Module 5.3: Arithmetic Brownian Motion-Based Binomial Models

Learning objectives

- Develop the binomial lattice approach to valuing ABM-based options
- Explore the challenge posed by complex probabilities of observing terminal nodes
- Computing European-style option values using one ABM binomial option valuation approach
- Computing American-style option values using one ABM binomial option valuation approach

Executive summary

Arithmetic Brownian motion (ABM) results in a normally distributed terminal distribution. In this module, we derive a binomial model that converges to the arithmetic Brownian motion option valuation model (ABMOVM).

A lattice approach to valuing various options consistent with a normal terminal distribution is presented in this module. Recall in the last module, the terminal distribution was lognormal. Again, a lattice refers to how some underlying instrument's value may change discretely over the next time step. The valuation approach is based on dynamic arbitrage. Dynamic arbitrage is based on the capacity to continuously rebalance a custom-designed portfolio.

In this module, we present the non-traditional binomial valuation model we refer to as the arithmetic Brownian motion binomial option valuation model or ABM-BOVM. The key weakness of the ABM-BOVM is the need to use backward recursion with European-style options. We argue that like tools in a toolbox for the quantitative analyst, the varied challenges analysts face will warrant the unique tool selected. Unorthodox tools often prove vital with particularly challenging tasks.

Central finance concepts

We again seek an exact option value based on a valuation model. In this module, the focus is on a model that converges to ABMOVM in the limit.

Our strategy here is to apply a lattice-based approach. Again, in finance, a lattice refers to how some underlying instrument's value may change discretely over the next time step. Here we present the ABM-BOVM in this module and illustrate valuing plain vanilla options as well as digital options. We also apply this model to both European-style and American-style calls and puts.

The binomial option framework presented in this module is designed to converge to a normal distribution. In the limit, the ABM-BOVM will not be consistent with the BSMOVM. Like the GBM-BOVM, this model has several objectives that will be further developed in this module, including

1. Additive,
2. Recombining,
3. Incorporate dividends (discrete and continuous), and
4. Address early exercise with American-style options.

Objectives 2-4 are like GBM-BOVM. Additive and recombining are incorporated with u and d parameters at each node. These parameters are no longer total return, but rather expressed in currency units such as dollars. As we will see, the discrete dividend payments are easily handled with ABM-BOVM.

ABM one period binomial lattice framework

Many of the more complex concepts can be easily understood within the context of a simple one period model. The technical details of the single period model presented below is not realistic, but it lays a solid framework for understanding the both the multiperiod model presented here as well as continuous models presented in Modules 5.4, 5.5, and 5.7.

ABM European-style option two period model

The single period model can be extended to multiple periods and thereby accommodate options with longer lives or smaller time steps. Note that by design, described in detail below, the lattice is recombining based on an additive process. Thus, in a single period model there are two potential future outcomes, whereas in the two period model there are three potential outcomes. There will be two different paths where one arrives at the middle node after two periods.

Two key features of the two period binomial model are the recombining nature of the tree and the underlying instrument change is driven by an additive process. The tree is recombining because the stock price is assumed to grow in an additive fashion. The additive approach presented in this module facilitates the convergence of the stock price to the normal distribution.

ABM American-style option two period model

With American-style options, a backward recursion approach is taken within the lattice. At each node where time remains on the option, three conditions are appraised where the highest value is selected. First, the value of the option assuming the option is not early exercised is computed. Second, the cash value of immediately exercising is computed. Finally, the lower boundary of the option is computed. The maximum of these three values is placed in the lattice and the evaluation continues.

Thus, American-style options will not trade for less than their European-style counterparts. Given the vast number of different binomial frameworks possible, we explore guidelines known as coherence conditions.

ABM coherence conditions

The ABM-BOVM presented technically below is designed to converge to a normal distribution. Again, it is inconsistent with the Black–Scholes–Merton option valuation model (BSMOVM). The lattice will be built additively. That is, the value of the underlying at some future date is found by adding certain parameters. In the previous module, we introduced another binomial model that converges to the lognormal distribution. In that case, the lattice was built multiplicatively. That is, the value of the underlying at some future date is found by multiplying certain parameters.

Vital to all lattice frameworks is the need to have the lattice recombine over maturity time. In the binomial cases, the goal is to have the number of futures states grow linearly. In the binomial framework, the number of future states increase by one with each additional point in time in the lattice.

There have been numerous lattice-based option valuation models posited over the past several decades. Many of these models are not internally coherent, often admitting simple arbitrage opportunities even within the sterile theoretical environment. Seeking to thwart that potential, a set of four coherence conditions have been offered. If all four of these coherence conditions are satisfied, then the lattice model is at least internally coherent.

Although presented in detail later, we briefly sketch the coherence conditions here. First, the no arbitrage boundary condition requires that the total return from investing in the risk-free interest rate be greater than the total return on the risky instrument if the down state occurs as well as total return from investing in the risk-free interest rate be less than the total return on the risky instrument if the up state occurs. Second, there is a technical condition on the probability of an up move that it cannot be too close to either zero or one. Third, there is a mathematical relationship between the assumed probability of an up move and the values of the up and down parameters. Finally, there is a technical requirement that forces the local variance within the lattice to exactly equal to the assumed variance of the normal distribution in the limit.

Though highly technical, the coherence conditions are deeply useful when exploring alternative models for actual implementation.

Dividends

Dealing with discrete cash flows paid to the underlying instrument is straightforward for additive lattice models that converge to the normal distribution in the limit. These lattice trees can be built to recombine posing significant benefits related to practical implementation.

Again, one easy approach is the escrow method introduced below for handling dividend payments simply divides the current stock price into the present value of the known discrete dividend payments and the remaining stock value including any potential dividend yield component. The term escrow suggests that the present value of known discrete dividends is placed in a bankruptcy-proof trust that will be paid for sure and the remaining stock value is stochastic. Companies do not actually do this, but it is a conceptual framework for dividends.

ABM European-style multiperiod option model

Several different ABM-based multiperiod binomial lattice frameworks are covered here. We review both the European-style and American-style models both with and without various forms of dividends.

