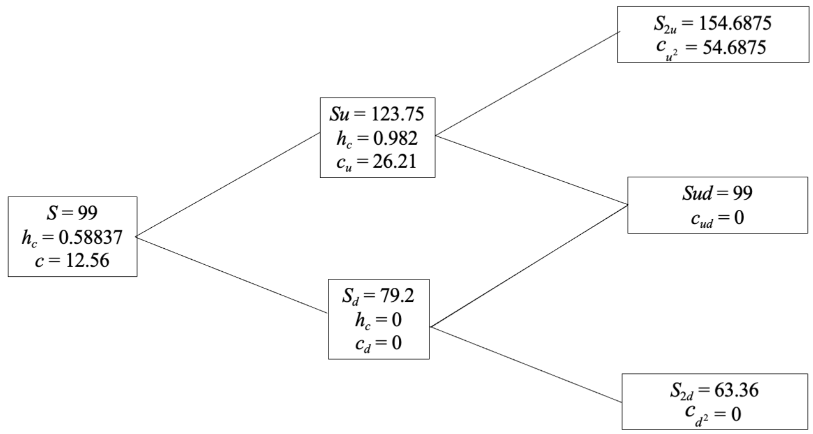
$$h_{c,0} = \frac{c_u - c_d}{S_0(u-d)}$$
$$= \frac{26.21 - 0}{99(1.25-0.8)} = 0.58837. \quad (5.2.36)$$

At time 0, the call value is

$$c_0 = \frac{\pi c_u + (1-\pi)c_d}{1+r}$$
$$= \frac{0.4889(26.21) + (1-0.4889)0}{1+0.02} = 12.56. \quad (5.2.37)$$

The call values using both techniques will result in the same value except for rounding error.

***Figure 5.2.11 Two period European-Style Binomial Call Model Example***




The put results are illustrated in Figure 5.2.12. Note that at node (1,1) both the put value and hedge ratio are zero because it is not possible that this option will end up in-the-money at time 2. Node (2,2) is out-of-the-money and node (2,1) is at-the-money. At node (1,0), the call value is

$$\begin{aligned} p_d &= \frac{\pi_{1,d}\, p_{ud} + \left(1 - \pi_{1,d}\right) p_{2d}}{1+r} \\ &= \frac{0.4889(1) + (1 - 0.4889)36.64}{1 + 0.02} = 18.84 \end{aligned}. \tag{5.2.38}$$

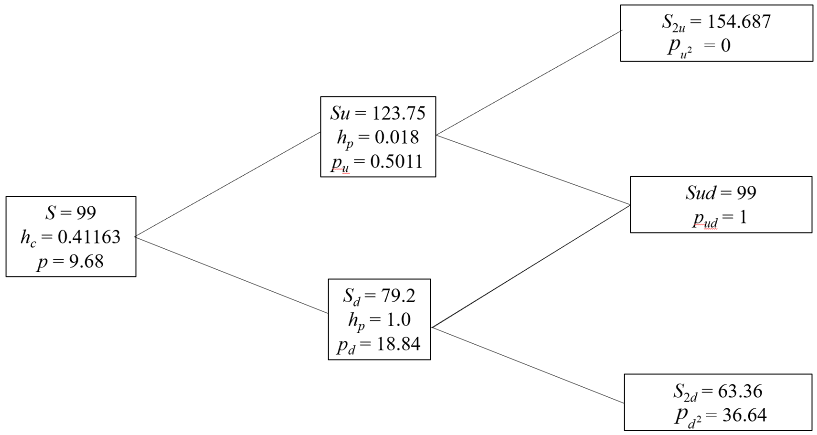
At time 0, the put hedge ratio is

$$\begin{aligned} h_{p,0} &= \frac{p_u - p_d}{S_0\left(u - d\right)} \\ &= \frac{0.50109 - 18.83922}{99(1.25 - 0.8)} = -0.41163 \end{aligned}. \tag{5.2.39}$$

At time 0, the put value is

$$\begin{aligned} p_0 &= \frac{\pi_0 p_u + \left(1 - \pi_0\right) p_d}{1+r} \\ &= \frac{0.4889(0.50109) + (1 - 0.4889)18.83922}{1 + 0.02} = 9.68 \end{aligned}. \tag{5.2.40}$$

*Figure 5.2.12 Two period European-Style Binomial Put Model Example*




We turn now to address American-style options where early exercise may enhance the worth of an option.

**GBM American-style option two period model**

If the options are American-style, they can be exercised early. Cash payments, such as dividends, will influence the early exercise decision. Thus, we first examine this influence.

*American-stye options and dividends*

It is well known that American call options will not be exercised early unless there is some cash or cash-equivalent amount paid by the underlying instrument, in which case early exercise could be justified immediately after the cash is paid. An example of a non-cash benefit is ski lift tickets given to stockholders of a ski company. The typical assumption is that any benefits of this nature are immediately sold for cash and this cash amount is included in any holding period return calculations. Obviously, one could go skiing but the financial analysis assumes that the lift tickets are sold. Note that cash dividend on the stock result in less equity per share remaining with the company and hence, the stock price should decline by the dividend amount. This stock price decline is detrimental to stockholders.

There are two primary methods for handling the underlying instrument paying out something of value, the yield method and the escrow method. We focus here on cash dividends on a stock. The yield method assumes the dividend is a constant rate of the value of the stock. This approach, however, would imply a very small dividend at every time step. Options on stock indexes come close to a continuous yield and can be approximated by a yield.

The escrow method assumes the present value of the dividends to be paid out over the life of the option is placed in a bankruptcy proof escrow account denoted *PVD*. The escrow account is then used to make the future dividend payments. Thus, the remaining stock value is simply based on subtracting the escrow amount from the current value of the underlying. The stock price minus the present value of dividends, $S' = S - PVD$, is modeled with the binomial tree according to the factors $u$ and $d$. At a given node at which the dividend is paid, we decide if the option is worth exercising just before the stock goes ex-dividend. If so, the exercise value replaces the value obtained using the formula.

For example, suppose at a point in the tree, we have a value of the stock price minus the present value of all remaining dividends over the life of the option of \$42. Suppose that using the binomial formula, we compute the value of the call at that point as \$2.25. Assume there is a \$3 dividend being paid at this time point. Then the stock price with the dividend is \$45. If the exercise price is \$42, we could exercise it and collect a value of \$3, which is more than its unexercised value of \$2.25. Thus, we would replace \$2.25 with \$3. This early exercise check would be done at all points in the tree in which the option is in-the-money.

