Module 5.5

Arithmetic Brownian Motion
Option Valuation Model
(ABMOVM)

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Overview

- Review assumptions of ABMOVM
- Explore role of dividends
- Identify different representations of ABMOVM
- Derive the ABMOVM
- Review selected plots
- Compare and contrast GBMOVM and ABMOVM

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Central Finance Concepts

- ABMOVM or GBMOVM
  - Contrast
  - Historical review
  - BSMOVM
  - Key issues
- Empirical evidence
- Limited liability issue

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ABMOVM or GBMOVM

- GBMOVM deeply embedded in practice
- GBMOVM deeply flawed
  - Lognormal distribution (portfolios intractable)
  - Probability(S_T = 0) = 0
- ABMOVM easily address GBMOVM flaws
  - Normal distribution (portfolios tractable)
  - Probability(S_T <= 0) > 0

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Historical Review

- Bachelier (1900) – ABMOVM (arithmetic drift)
- Kruizenga (1952) – foundation of delta hedging, empirical support for ABMOVM
- Osborne (1959) – detailed statistical analysis, could have supported either ABM or GBM, gave early support for GBM, but study deeply flawed

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Early GBM-Based Models

- Sprenkle (1961) – argues negative stock prices unwarranted, hence GBMOVM
- Alexander (1961) – fat tails overshadows any need to distinguish between ABM/GBM
- Boness (1964) – early GBMOVM
- Samuelson (1965) – coined "geometric Brownian motion", call option cannot violate upper bound

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BSMOVM

- Black and Scholes (1972, 1973)
  - Continuous rebalancing
  - No risk adjustment
  - Over (under) priced high (low) variance stocks
- Merton (1973)
  - Rational boundaries
  - Partial differential equation solution

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Zero Value Issue

- Bankruptcy rate = 0.7% per year
- Many underlying instruments can and do have negative values
  - Interest rates
  - Spreads (crack, crush, and various basis)
  - Measurement data (temperature)
- Options in France could have negative strike prices

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Zero Value Issue

- Common stock
  - Unincorporated assets
  - Zero strike put option
- ABMOVM handles negative values easily
  - Estimate probability of zero or below
  - Quantify economic value of limited liability

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Multiple Risk Factors

- Chief Risk Officer's task
  - Identify key set of risk factors
  - Map each balance sheet item to these factors
  - ABM well suited

$$dS = \mu(S,t)dt + \sum_{j=1}^{N_F} \sigma_j(t)dw_j$$

- GBM is not well suited for multiple factors

$$dS = \mu(S,t)dt + \sum_{j=1}^{N_F} \sigma_j(t)Sdw_j$$

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Portfolio Aggregation

- Multifactor ABM-based stock value

$$dS_i = \mu(S_i,t)dt + \sum_{j=1}^{N_F} \sigma_{i,j}(t)dw_j$$

- Portfolio of stocks

$$\Pi = \sum_{i=1}^{N_I} N_i S_i$$

- Diffusion of portfolio

$$d\Pi = \sum_{i=1}^{N_I} N_i dS_i = \sum_{i=1}^{N_I} N_i \left[\mu(S_i,t)dt + \sum_{j=1}^{N_F} \sigma_{i,j}(t)dw_j \right]$$

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Greeks (see Module 8.3, 8.4)

- Model Greeks based on empirical observations
  - Crash of October 1987, BSM suffered a permanent loss of confidence
  - Implied volatility surface emerged
  - No unanimity of opinion on best model
  - Best model is one that leads to best decision-making

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Relative or Absolute Risk

- Example: $100 stock falls to $50
  - Based on bad news
    - Relative volatility implies lower absolute risk
    - Absolute volatility implies higher relative risk
  - Based on 2 for 1 stock split
    - Relative volatility automatically handles
    - Absolute volatility must divide by 2
- GBM ignores the leverage effect of falling prices

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Extreme Volatilities

- Lognormal
  - Mean: $\exp(\mu + \sigma^2/2)$
  - Median: $\exp(\mu)$
  - Mode: $\exp(\mu - \sigma^2)$
- Normal
  - Mean = Median = Mode = μ
- High volatilities results in unstable lognormal distribution

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Binomial Convergence

- GBMOVM
  - Recombining
  - Multiplicative
- ABMOVM
  - Recombining
  - Additive
- Either model can handle early exercise

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Homogeneity of Degree 1

- GBMOVM with S and X
  - Stock splits automatically handled
  - 2S, 2X implies 2C and 2P
- ABMOVM with S, X and σ
  - Stock splits must adjust σ
  - 2S, 2X, 2σ implies 2C and 2P

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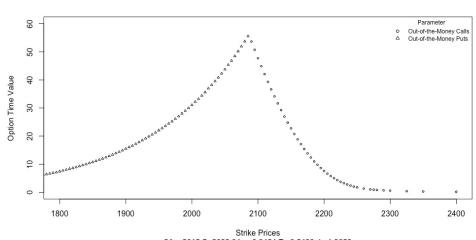
Empirical Evidence