Note that due to the use of backward recursion, we do not have to apply the log transformation introduced in the last module. Unfortunately, backward recursion requires more computational time.

ABM American-style multiperiod option model

With American-style options, a backward recursion approach is taken within the lattice. At each node where time remains on the option, three conditions are appraised where the highest value is selected. First, the value of the option assuming the option is not early exercised is computed. Second, the cash value of immediately exercising is computed. Finally, the lower boundary of the option is computed. Again, the maximum of these three values is placed in the lattice and the evaluation continues.

Backward recursion is necessary with both the European-style and American-style ABM-BOVM. At this time, there are no efficient ways to solve for European-style option values.

We now review selected graphical results based on the quantitative models developed below along with the corresponding R code.

ABM-BOVM European-style results

The time value plots highlight the normal distribution's zero skewness or symmetry. Recall in Module 5.1, we introduced lower and upper boundaries. These boundaries are independent of assumed underlying distribution. Figure 5.3.1 is based on an assumed stock price of 100, exercise price of 100, risk-free interest rate of 5% (continuously compounded), volatility of $29.8848, dividend yield of 0%, and time to maturity of 1 year.[1]

Panel A illustrates the convergence to the lower boundary as the stock price declines (zero for call and the present value of the exercise price less the stock price for the put) as well as the convergence to the lower boundary as the stock price increases (stock price less the present value of the exercise price for the call and zero for the put).

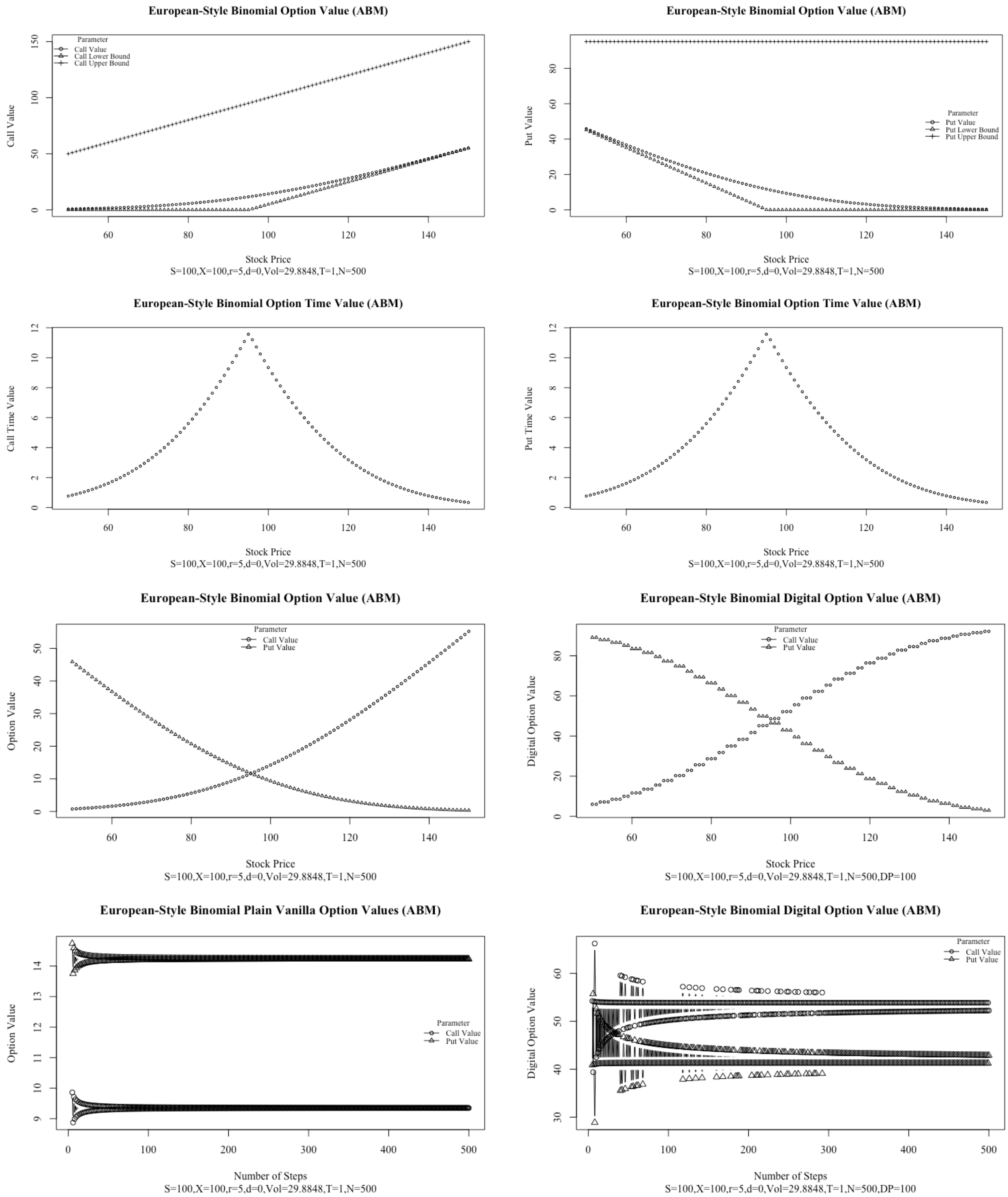
Panel B draws attention to just the time value. Upon careful inspection, we see the time values are identical and have no skewness. That is, the time values are symmetrical. The mode (peak) is technically at the present value of the exercise price. Recall based on actual option data, we documented significant observed negative skewness.

Panel C illustrates calls and put together for both plain vanilla options and digital options. The digital options are cash-or-nothing and pays the exercise price if the underlying is in-the-money at expiration. The stepped mapping with digitals reflects the use of 500 time steps introducing minor discontinuities.

Panel D shows the convergence properties for both plain vanilla and digital options as we increase the number of time steps. Note that digital options are a bit unstable for smaller number of steps.

[1]The volatility was selected to yield the same initial option values as 30% relative volatility with GBMOVM or BSMOVM.

Figure 5.3.1 Selected ABM-BOVM Results: European-Style with no Dividends



For comparison with the previous module, the dividends to be incorporated are based on a constant dividend yield—a common approach to handling interim cash flows of an underlying instrument. Figure 5.3.2 was computed based on the same parameters above, but the dividend yield equals the interest rate of 5% (annualized, continuously compounded).