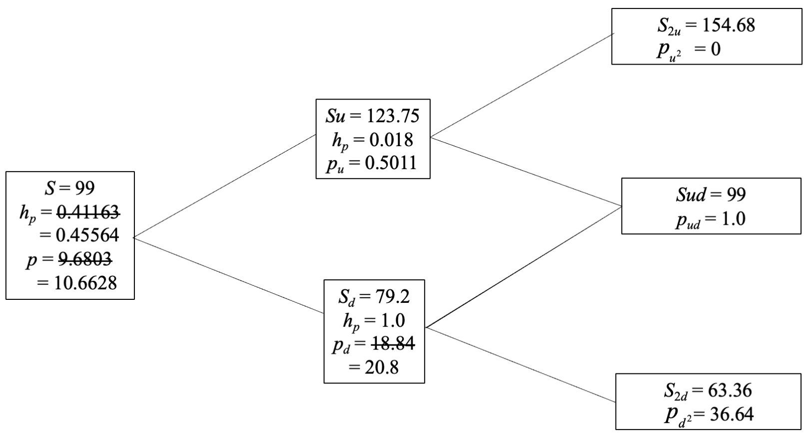
It is known that early exercise could occur regardless of a dividend for put options. At every in-the-money point in the binomial tree, we examine whether the put is worth more exercised or not. If it is worth more to early exercise, the exercise value is used at that point into the tree as the option value. If it is not worth more to early exercise, we simply continue to use the computed value obtained by the single period binomial formula. Dividends will reduce the frequency of early exercise since dividends drive the stock price down, which makes puts worth more. Exercising early negates this benefit. Early exercise generally occurs just after a dividend when the stock price falls.

*GBM two period American-style put option binomial model example*

Recall the data related to Figure 5.2.13. If this put option was American-style, we would exercise the put at node (1,0). Figure 5.2.8 illustrates this adjustment. Therefore, at time 0, the put value is

$$\begin{aligned} p &= \frac{\pi_0 p_u + (1-\pi_0) p_d}{1+r} \\ &= \frac{0.4889(0.50109)+(1-0.4889)20.8}{1+0.02} = 10.66 \end{aligned} . \quad (5.2.41)$$

***Figure 5.2.13 Two Period American-Style Binomial Put Model Example***




We now explore the important role of coherence conditions.

**GBM coherence conditions**

Coherence conditions are a set of assumptions require to assure the model does not allow for arbitrage opportunities within the theoretical model itself. Many lattice approaches unfortunately are not internally consistent and potentially allow for arbitrage. First, we examine the case where there are no dividends.

*No dividend coherence conditions*

We seek to build an option valuation model with certain assumption known as the coherent conditions. The coherent conditions comprise four assumptions:[5]

1) $0 < d < e^{r\Delta t} < u$ (no arbitrage boundary condition).
2) $0 << \pi << 1$ (probability condition, distribution independent, not "close" to 0 or 1).
3) $\pi = \dfrac{e^{r\Delta t} - d}{u - d}$ (no arbitrage condition, distribution independent).
4) $Var_\pi\left[\ln\left(\dfrac{S_{\Delta t}}{S_0}\right)\right] = \left[\ln\left(\dfrac{u}{d}\right)\right]^2 \pi(1-\pi) = \sigma^2 \Delta t$ (variance condition of log of price relative, distribution independent so long as $S_0 > 0$ and $S_T > 0$, $\sigma$ denotes the annualized volatility used in the BSMOVM).

We briefly comment on each coherent condition. The no arbitrage boundary condition is intuitive as the risk-free total return, $e^{r\Delta t}$, can neither be higher than $u$ (otherwise everyone will buy the risk-free instrument) nor be lower than $d$ (otherwise everyone will buy the risky instrument). More specifically, $d > 0$ based on the lognormal distribution assumption of $S_0 > 0$. Recall financial instruments have limited liability but zero is

[5]Based, in part, on Don Chance, "A Synthesis of Binomial Option Pricing Models for Lognormally Distributed Assets," *Journal of Applied Finance* (Spring/Summer 2008).

certainly a possible outcome in practice but not with this model. Note $u > d$ based on positive assumed volatility as financial instruments are risky. If $e^{r\Delta t} \geq u$, then you would buy risk-free instrument and short sell risky instrument. If $d \geq e^{r\Delta t}$, then you would buy the risky instrument and short sell the risk-free instrument.

The probability condition is required due to the potential for computational problems related to large scale calculations. If $\pi$ is "too close" to 0, then $u$ will tend to positive infinity. If $\pi$ is "too close" to 1, then $d$ will tend to 0. As will be illustrated later, both cases will cause stability problems with numerical implementation.

The no arbitrage condition is the result of potential arbitrage trading activities forcing specific relationship between the option and underlying stock. Under the equivalent martingale measure, the present value of an option ($O$) is

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = e^{-rT}\left[\pi O_u + \left(1-\pi\right)O_d\right]. \tag{5.2.42}$$

For a zero strike call (or the underlying instrument, $S$), we observe

$$S_0 = PV\left[E_\pi\left(S_T\right)\right] = e^{-rT}\left[\pi u S_0 + \left(1-\pi\right)dS_0\right]. \tag{5.2.43}$$

Dividing by $S_0$ and multiplying by $e^{rT}$, we have

$$e^{rT} = \pi u + \left(1-\pi\right)d. \tag{5.2.44}$$

Note that these equations hold only if the no arbitrage condition above is true.

The variance condition is required to converge to the BSMOVM (lognormal distribution) as well as be consistent at each node. The variance of the natural log of the price relative ($S_0 > 0$) can be expressed as

$$Var_\pi\left[\ln\left(\frac{S_T}{S_0}\right)\right] = E\left[\left\{\ln\left(\frac{S_T}{S_0}\right) - E\left[\ln\left(\frac{S_T}{S_0}\right)\right]\right\}^2\right]. \tag{5.2.45}$$

Substituting the results for the single period binomial model, we have

$$\begin{aligned}
Var_\pi\left[\ln\left(\frac{S_T}{S_0}\right)\right] &= \left\{\ln u - E\left[\ln\left(\frac{S_T}{S_0}\right)\right]\right\}^2 \pi + \left\{\ln d - E\left[\ln\left(\frac{S_T}{S_0}\right)\right]\right\}^2 \left(1-\pi\right) \\
&= \left\{\ln u - \left[\pi \ln u + \left(1-\pi\right)\ln d\right]\right\}^2 \pi + \left\{\ln d - \left[\pi \ln u + \left(1-\pi\right)\ln d\right]\right\}^2 \left(1-\pi\right). \\
&= \left(1-\pi\right)^2\left(\ln u - \ln d\right)^2 \pi + \pi^2\left(\ln u - \ln d\right)^2\left(1-\pi\right) = \left[\ln\left(\frac{u}{d}\right)\right]^2 \pi\left(1-\pi\right)
\end{aligned} \tag{5.2.46}$$

With these four coherence conditions, we can demonstrate the functional form for $u$ and $d$.

*No dividend u and d conditions*

With these coherence conditions, we can establish the following coherence conditions for $u$ and $d$:

$$u = \frac{e^{r\Delta t + \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + \left(1-\pi\right)}, \tag{5.2.47}$$

and

$$d = \frac{e^{r\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}. \tag{5.2.48}$$

We now prove these two conditions. Isolating *u* based on the no arbitrage condition, we have

$$u = \frac{e^{r\Delta t} - d}{\pi} + d = \frac{e^{r\Delta t} - d(1-\pi)}{\pi}. \tag{5.2.49}$$

Substituting this result into the variance condition, we have

$$\begin{gathered} \left[\ln\left(\frac{u}{d}\right)\right]^2 \pi(1-\pi) = \sigma^2 \Delta t \\ \left[\ln\left(\frac{\frac{e^{r\Delta t} - d(1-\pi)}{\pi}}{d}\right)\right]^2 \pi(1-\pi) = \sigma^2 \Delta t \end{gathered}. \tag{5.2.50}$$