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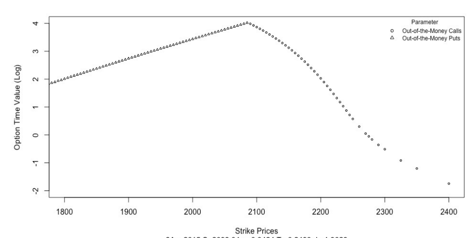
Log Transform of Time Value



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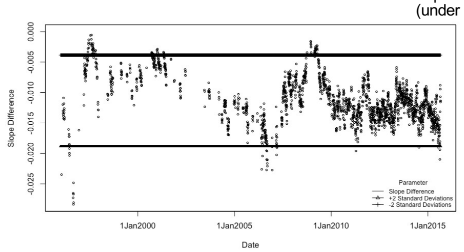
Analysis of Slopes

H0: Slope Diff > 0
(under GBM)



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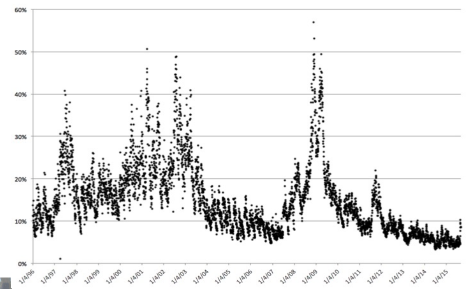
Analysis of Implied Volatility ABM % Improvement Over GBM



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ABMOVM's Rebuttal Addressing Limited Liability

- S_{UL} – underlying instrument without LL
- $S = S_{LL}$ – underlying instrument with LL
- $S_{LL} = S_{UL} + P(S, X = 0)$
- Example: S_{UL} = \$10, σ_A = \$30, $T - t$ = 1.0, r = 5% and δ = 0%
  - $P(S, X = 0)$ = \$7.35
  - $S_{LL} = S_{UL} + P(S, X = 0)$ = \$17.35

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Example

- S_{UL}=\$10; X=\$10; T–t=1; r=5%; σ_{UL}=\$30
  - C = \$11.92; P = \$11.43
  - Apparent violation of upper bounds
- S_{LL}=\$17.35 [$P(S_{UL}, X$=0)=\$7.35]
  - X=\$10 Call: Short X=0 put and long X=\$10 call with cost of \$4.57
  - X=\$10 Put: Bear spread (Short X=0 put and long X=\$10 put) with cost of \$4.08

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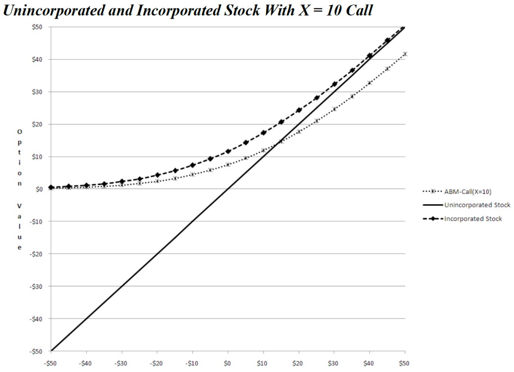
Addressing Limited Liability

Figure 5.5.5 Unincorporated and Incorporated Stock With X = 10 Call



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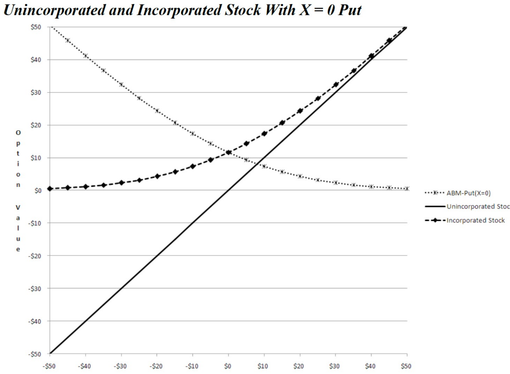
Addressing Limited Liability

Figure 5.5.6 Unincorporated and Incorporated Stock With X = 0 Put



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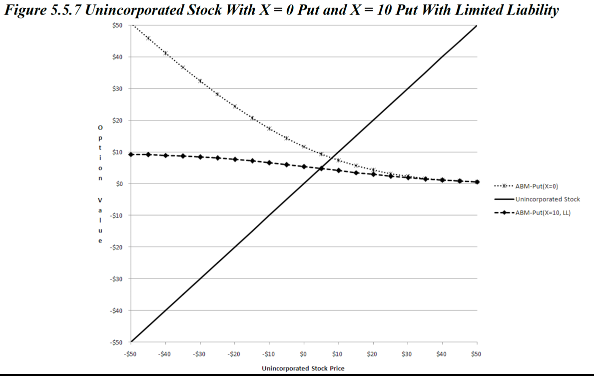
Addressing Limited Liability

Figure 5.5.7 Unincorporated Stock With X = 0 Put and X = 10 Put With Limited Liability