Comparing Panel A here with the prior no dividend case we see the lower boundary now changes at the current exercise price rather than at the present value of the exercise price. This effect is due to arbitrageurs being able to purchase less stocks due to dividend receipts to hedge their position.

Panel B is also like the no dividend case, except the mode is at the current exercise price. Compared to GBM-BOVM, the time value is symmetric and not positively skewed.

Panel C is also similar, but the intersection point for both the plain vanilla and digital options is at the current exercise price.

Finally, Panel D shows eventual convergence, but it is much less stable in both cases.

Figure 5.3.2 Selected ABM-BOVM Results: European-Style with Dividends

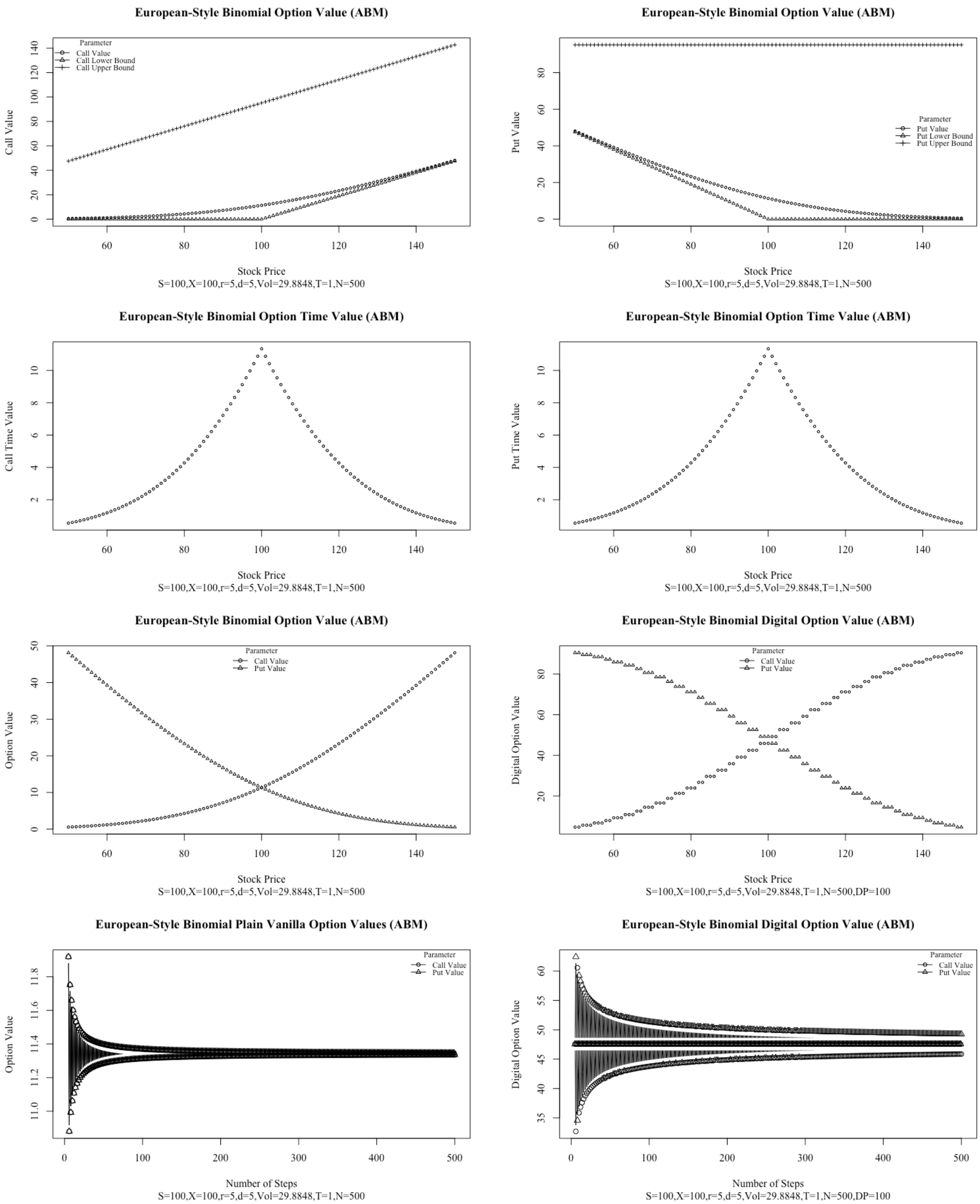


Figure 5.3.3 presents values derived from both the European-style and American-style option valuation model with no dividends. Note that without dividends, we see from Panel A that call options are never exercised early; hence, the American-style (AS) call value is identical to the European-style (ES) call value. The lower boundary conditions for AS calls and ES calls are also the same. The same cannot be observed for

puts. Due to arbitrage forces, AS put values are worth more than ES put values, particularly noticeable when the put options are in-the-money. Further, notice that put values converge to the appropriate lower boundary conditions.

Panel B provides the same format, except focused solely on option time value. Again, we see there is no difference between AS and ES call option time values whereas there is significant difference between AS and ES put option time values. Although, ES put values are lower, due to the lower boundary effect, the ES put time values are higher than AS put time values. Like GBM-BOVM, the early exercise feature has a material effect on non-dividend paying put options based on the ABM-BOVM. We again observe the symmetric pattern of time values with ABM-BOVM.

Panel C combines AS and ES as well as puts and calls. The left-hand side shows the plain vanilla options, and the right hand side shows the digital cash-or-nothing options. Obviously, the early exercise feature of AS digital options has a profound impact on option values.