Solving for *d*,

$$\begin{gathered} \left\{\ln\left[\frac{e^{r\Delta t}}{d\pi} - \frac{(1-\pi)}{\pi}\right]\right\}^2 = \frac{\sigma^2 \Delta t}{\pi(1-\pi)} \\ \ln\left[\frac{e^{r\Delta t}}{d\pi} - \frac{(1-\pi)}{\pi}\right] = \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}} \\ \frac{e^{r\Delta t}}{d\pi} = e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + \frac{(1-\pi)}{\pi} \end{gathered}. \tag{5.2.51}$$

Thus, the lognormal coherent binomial down move for single period is

$$d = \frac{e^{r\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}. \tag{5.2.52}$$

Solving for the up move, we have

$$\begin{aligned} u &= \frac{e^{r\Delta t} - d(1-\pi)}{\pi} = \frac{e^{r\Delta t} - \frac{e^{r\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}(1-\pi)}{\pi} \\ &= \frac{\frac{e^{r\Delta t}\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + e^{r\Delta t}(1-\pi) - e^{r\Delta t}(1-\pi)}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}}{\pi} \end{aligned}. \tag{5.2.53}$$

Thus, the lognormal coherent binomial up move for single period

$$u = \frac{e^{r\Delta t + \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}. \qquad (5.2.54)$$

Note $\pi$ is arbitrary. As $\pi$ tends to 1 from below, note that $u$ tends to $e^{r\Delta t}$ and $d$ tends to 0. As $\pi$ tends to 0 from above, note that $u$ tends to positive infinity and $d$ tends to $e^{r\Delta t}$. Thus, $\pi$ cannot be too 'close' to either 0 or 1. So long as $\pi$ is in a reasonable range, then numerically $\pi = \frac{e^{r\Delta t} - d}{u - d}$ exactly. Because $\pi$ is arbitrary, the coherence conditions comprise a family of binomial option valuation models.

Table 5.2.14 illustrates the relationship between $u$, $d$, and $\pi$. The first column is selected values for the equivalent martingale probability of up move. The values for u and d are computed based on Equations (5.2.54) and (5.2.52), respectively. Finally, the fourth column (Prob Check) recomputes the equivalent martingale probability of up move based on the coherence condition 3 (no arbitrage distribution independent condition) as well as the computed values for $u$ and $d$. Note that there are computational problems when $\pi$ is too close to zero or one. Clearly, selecting $\pi$ close to 0.5 results in stable values for $u$ and $d$. Thus, we will use 0.5 when implementing this model.

***Table 5.2.14. Relationship between u, d, and π***

| Probability | u | d | Prob Check |
|---|---|---|---|
| 0 | #DIV/0! | #DIV/0! | #DIV/0! |
| 0.00000001 | #NUM! | #NUM! | #NUM! |
| 0.0000001 | #NUM! | #NUM! | #NUM! |
| 0.000001 | 1051271.096 | 5.411E-125 | 0.000001 |
| 0.00001 | 105127.1096 | 6.6178E-37 | 0.00001 |
| 0.0001 | 10512.71095 | 9.8227E-10 | 0.0001 |
| 0.001 | 977.554438 | 0.07379045 | 0.001 |
| 0.01 | 17.95514001 | 0.88052495 | 0.01 |
| 0.1 | 2.438626379 | 0.89712051 | 0.1 |
| 0.2 | 1.819144125 | 0.85930284 | 0.2 |
| 0.3 | 1.58387078 | 0.82301409 | 0.3 |
| 0.4 | 1.449553247 | 0.78574966 | 0.4 |
| 0.5 | 1.357519626 | 0.74502257 | 0.5 |
| 0.6 | 1.287020681 | 0.69764672 | 0.6 |
| 0.7 | 1.228283162 | 0.63824294 | 0.7 |
| 0.8 | 1.175296215 | 0.55517062 | 0.8 |
| 0.9 | 1.122208183 | 0.41283732 | 0.9 |
| 0.99 | 1.061364243 | 0.05204959 | 0.99 |
| 0.999 | 1.05232334 | 7.9434E-05 | 0.999 |
| 0.9999 | 1.051376234 | 9.8236E-14 | 0.9999 |
| 0.99999 | 1.051281609 | 6.6179E-42 | 0.99999 |
| 0.999999 | 1.051272148 | 5.411E-131 | 0.999999 |
| 0.9999999 | #NUM! | #NUM! | #NUM! |
| 0.99999999 | #NUM! | #NUM! | #NUM! |
| 1 | #DIV/0! | #DIV/0! | #DIV/0! |

Note: #DIV/0! denotes division by zero and #NUM! denotes here a number that is too small to be represented in this spreadsheet.

Because $\pi$ is arbitrary, the coherence conditions comprise a family of binomial option valuation models. These models converge to the geometric Brownian motion option valuation model in the limit as the number of time steps tends to infinity (or the step size tends to zero). Specifically, based on the use of $u$ and $d$ above, the coherent lognormal binomial model converges to the geometric Brownian option valuation model presented in Module 5.4.

We now introduce multiperiod binomial models. These GBM-based models converge to the BSMOVM in the limit as the number of time steps tends to infinity (or the step size tends to zero). Before addressing dividends, we illustrate the no dividend case.

**GBM-based binomial option valuation model: No dividends**

The current value of an option is equal to the present value of the expected terminal payout as we assume European-style options. The multiperiod binomial valuation equation can be expressed as

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U S_0 Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U}, \tag{5.2.55}$$

where the binomial summations are

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{5.2.56}$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{5.2.57}$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{5.2.58}$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{5.2.59}$$

where the indicator function denotes

$$\iota_U = \begin{cases} +1 \text{ if call option} \\ -1 \text{ if put option} \end{cases}, \tag{5.2.60}$$

$$\Delta t = \frac{T}{n}, \tag{5.2.61}$$

$$\pi = \frac{e^{r\Delta t} - d}{u - d}, \tag{5.2.62}$$

$$A \equiv \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}, \tag{5.2.63}$$

$$Den \equiv \pi e^A + \left(1-\pi\right), \tag{5.2.64}$$

$$\inf\left\{\text{int } j : u^j d^{n-j} S_0 > X\right\} > a = \frac{-\ln\left(\frac{S}{X}\right) - rT + n\ln\left(Den\right)}{A}, \tag{5.2.65}$$

$$\pi_1 = \frac{\pi e^A}{Den}, \text{ and} \tag{5.2.66}$$

$$\pi_2 = \pi = \frac{e^{r\Delta t} - d}{u - d}. \tag{5.2.67}$$

Alternatively, the binomial option valuation model can be expressed as

$$O_0 = PV_r\left[\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0,\iota_U u^j d^{n-j} S_0 - \iota_U X\right)\right].$$

where u and d are defined as

$$u = \frac{e^{r\Delta t + A}}{Den} \text{ and} \tag{5.2.68}$$

$$d = \frac{e^{r\Delta t}}{Den}. \tag{5.2.69}$$

*Multi-period binomial option valuation model*

For completeness, we document the current value for several specific option contracts. The multi-period binomial option valuation model is simply the present value of the expected terminal payout. For plain vanilla European-style call and put options, we have

$$c = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0,u^j d^{n-j} S_0 - X\right) \text{ and} \tag{5.2.70}$$

$$p = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0,X - u^j d^{n-j} S_0\right). \tag{5.2.71}$$

For cash-or-nothing digital call and put options, we have

$$c_{CoN} = e^{-rT} DP \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} I_{u^j d^{n-j} S_0 > X} \text{ and} \tag{5.2.72}$$

$$p_{CoN} = e^{-rT} DP \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} I_{u^j d^{n-j} S_0 < X}. \tag{5.2.73}$$

where *DP* denotes the digital cash payout if the option expires in-the-money.