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Limited Liability

- S_{UL} – stock with unlimited liability
- S_{LL} – stock with limited liability
  - $S_{LL} = S_{UL} + P(S_{UL}, X=0)$
  - GBM: $S_{LL} = S_{UL}$ or $P(S_{UL}, X=0) = 0$
  - ABM: $P(S_{UL}, X=0) > 0$ or $S_{LL} > S_U$
- When $S_{UL}/\sigma < 3.1 \Rightarrow P(S_{UL}, X=0) < 0.01$

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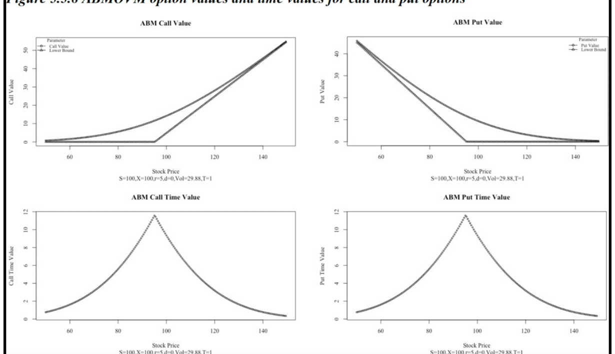
Figure 5.5.8 ABMOVM option values and time values for call and put options



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Quantitative Finance Materials

- Review key assumptions
- Introduce ABMOVM
- Dividends
- Derive ABMOVM
- Illustrative graphs

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ABMOVM Key Assumptions

- Terminal distribution is *normal*
- Risk-free rate is constant, borrowing and lending allowed
- Volatility of the underlying instrument's annualized dollar change is known, positive, and constant
- European-style options only

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Dividends (Same as GBM)

- Escrow method assumed
- All dividends over option life extracted

$$PV_T(\underline{D}) = S_0\left(1 - e^{-\delta T}\right) + \sum_{i=1}^{N} PV_{\tau_i}(D_i)$$

- Underlying instrument sans dividends

$$S_0' = S_0 - PV_T(\underline{D}) = S_0 - \left[S_0\left(1 - e^{-\delta T}\right) + \sum_{i=1}^{N} PV_{\tau_i}(D_i)\right] = B_\delta S_0 - \sum_{i=1}^{N} PV_{\tau_i}(D_i)$$

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ABMOVM

$$O(S_0',t;\iota_U,X,T,r,\sigma)=PV_r\left\{E_0\left[O\left(FV_rS_t',T\right)\right]\right\}$$
$$=B_r\left[\iota_U\left(S_0'B_{-r}-X\right)N\left(\iota_Ud_n\right)+\sigma_An\left(d_n\right)\right]=\iota_U\left(S_0'-B_rX\right)N\left(\iota_Ud_n\right)+B_r\sigma_An\left(d_n\right)$$

$$\iota_U=\begin{cases}+1 \text{ if underlying call option}\\ -1 \text{ if underlying put option}\end{cases}\qquad B_r=e^{-r}\qquad N(d)=\int_{-\infty}^{d}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$$

$$\sigma_A^2=\sigma^2\frac{B_{-2r}-1}{2r}\qquad d_n=\frac{S_t'B_{-r}-X}{\sigma_A}$$

Dividend Yield Only

- Call model

$$C_0=\left(S_0e^{-\delta T}-e^{-rT}X\right)N\left(d_n\right)+e^{-rT}\sigma_An\left(d_n\right)$$

- Put model

$$P_0=\left(e^{-rT}X-S_0e^{-\delta T}\right)N\left(d_n\right)+e^{-rT}\sigma_An\left(d_n\right)$$

Derivation of ABMOVM

- Arithmetic Brownian motion with geometric drift

$$dS=\mu Sdt+\sigma dw$$

- Itô's lemma, C(S,t),

$$dC=\left(\mu S\frac{\partial C}{\partial S}+\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2}\right)dt+\frac{\partial C}{\partial S}\sigma dw$$

Risk-free portfolio design

- Sell 1 call and buy delta stock

$$\Pi=-C+\frac{\partial C}{\partial S}S\qquad d\Pi=-dC+\frac{\partial C}{\partial S}dS+q\frac{\partial C}{\partial S}Sdt$$

- Hedged portfolio result (SFDR)

$$d\Pi=-\left(\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2}-q\frac{\partial C}{\partial S}S\right)dt$$

- Risk free growth implied

$$d\Pi=r\Pi dt=r\left(-C+\frac{\partial C}{\partial S}S\right)dt$$

ABM PDE

- ABM partial differential equation

$$rC=\frac{\partial C}{\partial t}+\left(r-q\right)\frac{\partial C}{\partial S}S+\frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2}$$

- Boundary condition

$$C\left(S,t=T\right)=\max\left(0,S_T-X\right)$$

Summary

- Review assumptions of ABMOVM
  - Normal distribution
  - Arbitrage (SFDR)
- Explored role of dividends
- Identified different representations of ABMOVM
- Derived the ABMOVM
- Reviewed selected plots