Figure 5.3.3 Selected ABM-BOVM Results: American-Style with no Dividends

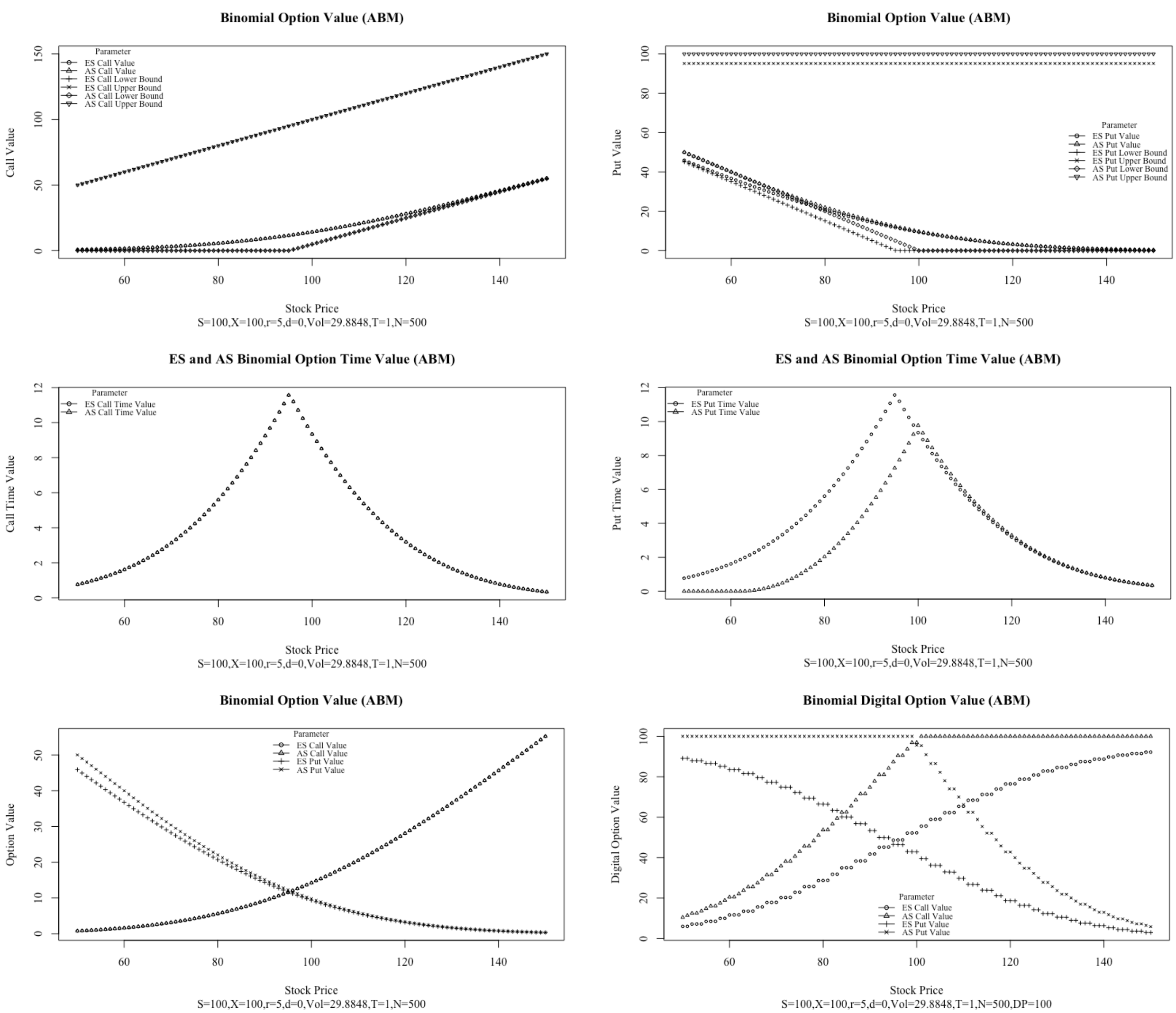


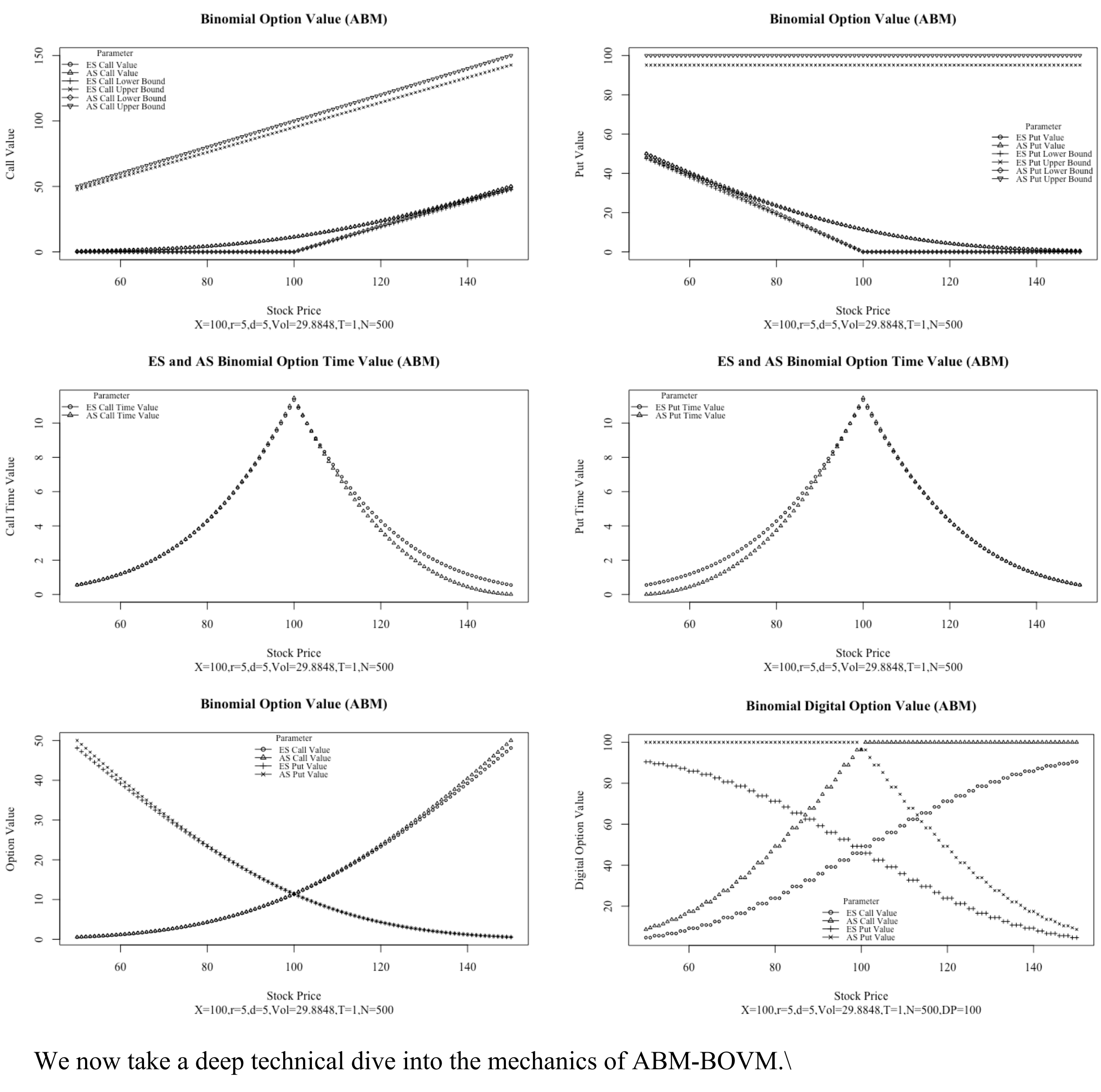
Figure 5.3.4 illustrates option values derived from both the European-style and American-style option valuation model with dividends. Here we assume the dividend yield equals the interest rate of five percent.