For asset-or-nothing digital call and put options, we have

$$c_{AoN} = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} u^j d^{n-j} S_0 I_{u^j d^{n-j} S_0 > X} = c + c_{CoN}\left(DP = X\right) \text{ and} \tag{5.2.74}$$

$$p_{AoN} = e^{-rT}\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j} u^j d^{n-j} S_0 I_{u^j d^{n-j} S_0 < X} = p + p_{CoN}\left(DP = X\right). \tag{5.2.75}$$

where *DP* denotes the digital cash payout if the option expires in-the-money.

*Log transformation of binomial probabilities*

One of the implementation difficulties when computing binomial values is the explosive nature of $j!$ and implosive nature of $\pi^j$. For example, $60! = 8.32\text{x}10^{81}$ and $0.5^{60} = 8.67\text{x}10^{-19}$. Thus, machine error will become a significant problem. There is, however, an elegant solution to this problem. Note that the combination of an exploding number and an imploding number may remain reasonable. Note the probability of observing the $j^{\text{th}}$ node can be expressed as

$$\Pr\left(j\right) = \left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}. \tag{5.2.76}$$

If we take the natural log, we have

$$\begin{aligned}\ln\left[\Pr(j)\right] &= \ln\left[\left(\frac{n!}{j!(n-j)!}\right)\pi^j(1-\pi)^{n-j}\right] \\ &= \ln(n!) - \ln(j!) - \ln\left[(n-j)!\right] + j\ln(\pi) + (n-j)\ln(1-\pi) \quad . \\ &= \sum_{k=1}^{n}\ln(k) - \sum_{k=1}^{j}\ln(k) - \sum_{k=1}^{n-j}\ln(k) + j\ln(\pi) + (n-j)\ln(1-\pi)\end{aligned} \tag{5.2.77}$$

because $\ln(a/b) = \ln(a) - \ln(b)$, $\ln(a^b) = b\ln(a)$, and $\ln(ab) = \ln(a) + \ln(b)$. Finally, we take advantage of the partial cancellation of the sums depending on the value of $j$, thus

$$\ln\left[\Pr(j)\right] = \sum_{k=j+1}^{n}\ln(k) - \sum_{k=1}^{n-j}\ln(k) + j\ln(\pi) + (n-j)\ln(1-\pi). \tag{5.2.78}$$

**American-style options**

We now explore some of the issues surrounding that American-style. We explain both the case with no dividends as well as the case with dividends.

*No dividends*

The current value of an option is no longer equal to the present value of the expected terminal payout with American-style options. The early exercise potential must be incorporated. The approach typically taken is known as backward induction. "Backward induction is the process of reasoning backwards in time, from the end of a problem or situation, to determine a sequence of optimal actions. It proceeds by first considering the last time a decision might be made and choosing what to do in any situation at that time. Using this information, one can then determine what to do at the second-to-last time of decision. This process continues backwards until one has determined the best action for every possible situation (i.e. for every possible information set) at every point in time."[6] Thus, at the maturity of the option, we know

$$O_{n,j} = \max\left[0, \iota_U\left(S_{n,j} - X\right)\right] = \max\left[0, \iota_U\left(u^j d^{n-j} S_0 - X\right)\right] : j = 0,\ldots,n, \tag{5.2.79}$$

where $j$ denotes the number of up moves for the underlying over the option life. The indicator function denotes

$$\iota_U = \begin{cases} +1 \text{ if call option} \\ -1 \text{ if put option} \end{cases} \text{ and} \tag{5.2.80}$$

$$n = \frac{T}{\Delta t}. \text{ (total number of time periods over option life)} \tag{5.2.81}$$

Based on our single period results, we know that at time $i$ for $j$ up moves, the binomial model value (denoted with $B$ superscript) can be expressed as

$$O_{i,j}^{B} = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + (1-\pi)O_{i+1,j}\right], \tag{5.2.82}$$

where $PV_{r,i,\Delta t}(\ )$ denotes the present value at time $i$ for the next $\Delta t$ period based on the continuously compounded rate $r$ and as defined before $\pi = \frac{e^{r\Delta t} - d}{u - d}$, $A = \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}$, $Den = \pi e^{A} + (1-\pi)$, $u = \frac{e^{r\Delta t + A}}{Den}$ and $d = \frac{e^{r\Delta t}}{Den}$. With constant interest rates, we have $PV_{r,i,\Delta t}(1) = e^{-r\Delta t}$. The binomial model value, however, may be lower than the early exercise value (denoted with superscript $X$) that can be expressed as

[6]Wikipedia, "Backward Induction," observed on February 20, 2017.

$$O_{i,j}^{X} = \max\left[0, \iota_U\left(S_{i,j} - X\right)\right]. \tag{5.2.83}$$

Recall the lower boundary condition (denoted with superscript *L*) is

$$O_{i,j}^{L} = \max\left\{0, \iota_U\left[S_{i,j} - PV_{r,i,n-i}\left(X\right)\right]\right\}. \tag{5.2.84}$$

Thus, the fair value of the option at time *i* with *j* up moves is

$$O_{i,j} = \max\left[O_{i,j}^{B}, O_{i,j}^{X}, O_{i,j}^{L}\right]. \tag{5.2.85}$$

Note assuming positive interest rates and no dividends $O_{i,j}^{L} \geq O_{i,j}^{X}$ for call options and $O_{i,j}^{L} \leq O_{i,j}^{X}$ for put options. The initial option value is obtained through backward induction along the binomial lattice for the underlying instrument.

## Summary

A lattice approach to valuing various options consistent with a lognormal terminal distribution was presented in this module. The valuation approach is based on dynamic arbitrage. Dynamic arbitrage is based on the capacity to continuously rebalance a custom-designed portfolio.

We presented the traditional binomial valuation model or GBM-BOVM. In the next module, we introduce an unorthodox binomial valuation model consistent with a normal terminal distribution or ABM-BOVM. Like tools in a toolbox for the quantitative analyst, the varied challenges analysts face will warrant the unique tool selected. Unorthodox tools often prove vital with particularly challenging tasks.

## References


Brennan, Michael J., "The Pricing of Contingent Claims in Discrete Time Models," *The Journal of Finance* 34 (1979), 53-68.

Chance, Don M., "A Synthesis of Binomial Option Pricing Models for Lognormally Distributed Assets," *Journal of Applied Finance* 18 (2008), 38-56.

Chance, Don and Robert Brooks, *Foundations of the Pricing of Financial Derivatives: A Primer for Graduate Students in Finance*, Chapter 7, forthcoming.

Chance Don M. and Robert Brooks, *An Introduction to Derivatives and Risk Management*, 10th ed. (2006) Mason, Ohio: Cengage, Chapter 4.

Cox, J. C., S. A. Ross, and M. Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics* 7 (1979), 229-263.

Rendleman, Richard J., Jr. and B. J. Bartter, "Two-State Option Pricing," *The Journal of Finance* 34 (1979), 1093-1110.

Trigeorgis, Lenos, "A Log-Transformed Binomial Numerical Analysis Method for Valuing Complex Multi-Option Investments," *Journal of Financial and Quantitative Analysis* 26(3), (September 1991), 309-326.


## Appendices for Module 5.2.

Several technical issues are covered in these appendices.

### Appendix 5.2A Single period binomial valuation equation

Within the single period binomial framework, we provide mathematical details related to the binomial option valuation model both without and with dividends.

*No dividend single period valuation equation*

The current value of an option is equal to the present value of the expected terminal payout or

$$O_0 = PV\left[E_\pi\left(O_{\Delta t}\right)\right]. \tag{5.2.86}$$

Two natural questions arise: What is the appropriate probability distribution to compute the expected value? What is the appropriate discount rate? The following sketch answers both questions.

*No arbitrage sketch*: Consider a portfolio of long $\Delta$ stock and borrow $B_0$ (or short a risk-free bond) and the next time period has length $\Delta t$. Thus,

$$\Pi_0 = \Delta S_0 - B_0. \tag{5.2.87}$$

at expiration

$$\Pi_u = \Delta u S_0 - B_{\Delta t} = O_u \text{ and} \tag{5.2.88}$$

$$\Pi_d = \Delta d S_0 - B_{\Delta t} = O_d. \tag{5.2.89}$$

Making the value of the portfolio risk-free at $\Delta t$ implies

$$B_{\Delta t} = \Delta u S_0 - O_u = \Delta d S_0 - O_d. \tag{5.2.90}$$

Solving for $\Delta$,

$$\Delta = \frac{O_u - O_d}{S_0(u-d)}. \tag{5.2.91}$$

Note that for call options, $\Delta \geq 0$ and for put options, $\Delta \leq 0$. Entering $\Delta$ shares of stock and writing one option results in the payoff of the risk-free portfolio worth $B_T$. Note that $\Delta \geq 0$ for calls implies purchasing stock and $\Delta \leq 0$ for puts implies short selling stock. Thus,

$$FV(\Delta S_0 - O_0) = B_{\Delta t} = \Delta u S_0 - O_u = \Delta d S_0 - O_d, \tag{5.2.92}$$

or

$$FV(O_0 - \Delta S_0) = O_u - \Delta u S_0. \tag{5.2.93}$$

Isolating the initial call option value with have the no arbitrage model expression as

$$O_0 = \Delta S_0 - PV(\Delta u S_0 - O_u). \text{ (no arbitrage model)} \tag{5.2.94}$$

For justifying the equilibrium martingale measure model, we introduce the following lemma.

*Lemma*:

$$\begin{aligned} O_0 &= \Delta S_0 - PV(\Delta u S_0 - O_u) \\ &= PV\{\pi O_u + (1-\pi) O_d\} = PV\left[E_\pi(O_T)\right] \end{aligned} \text{ (equilibrium martingale measure model)} \tag{5.2.95}$$

*Lemma proof*: Note from the future investment value expression above rearranged

$$FV(\Delta S_0 - O_0) = \Delta u S_0 - O_u. \tag{5.2.96}$$

Substituting for $\Delta$ and solving for $O_0$, we have

$$FV\left(\frac{O_u - O_d}{uS_0 - dS_0} S_0 - O_0\right) = \frac{O_u - O_d}{uS_0 - dS_0} u S_0 - O_u. \tag{5.2.97}$$

Canceling $S_0$,

$$FV\left(\frac{O_u - O_d}{u-d} - O_0\right) = \frac{O_u - O_d}{u-d} u - O_u. \tag{5.2.98}$$

Taking the present value and rearranging,

$$O_0 = \frac{O_u - O_d}{u-d} - PV\left(\frac{O_u - O_d}{u-d}u - O_u\right). \tag{5.2.99}$$

Factoring out the present value factor, multiplying and dividing by u – d, and rearranging,

$$O_0 = PV\left[FV\left(\frac{O_u - O_d}{u-d}\right) - \left(\frac{O_u - O_d}{u-d}u - \frac{u-d}{u-d}O_u\right)\right]. \tag{5.2.100}$$

Rearranging once again,

$$O_0 = PV\left[\frac{FV\left(O_u\right) - FV\left(O_d\right) - uO_u + uO_d + uO_u - dO_u}{u-d}\right]. \tag{5.2.101}$$

Canceling terms once again and substituting for FV(),

$$O_0 = PV\left[\frac{\left(e^{r\Delta t} - d\right)O_u + \left(u - e^{r\Delta t}\right)O_d}{u-d}\right]. \tag{5.2.102}$$

Let,

$$\pi = \frac{e^{r\Delta t} - d}{u-d}, \tag{5.2.103}$$

and

$$1 - \pi = 1 - \frac{e^{r\Delta t} - d}{u-d} = \frac{u-d}{u-d} - \frac{e^{r\Delta t} - d}{u-d} = \frac{u - e^{r\Delta t}}{u-d}. \tag{5.2.104}$$

Thus,

$$O_0 = PV\left(\frac{e^{r\Delta t} - d}{u-d}O_u + \frac{u - e^{r\Delta t}}{u-d}O_d\right) = PV\left[\pi O_u + \left(1-\pi\right)O_d\right]. \tag{5.2.105}$$

Note,

$$O_0 = PV\left[E_\pi\left(O_T\right)\right], \tag{5.2.106}$$

where $E_\pi\left(\ \right)$ denotes taking the expected value based on the equivalent martingale measure, $\pi$, defined above. Thus, the appropriate probability distribution within the binomial framework has the probability of an up arc occurring is

$$\pi = \frac{FV\left(1\right) - d}{u-d} = \frac{e^{r\Delta t} - d}{u-d}, \tag{5.2.107}$$

and the appropriate discount rate is the "risk-free" interest rate because the hedged portfolio is without risk.

We turn now to explore how dividends influence these results.

*Dividend yield adjusted valuation equation*

The current value of an option is equal to the present value of the expected terminal payout or

$$O_0 = PV\left[E_\pi\left(O_T\right)\right]. \tag{5.2.108}$$

*No arbitrage sketch*: Consider a portfolio of long $\Delta$ stock and borrow $B_0$ (or short a risk-free bond)

$$\Pi_0 = \Delta S_0 - B_0. \tag{5.2.109}$$

At expiration, recall the number of shares of the underlying instrument will grow at $e^{\delta\Delta t}$,

$$\Pi_u = \Delta e^{\delta\Delta t} u S_0 - B_T = O_u \text{ and} \tag{5.2.110}$$

$$\Pi_d = \Delta e^{\delta\Delta t} d S_0 - B_T = O_d. \tag{5.2.111}$$

Thus, rather than owning $\Delta$ shares at expiration, the arbitrageur will own $\Delta e^{\delta\Delta t}$. Making the value of the portfolio risk-free at $T$ implies

$$B_T = \Delta e^{\delta\Delta t} u S_0 - O_u = \Delta e^{\delta\Delta t} d S_0 - O_d. \tag{5.2.112}$$