Note that with dividends, we see a similar result as observed with GBM-BOVM. In Panel A, note that call options are potentially exercised early; hence, the American-style (AS) call values are no longer identical to the European-style (ES) call values when the call options are deep in-the-money. The lower boundary conditions for AS calls and ES calls are no longer the same. Due to arbitrage forces, both AS call values and AS put values are worth more than ES call values and ES put values, respectively. This difference is noticeable when the options are in-the-money. Also, notice that both the call and put valuation models for both AS and ES options converge to their appropriate lower boundary conditions.

Panel B provides the same format, except focused solely on option time value. We see here that both calls and puts differ between AS and ES option time values when the options are deep in-the-money. Although, ES option values are lower, due to the lower boundary effect, the ES time values are higher than AS time values. Although the early exercise feature has a material effect on dividend paying options based on the ABM-BOVM, the impact on puts is diminished because dividends have on the lower boundary condition.

Panel C combines AS and ES as well as puts and calls. Again, the left-hand side shows the plain vanilla options, and the right hand side shows the digital cash-or-nothing options. As before, the early exercise feature of AS digital options has a profound impact on option values.

Figure 5.3.4 Selected ABM-BOVM Results: American-style with Dividends



We now take a deep technical dive into the mechanics of ABM-BOVM.\

Quantitative finance materials

We repeat the notation given in Module 5.2 with some modifications (u and d). Like the GBM-based binomial model, the notation used in this module is extensive, so we first explicitly define all the variables used.

ABM notation review

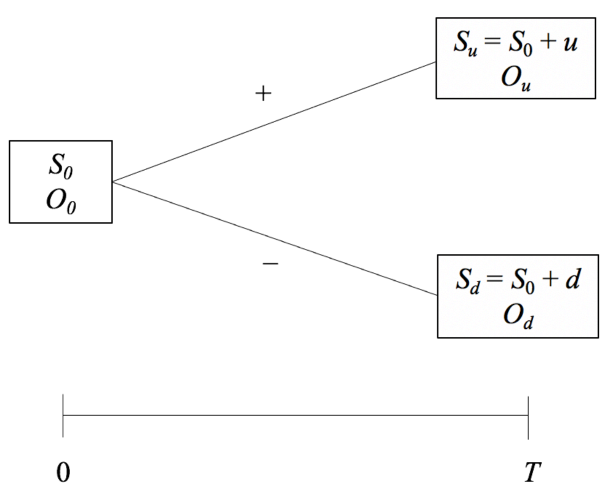
$0, T, \Delta t$	initial trade date, time 0; expiration or maturity date, time T; next time step,
S_0, S_T	value of underlying instrument, e.g., stock, at time 0 and at time T,
u, d	up, dollar change in S, if up occurs ($u > 0$) and if down occurs ($d < 0$),
B_0, B_T	bond, value of risk-free investment at time 0 and at time T,
V_0, V_T	portfolio, value of some financial instrument portfolio at time 0 and at time T,
ι	indicator function, +1 for calls and –1 for puts,
O_0	option, value of options, either call or put, at time 0,
O_u, O_d	option, value of option at time T if up occurs and if down occurs,
Δ	delta, hedge ratio, units of the financial instrument to enter to hedge option position,
$FV()$	future value based on risk-free interest rate,
$PV()$	present value based on risk-free interest rate,
π	equivalent martingale probability of up move,
$E_\pi()$	expectation under equivalent martingale probability,
r	discretely compounded, periodic "risk-free" interest rate,
r_c	continuously compounded, annualized, "risk-free" interest rate,
δ	continuously compounded, annualized, dividend yield, and
D_T	known discrete dividend amount paid at time T (ex-dividend the instant *before* the next binomial point in time).

ABM one period binomial option model

We specify an underlying instrument priced at S_0 that can go up to $S_0 + u$ (up state) or down to $S_0 + d$ (down state), where $u > 0$ and $d < 0$. Given the desire to converge to a normal distribution and not the lognormal distribution, we pursue an additive binomial approach. Note that u and d are expressed in currency units, such as dollars, and not total return as with the GBM-based approaches.

Figure 5.3.5 illustrates the additive single period binomial framework where O denotes a generic (call or put) option value. At the initial point in time, there is only one node whereas at the next point in time there are only two nodes. Also, at the initial point in time, there are two arcs emanating from the initial node, hence the name binomial.