Solving for $\Delta$, we have

$$\Delta = e^{-\delta\Delta t} \frac{O_u - O_d}{S_0 (u - d)}. \tag{5.2.113}$$

Note that the dividend yield adjusted delta does not significantly change its properties. Again, for call options, $\Delta \geq 0$ and for put options, $\Delta \leq 0$. Entering $\Delta$ shares of stock and writing one option results in the payoff of the risk-free portfolio worth $B_T$. Note that $\Delta \geq 0$ for calls implies purchasing stock and $\Delta \leq 0$ for puts implies short selling stock. Thus, the future value of the initial investment is

$$FV\left(\Delta S_0 - O_0\right) = B_T = \Delta e^{\delta\Delta t} u S_0 - O_u = \Delta e^{\delta\Delta t} d S_0 - O_d, \tag{5.2.114}$$

or

$$FV\left(O_0 - \Delta S_0\right) = O_u - \Delta e^{\delta\Delta t} u S_0. \text{ (future investment value)} \tag{5.2.115}$$

Isolating the initial call option value, we have

$$O_0 = \Delta S_0 - PV\left(\Delta e^{\delta\Delta t} u S_0 - O_u\right). \text{ (dividend yield adjusted no arbitrage model)} \tag{5.2.116}$$

Lemma:

$$\begin{aligned} O_0 &= \Delta S_0 - PV\left(\Delta u e^{\delta\Delta t} S_0 - O_u\right) \\ &= PV\left[\frac{e^{(r-\delta)\Delta t} - d}{u - d} O_u + \frac{u - e^{(r-\delta)\Delta t}}{u - d} O_d\right] \\ &= PV\left[\pi O_u + (1 - \pi) O_d\right] = PV\left[E_\pi\left(O_{\Delta t}\right)\right] \end{aligned} \text{. (equilibrium martingale measure model)} \tag{5.2.117}$$

*Lemma sketch*: Note from the future investment value expression above rearranged

$$FV\left(\Delta S_0 - O_0\right) = \Delta e^{\delta\Delta t} u S_0 - O_u. \tag{5.2.118}$$

Substituting for $\delta$ and solving for $O_0$, we have

$$FV\left(e^{-\delta\Delta t} \frac{O_u - O_d}{u S_0 - d S_0} S_0 - O_0\right) = e^{-\delta\Delta t} \frac{O_u - O_d}{u S_0 - d S_0} e^{\delta\Delta t} u S_0 - O_u. \tag{5.2.119}$$

Canceling $S_0$,

$$FV\left(e^{-\delta\Delta t}\frac{O_u - O_d}{u-d} - O_0\right) = \frac{O_u - O_d}{u-d}u - O_u. \qquad (5.2.120)$$

Taking the present value and rearranging,

$$O_0 = e^{-\delta\Delta t}\frac{O_u - O_d}{u-d} - PV\left(\frac{O_u - O_d}{u-d}u - O_u\right). \qquad (5.2.121)$$

Factoring out the present value factor, multiplying and dividing by $u - d$, and rearranging,

$$O_0 = PV\left[FV\left(e^{-\delta\Delta t}\frac{O_u - O_d}{u-d}\right) - \left(\frac{O_u - O_d}{u-d}u - \frac{u-d}{u-d}O_u\right)\right]. \qquad (5.2.122)$$

Rearranging once again,

$$O_0 = PV\left[\frac{FV\left(e^{-\delta\Delta t}O_u\right) - FV\left(e^{-\delta\Delta t}O_d\right) - uO_u + uO_d + uO_u - dO_u}{u-d}\right]. \qquad (5.2.123)$$

Canceling terms once again and substituting for $FV()$,

$$O_0 = PV\left\{\frac{\left[e^{(r-\delta)\Delta t} - d\right]O_u + \left[u - e^{(r-\delta)\Delta t}\right]O_d}{u-d}\right\}. \qquad (5.2.124)$$

Let,

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u-d}, \qquad (5.2.125)$$

and

$$1 - \pi = 1 - \frac{e^{(r-\delta)\Delta t} - d}{u-d} = \frac{u-d}{u-d} - \frac{e^{(r-\delta)\Delta t} - d}{u-d} = \frac{u - e^{(r-\delta)\Delta t}}{u-d}. \qquad (5.2.126)$$

Thus,

$$O_0 = PV\left[\frac{e^{(r-\delta)\Delta t} - d}{u-d}O_u + \frac{u - e^{(r-\delta)\Delta t}}{u-d}O_d\right] = PV\left[\pi O_u + (1-\pi)O_d\right]. \qquad (5.2.127)$$

Note,

$$O_0 = PV\left[E_\pi\left(O_T\right)\right], \qquad (5.2.128)$$

where $E_\pi(\ )$ denotes taking the expected value based on the equivalent martingale measure, $\pi$, defined above. Thus, the appropriate probability distribution within the binomial framework has the probability of an up arc occurring is

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u-d}, \qquad (5.2.129)$$

and the appropriate discount rate is the "risk-free" interest rate less the dividend yield because the hedged portfolio is without risk.

In the next appendix, we illustrate capturing arbitrage profits within the one period GBM-BOVM.

**Appendix 5.2B. Arbitraging price discrepancies within a one period model**
If the actual market price of the option differs from the model price, an arbitrage is possible. Consider the call option case. If the call can be sold for more than the formula value, Equation (5.2.8), the call is overpriced. Overpriced instruments should be sold. Simply selling the call, however, hardly qualifies as an arbitrage. If the call expires in-the-money, one could incur a significant loss, even though the call were underpriced. Instead, the arbitrage should be completed, and the risk eliminated by holding an offsetting number of units of the stock.

The arbitrageur would, thus, buy $h_c$ units of the stock for each call sold and borrow $B_c$. It should be easy to see that the investment required would be less than what is received from the written call. Convergence of the option value to its exercise value is assured one period later, as the option is expiring and can clearly be worth only its exercise value. With less money invested and the same payoff as before, the rate of return clearly exceeds the risk-free rate. If the option trades at below the formula price, it would be purchased and $h_c$ units of the stock would be sold, creating a net short position. The proceeds would be invested in risk-free bonds to earn the rate $r$. With the option purchased at a lower than fair price, the stock and option would finance the purchase of the risk-free instrument at a lower cost than it should if correctly priced, so the investor would earn an arbitrage profit.

Based on the information given in the module, suppose we have the following market quotes, $c_Q$ = \$11.43 and $p_Q$ = \$10.37. Recall $S_0$ = \$99, $X$ = \$100, $r$ = 0.02, $\tau$ = 1, $u$ = 1.25, and $d$ = 0.8. In equilibrium, we found $c_0$ = \$11.38 and $p_0$ = \$10.42, thus the call price is too high and the put price is too low. Arbitrageurs typically prefer to receive positive cash flow today with no chance of any future liability.

Because the quoted call price is too high, the arbitrageur would sell it and buy the synthetic call option. Buying the synthetic call entails buying the stock with borrowed money. Table 5.2B.1 illustrates capturing the arbitrage profit available with the call option.

***Table 5.2B.1. Cash Flow Table for Single Period ABM Model Applied to Call Options***

| Strategy | Today | Down Event at Expiration | Up Event at Expiration |
|---|---|---|---|
| Sell Call | $+c_{0,q} = +11.43$ | $-\max(0,dS_0 - X) = 0$ | $-\max(0,uS_0 - X) = -23.75$ |
| Buy $h_c$ Shares | $-h_cS_0 = -52.78$ | $+h_c(dS_0) = +42.22$ | $+h_c(uS_0) = +65.97$ |
| Borrow | $+B_c = +41.39$ | $-B_c(1 + r) = -42.22$ | $-B_c(1 + r) = -42.22$ |
| Net Cash Flow | +0.04 | 0 | 0 |

Thus, the arbitrageur receives \$0.04 today with no chance of a future liability. Within this simple one period binomial world, trading pressure will drive down the quoted call price and drive up the quoted stock price until the net cash flow is zero.