Figure 5.3.5 Additive One Period Binomial Framework

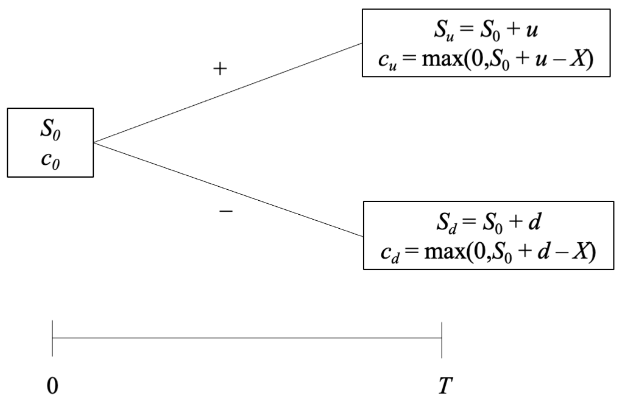


Again, changes in S are additive; hence, this process is called an additive binomial tree. Consider a generic option with exercise price X that expires in one period. The two possible values for the generic option at expiration are

$$\begin{aligned} O_u &= \max\left[0, \iota\left(S_0 + u - X\right)\right] \\ O_d &= \max\left[0, \iota\left(S_0 + d - X\right)\right] \end{aligned}. \tag{5.3.1}$$

Of course, our objective is to determine the current option value denoted generically as O.
ABM one period call option binomial model
The basic layout with the corresponding call option prices inserted at each node is in Figure 5.3.6.

Figure 5.3.6 Additive One Period Call Option Binomial Framework



A portfolio consisting of the option and the underlying instrument is created in such a way that it is hedged. That is, the future value is known for certain and therefore should earn the risk-free rate. We can then solve for the price of the call option that is consistent with a risk-free return. Let us buy h_c units of the instrument and sell one call. The value of this portfolio today (V_0) is

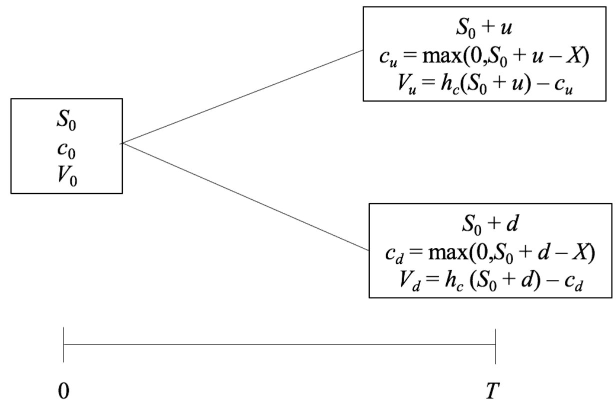
$$V_0 = h_c S_0 - c_0. \tag{5.3.2}$$

The value at expiration of this portfolio in the two future states is

$$\begin{aligned} V_u &= h_c\left(S_0 + u\right) - \max\left(0, S_0 + u - X\right) = h_c\left(S_0 + u\right) - c_u \\ V_d &= h_c\left(S_0 + d\right) - \max\left(0, S_0 + d - X\right) = h_c\left(S_0 + d\right) - c_d \end{aligned}. \tag{5.3.3}$$

Figure 5.3.7 illustrates the process thus far. The top row is the underlying instrument's price process. The second row illustrates the call option's price process. Finally, the third row illustrates the portfolio value's process, where the portfolio is comprised of h_c units of the underlying instrument and short one call option.

Figure 5.3.7 ABM Binomial Process for Underlying Instrument, Call Option, and Hedge Portfolio



Up to this point, we have two instruments and have entered opposite exposures. Specifically, we are long the underlying instrument and short the call. We now introduce a third instrument, the risk-free instrument. If the portfolio represented by V can exactly replicate a risk-free instrument, it must produce a risk-free return, meaning that these two outcomes are the same, as specified by the terminal value condition,

$$V_u = V_d. \tag{5.3.4}$$

If we set the terminal portfolio values equal to each other, we have one equation with only one unknown, $h_c(S_0 + u) - c_u = h_c(S_0 + d) - c_d$ whose solution can be expressed as

$$h_c = \frac{c_u - c_d}{S_0 + u - \left(S_0 + d\right)} = \frac{c_u - c_d}{u - d}. \tag{5.3.5}$$

This result is known as the optimal hedge ratio. Specifically, it tells us how many underlying instruments to buy for every call written. The sign of h_c will be positive as $c_u > c_d$ and $u > d$. Recall we assume $u > 0$ and $d < 0$, hence $u - d > 0$. Thus, if the number of units of the underlying instrument that we hold is set to h_c, the two future values of the instrument will be identical. Hence, the portfolio is risk-free. To avoid arbitrage, the portfolio must be priced to earn the risk-free rate. The discretely compounded periodic risk-free rate is denoted r. Thus, the following condition must hold:

$$V_0 = \frac{V_u}{1 + r} = \frac{V_d}{1 + r}. \tag{5.3.6}$$

Consequently, we can substitute into Equation (5.3.6), using either V_u or V_d. We will choose V_u, thus

$$\frac{h_c\left(S_0 + u\right) - c_u}{1 + r} = h_c S_0 - c_0. \tag{5.3.7}$$

Therefore, the initial call price can be represented based on the *no arbitrage model* as

$$c_0 = h_c S_0 - B_{0,c}, \tag{5.3.8}$$

where

$$B_{0,c} = \frac{h_c\left(S_0 + u\right) - c_u}{1 + r}. \tag{5.3.9}$$

Thus, a call option can be replicated by purchasing h_c units of the underlying instrument partially financed through borrowing of $B_{0,c}$. From this analysis, a call option is simply a leveraged position in the underlying instrument.

To solve for the *equivalent martingale measure model*, the next step is to insert the solution for h_c, Equation (5.3.8), and solve for c_0:

$$c_0 = PV\left[E\left(c_T\right)\right] = \frac{\pi_0 c_u + \left(1-\pi_0\right)c_d}{1+r}, \tag{5.3.10}$$

where the equivalent martingale measure probability is time and state dependent[2]

$$\pi_0 = \frac{S_0 r - d}{u-d}. \tag{5.3.11}$$

The derivation of Equation (5.3.10) is provided in Appendix 5.3A. We emphasize that the equivalent martingale measure probability is dependent on time and state due to its dependence on the underlying instrument's price. This will result in a less efficient valuation procedure when compared to GBM-BOVM.