If the quoted put price, however, is too low, the arbitrageur would buy it and sell the synthetic put option. Selling the synthetic put entails buying the stock with borrowed money. Table 5.2B.2 illustrates capturing the arbitrage profit available with the put option.

***Table 5.2B.2. Cash Flow Table for Single Period ABM Model Applied to Put Options***

| Strategy | Today | Down Event at Expiration | Up Event at Expiration |
|---|---|---|---|
| Buy Put | $-p_{0,q} = -10.37$ | $+\max[0,X - (S_0 + d)] = +20.80$ | $+\max[0,X - (S_0 + u)] = 0$ |
| Buy $h_p$ Shares | $-h_pS_0 = -46.22$ | $+h_p(dS_0) = +36.98$ | $+h_p(uS_0) = +57.78$ |
| Borrow | $+B_c = +56.65$ | $-B_c(1 + r) = -57.78$ | $-B_c(1 + r) = -57.78$ |
| Net Cash Flow | +0.06* | 0 | 0 |

* Note the quoted price is \$10.37 and the model price is \$10.42, a difference of \$0.05. The table reports an arbitrage profit of \$0.06 The 0.01 discrepancy is simply rounding error.

Thus, the arbitrageur receives \$0.06 today with no chance of a future liability. Within this simple one period binomial world, trading pressure may simply drive up the quoted put price. Alternatively, buying shares may drive up the quoted stock price with some influence on the put price. Ultimately, the initial net cash flow must be zero. There is another arbitrage opportunity based on put call parity, but we will not address it here.

Regardless of the direction of the mispricing, the ability to earn an arbitrage profit would force a price alignment until the option price conforms to the model price.

In the next appendix, we explore various other issues related to dividends.

**Appendix 5.2C Dividends and the binomial lattice valuation approach**

We briefly sketch some of the issues related to the binomial model with a dividend yield.

*Dividend yield adjusted coherence conditions*

The non-dividend binomial option valuation models are derived from this framework and are usually based on the following coherent set of assumptions:

1) $0 < d < e^{(r-\delta)\Delta t} < u$ (no boundary arbitrage condition)
2) $0 << \pi << 1$ (probability condition, distribution independent, not "close" to 0 or 1)
3) $\pi = \frac{e^{(r-\delta)\Delta t} - d}{u - d}$ (no arbitrage condition, distribution independent)
4) $Var_\pi\left(\ln\left(\frac{S_{\Delta t}}{S_0}\right)\right) = \left[\ln\left(\frac{u}{d}\right)\right]^2 \pi(1-\pi)$ (variance condition of log of price relative, distribution independent so long as $S_0 > 0$ and $S_T > 0$)

*u and d conditions*

With these coherence conditions, we can establish the following coherence conditions for u and d:

$$u = \frac{e^{(r-\delta)\Delta t + \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}, \tag{5.2.130}$$

and

$$d = \frac{e^{(r-\delta)\Delta t}}{\pi e^{\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}} + (1-\pi)}. \tag{5.2.131}$$

Based on the use of *u* and *d* above, the coherent lognormal binomial model converges to the dividend yield-adjusted Black–Scholes–Merton option valuation model. Within this framework, we can illustrate the binomial option valuation model with the following simple one period model.

*Dividend yield multiperiod period valuation equation*

As before, the current value of an option is equal to the present value of the expected terminal payout as we assume European-style options where the underlying instrument is adjusted for a continuously compounded cash flow yield.

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U S e^{-\delta T} Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U}, \tag{5.2.132}$$

where the binomial summations are

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{5.2.133}$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{5.2.134}$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_1^j\left(1-\pi_1\right)^{n-j}, \tag{5.2.135}$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a}\left(\frac{n!}{j!(n-j)!}\right)\pi_2^j\left(1-\pi_2\right)^{n-j}, \tag{5.2.136}$$

where the terms are as defined before except

$$\pi = \frac{e^{(r-\delta)T} - d}{u-d}. \tag{5.2.137}$$

Generically, the binomial option valuation model can be expressed as

$$O_0 = PV_r\left[\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left(0,\iota_U u^j d^{n-j}S_0 - \iota_U X\right)\right], \tag{5.2.138}$$

where u and d are defined as

$$u = \frac{e^{(r-\delta)\Delta t + A}}{Den} \text{ and} \tag{5.2.139}$$

$$d = \frac{e^{(r-\delta)\Delta t}}{Den}. \tag{5.2.140}$$

**Binomial option valuation theorem—discrete dividend payment**

We briefly sketch some of the issues related to the binomial model with discrete dividend payments. Mechanically, when a stock goes ex-dividend then the stock price generally falls by the dollar amount of the dividend payment. Ex-dividend refers to the first day that a stock is trading such that if you purchase it, you are not entitled to the dividend payment. The dividend is typically paid a few weeks after the ex-dividend date.

Figure 5.2C.1 reminds us of the recombining nature of the binomial lattice without dividends. We can express the terminal option value generically based on the indicator function, $\iota$.

***Figure 5.2C.1. Multiplicative Two Period Binomial Framework Without Dividends***

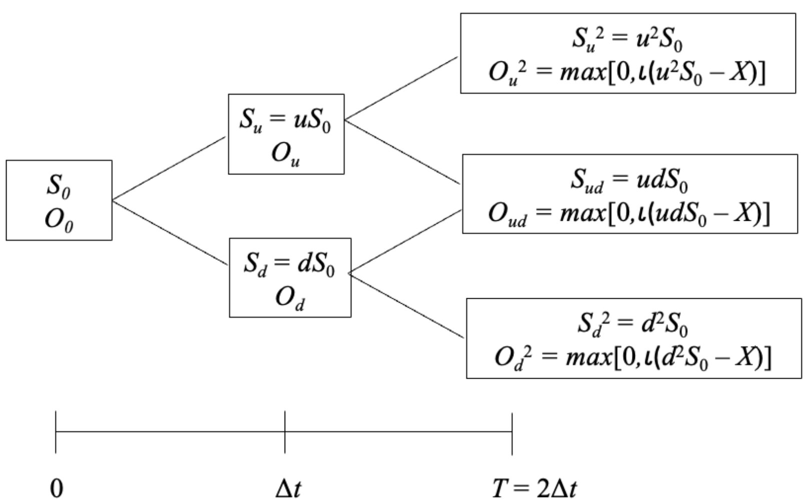
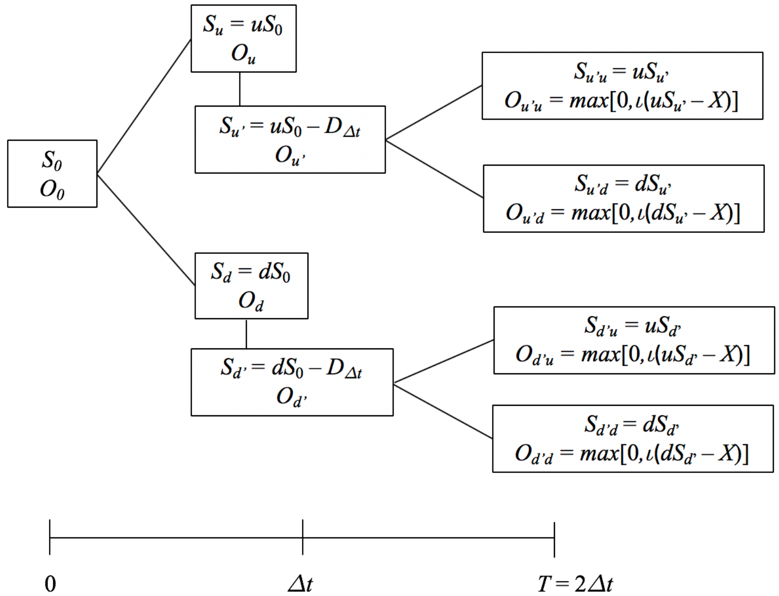



Figure 5.2C.2 illustrates the consequences of discrete dividends within the binomial framework. If we model the underlying stock price, then after the ex-dividend date the lattice will no longer combine. Given that the node count explodes with non-recombining lattices, an alternative solution is sought.

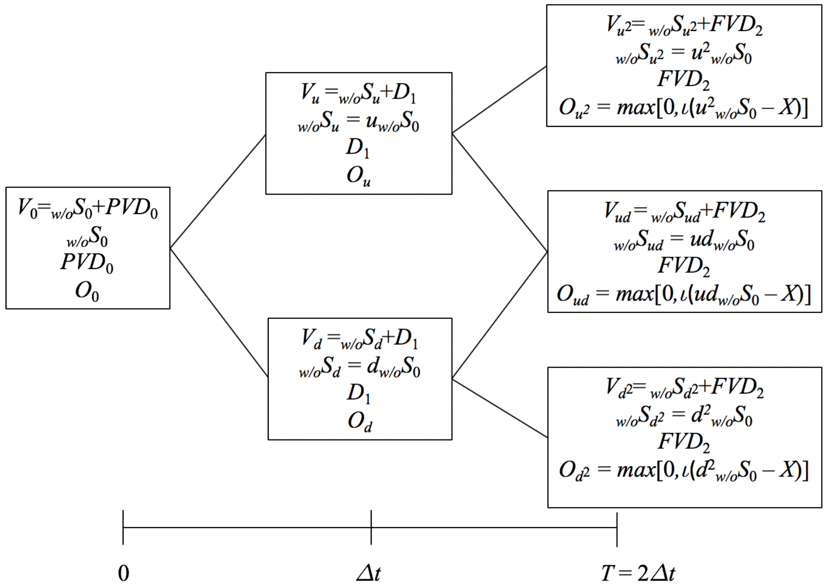
***Figure 5.2C.2. Multiplicative Two Period Binomial Framework with Discrete Dividend***




There is no solution to the discrete dividend problem that addresses all the known issues. The approach adopted here is known as the escrow method. The idea is that we first estimate the discrete dividends that are expected to be paid over the life of the specific option to be valued. Second, we assume that these dividends are known for sure—both timing and dollar amount. Third, using the risk-free discount rate, we estimate the present value of these dividend payments, $PVD_0$.

The escrow method implicitly assumes the company places $PVD_0$ in a bankruptcy-proof trust, guaranteeing its future payment. The remaining stock value without dividends, denoted ${}_{w/o}S_0$, is then modeled within the binomial framework. Thus, purchasing the stock can be viewed as purchasing a portfolio of $V_0 = {}_{w/o}S_0 + PVD_0$. The present value of dividends is assumed to grow at the risk free rate. Once the dividends are paid, the dividend amount is assumed to be invested in the risk-free instrument. Hence, at maturity of the option, the value of an actual stock purchase would be the terminal value of the stock as there are no remaining dividends by definition plus the future value of all dividends paid. The option value at maturity is simply the dollar amount the stock is in-the-money or zero. Figure 5.2C.3 illustrates the application of the escrow method. Notice that the lattice now recombines.

***Figure 5.2C.3. Multiplicative Two Period Binomial Framework With Discrete Dividend (Escrow Method)***




Thus, one of the most significant challenges within a binomial framework is handling any cash flows related to the underlying instrument. We focused here on stocks. If we assume a continuously compounded dividend yield, $\delta$, then the value of a stock investment at time 0 ($V_0 = S_0$) will be worth after time period $\Delta t$, $V_u = ue^{\delta \Delta t}S_0$ and $V_d = de^{\delta \Delta t}S_0$. Thus, the future value of the dividend payment is $D_u = u(e^{\delta \Delta t} - 1)S_0$ and $D_d = d(e^{\delta \Delta t} - 1)S_0$. Alternatively, the stock investment value could be expressed in discrete dollar terms as $V_u = uS_0 + D_u$ and $V_d = dS_0 + D_d$. With the continuous dividend yield assumption, note that $D_u \neq D_d$.

If we assume known discrete dollar dividend at the next point in time, $D_{\Delta t}$, then the value of a stock investment at time 0 ($V_0 = S_0$) will be $V_u = uS_0 + D_{\Delta t}$ and $V_d = dS_0 + D_{\Delta t}$. Note with the discrete dividend assumption, $D_{\Delta t} = D_u = D_d$.

The escrow method simply bifurcates the current stock price into the present value of the known discrete dividend payments and the remaining stock value including the dividend yield component. The escrow method can be thought of as the present value of known discrete dividends is placed in a bankruptcy-proof trust that will be paid for sure and the remaining stock value is stochastic.

*Dividends and American-style options*

Recall, the process to compute the option value is the same at the no dividend case except

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u - d}. \tag{5.2.141}$$

$$u = \frac{e^{(r-\delta)\Delta t + A}}{Den} \text{ and} \tag{5.2.142}$$

$$d = \frac{e^{(r-\delta)\Delta t}}{Den}. \tag{5.2.143}$$

Based on our single period results, we know that at time i for j up moves, the binomial model value (denoted with B superscript) can be expressed as

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + (1-\pi) O_{i+1,j}\right], \tag{5.2.144}$$

The binomial model value, however, may be lower than the early exercise value (denoted with superscript X) that can be expressed as

$$O_{i,j}^{X} = \max\left[0, \iota_U\left({}_{w/o}S_{i,j} + PV_{r,i,n-i}\left(\underline{D}\right) - X\right)\right], \tag{5.2.145}$$

where $\underline{D}$ denotes the vector of future dividend payments and $PV_{r,i,n-i}\left(\underline{D}\right)$ denotes its present value i periods from time 0. Recall the lower boundary condition (denoted with superscript *L*) is

$$O_{i,j}^{L} = \max\left\{0, \iota_U\left[PV_{\delta,i,n-i}\left(S_{i,j}\right) + PV_{r,i,n-i}\left(\underline{D}\right) - PV_{r,i,n-i}\left(X\right)\right]\right\}. \tag{5.2.146}$$

Thus, the fair value of the option at time *i* with *j* up moves is

$$O_{i,j} = \max\left[O_{i,j}^{B}, O_{i,j}^{X}, O_{i,j}^{L}\right]. \tag{5.2.147}$$

The initial option value is obtained through backward induction along the binomial lattice for the underlying instrument.