Again, another view is that the call price is simply the present value of the expected future call payoffs discounted at the risk free rate. The probabilities used in forming the expectations, however, are not the real probabilities. They are based on the equivalent martingale measure or the risk neutral probabilities.

ABM one period call option binomial model example

For example, suppose the current stock price is \$99, the strike price is \$100, the annual, discretely compounded, risk free rate is 2%, the time to expiration is one year, u = \$24.75, and d = –\$19.8. We can compute the call price in two ways. First, note:

$$c_u = \max(0, 123.75 - 100) = 23.75 \text{ and}$$
$$c_d = \max(0, 79.2 - 100) = 0.$$

For the no arbitrage model, we first find the hedge ratio

$$h_c = (c_u - c_d)/(u - d) = (23.75 - 0)/(24.75 - -19.8) = 23.75/44.55 = 0.5331.$$

Therefore, based on Equation (5.3.8), we have

$$\begin{aligned} c_0 &= h_c S_0 - \frac{h_c\left(S_0+u\right)-c_u}{1+r} \\ &= 0.5331\left(99\right) - \frac{0.5331\left(99+24.75\right)-23.75}{1+0.02} \\ &= 52.7769 - 41.3933 = 11.38 \end{aligned}$$

Alternatively, we can use apply the risk neutral model. The binomial probability of an up move is

$$\begin{aligned} \pi &= \frac{1+r-d}{u-d} \\ &= \frac{99\left(0.02\right)-\left(-19.8\right)}{24.75-\left(-19.8\right)} = \frac{21.78}{44.55} = 0.488889 \end{aligned}.$$

Therefore, based on Equation (5.3.10), we find the same results or

$$c_0 = \frac{0.4889\left(23.75\right)+\left(1-0.4889\right)0}{1+0.02} = 11.38.$$

ABM one period put option binomial model

Following the structure from the previous sections on calls, the basic layout with the corresponding put option prices inserted at each node is in Figure 5.3.8.

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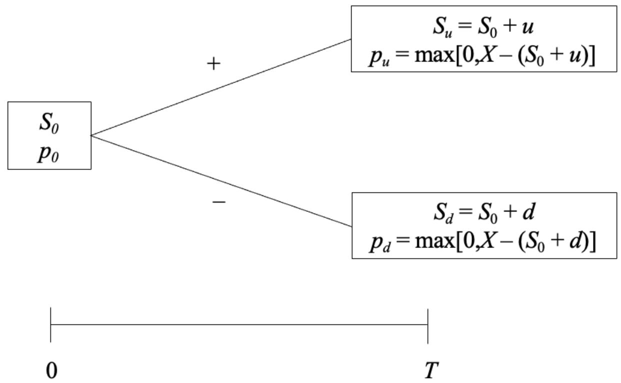
[2]The dependency on the level of S will prove a bit challenging when deriving European-style option values.

Figure 5.3.8 Additive One Period Put Option Binomial Framework



As with calls, a portfolio consisting of the put option and the underlying instrument is created in such a way that it is hedged. That is, the future value is known for certain and therefore should earn the risk-free rate. We can then solve for the price of the put option that is consistent with a risk-free return. Let us buy h_p units of the underlying instrument and buy one put. Note that to hedge, we need to be on the same side of the market. Here, we show buying both the underlying instrument and buying the put. The value of this portfolio today (V_0) is

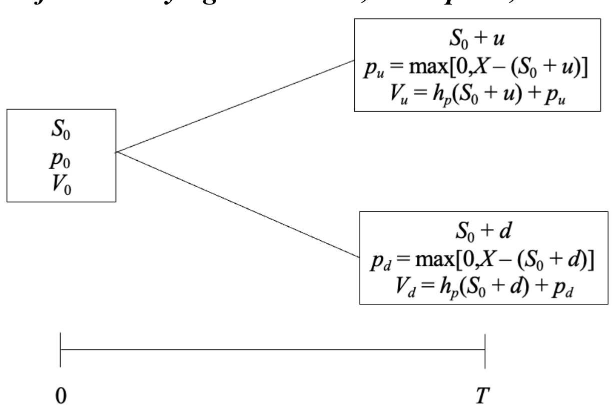
$$V_0 = h_p S_0 + p_0. \tag{5.3.12}$$

The value at expiration of this portfolio in the two future states are

$$\begin{aligned} V_u &= h_p\left(S_0 + u\right) + \max\left[0, X - \left(S_0 + u\right)\right] = h_p\left(S_0 + u\right) + p_u \\ V_d &= h_p\left(S_0 + d\right) + \max\left[0, X - \left(S_0 + d\right)\right] = h_p\left(S_0 + d\right) + p_d \end{aligned}. \tag{5.3.13}$$

Figure 5.3.9 illustrates the process thus far. The top row is the underlying instrument's price process. The second row illustrates the put option's price process. Finally, the third row illustrates the portfolio value's process, where the portfolio is comprised of h_p units of the underlying instrument, and one put option.

Figure 5.3.9 Binomial Process for Underlying Instrument, Put Option, and Hedge Portfolio



Up to this point, we have two instruments and have entered similar (long) exposures. Specifically, we are long the underlying instrument and long the put. We now introduce a third instrument, the risk-free instrument. If the portfolio represented by V can exactly replicate a risk-free instrument, it must produce a risk-free return, meaning that these two outcomes are the same, as specified by the terminal value condition,

$$V_u = V_d. \tag{5.3.14}$$

If we set the terminal portfolio values equal to each other, we have one equation with only one unknown,

$$h_p\left(S_0+u\right)+p_u=h_p\left(S_0+d\right)+p_d, \tag{5.3.15}$$

whose solution can be expressed as

$$h_p=\frac{p_d-p_u}{u-d}. \tag{5.3.16}$$

This result is known as the optimal hedge ratio. Specifically, it tells us how many underlying instruments to buy for every put purchased. The sign of h_p will be positive as $p_d > p_u$ and $u > d$. Recall we assume $u > rS_0 > d$. Thus, if the number of units of the underlying instrument that we hold is set to h_p, the two future values of the underlying instrument will be identical. Hence, the portfolio is risk-free. To avoid arbitrage, the portfolio must be priced to earn the risk-free rate. Again, the discretely compounded periodic risk-free rate is denoted r. Thus, the following condition must hold:

$$V_0=\frac{V_u}{1+r}=\frac{V_d}{1+r}. \tag{5.3.17}$$

Consequently, we can substitute into Equation (5.3.17) using either V_u or V_d. We choose V_d, thus

$$\frac{h_p\left(S_0+d\right)+p_d}{1+r}=h_pS_0+p_0. \tag{5.3.18}$$

Therefore, the initial put price can be represented based on the *no arbitrage model* as

$$p_0=B_{0,p}-h_pS_0, \tag{5.3.19}$$

where

$$B_{0,p}=\frac{h_p\left(S_0+d\right)+p_d}{1+r}. \tag{5.3.20}$$

Thus, a put option can be replicated by short selling h_p units of the underlying instrument and lending of $B_{0,p}$. From this analysis, a put option is simply shorting a stock with lending.

To solve for the *equivalent martingale measure model*, the next step is to insert the solution for h_p into Equation (5.3.20), and solve for p_0:

$$p_0=PV\left[E\left(p_T\right)\right]=\frac{\pi p_u+\left(1-\pi\right)p_d}{1+r}, \tag{5.3.21}$$

where the equivalent martingale measure probability is, unfortunately, time and state dependent[3]

$$\pi=\frac{rS_0-d}{u-d}. \tag{5.3.22}$$

The derivation of Equation (5.3.21) is provided in Appendix 5.3A. Thus, another view is that the put price is simply the present value of the expected future put payoffs discounted at the risk free rate. The probabilities used in forming the expectations, however, are not the investor's subjective probabilities. They are based on the equivalent martingale measure or the risk neutral probabilities.

ABM one period put option binomial model example

Again, suppose the current stock price is \$99, the strike price is \$100, the annual, discretely compounded, risk free rate is 2%, the time to expiration is one year, u = \$24.75, and d = –\$19.8. Like the call, we can

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[3]This independence is an important feature for optimizing calculations of European-style option values. As we will see in the next module, arithmetic Brownian motion-based binomial valuation models will have dependent equivalent martingale measure probabilities requiring a bit more effort to build binomial models.

compute the put price in two ways. First, note that p_u = max[0, 100 – (99 + 24.75)] = 0 and p_d = max[0, 100 – (99 + –19.8) = 20.8. For the no arbitrage model, we again find the hedge ratio $h_p = (p_d - p_u)/(u - d)$ = (20.8 – 0)/(24.75 – –19.8) = 20.8/44.55 = 0.4669. Therefore, based on Equation (5.3.19), we have

$$\begin{aligned} p_0 &= \frac{h_p\left(S_0 + d\right) + p_d}{1 + r} - h_p S_0 \\ &= \frac{0.4669\left(99 - 19.8\right) + 20.8}{1 + 0.02} - 0.4669\left(99\right). \\ &= 56.6456 - 46.2231 = 10.42 \end{aligned}$$

Alternatively, we can use the risk neutral model. Again, we have

π = [99(0.02) – (–19.8)]/[24.75 – (–19.8)] = 48.8889%.

Therefore, based on Equation (5.3.21), we find the same results or

$$p_0 = \frac{0.4889\left(0\right) + \left(1 - 0.4889\right)20.8}{1 + 0.02} = 10.42.$$

When market prices deviate from these model prices, then arbitrage opportunities exist. We explore capturing arbitrage profits in Appendix 5.3B.

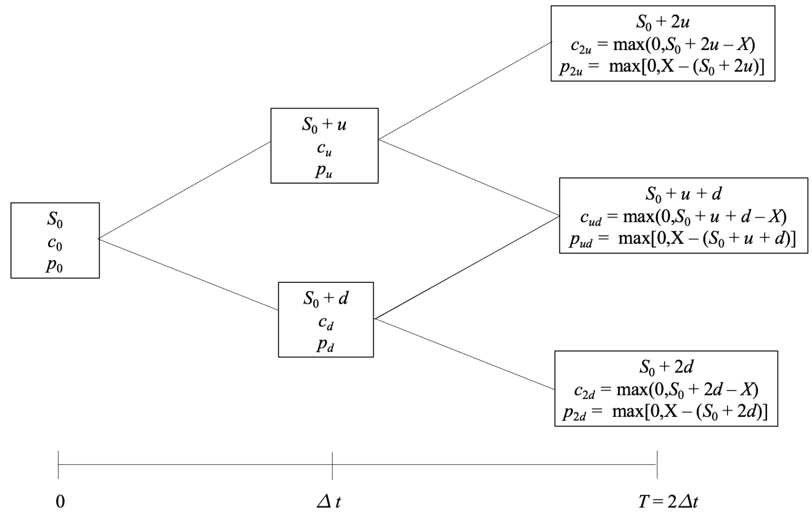
ABM European-style option two period model

The model can be extended to multiple periods and thereby accommodate options with longer lives or smaller time steps. For example, we can let the underlying instrument move from $S + u$ to $S + 2u$ or $S + u + d$. From $S + d$, the underlying instrument can move to $S + d + u$ or $S + 2d$. Note that $S + u + d = S + d + u$, so over two periods, there are only three possible outcomes. The underlying instrument can go up twice to $S_0 + 2u$, up and then down or down and then up to $S_0 + u + d$, or down twice to $S_0 + 2d$. The call and put option payoffs in those states are

$$\begin{aligned} c_{2u} &= \max\left(0, S_0 + 2u - X\right) & p_{2u} &= \max\left[0, X - \left(S_0 + 2u\right)\right] \\ c_{ud} &= \max\left(0, S_0 + u + d - X\right) \text{ and } & p_{ud} &= \max\left[0, X - \left(S_0 + u + d\right)\right]. \\ c_{2d} &= \max\left(0, S_0 + 2d - X\right) & p_{2d} &= \max\left[0, X - \left(S_0 + 2d\right)\right] \end{aligned} \qquad (5.3.23)$$

The layout is illustrated in Figure 5.3.9. The illustration is looking more like a branching tree or lattice. Two key features of the binomial model here is the recombining nature of the tree, and the growth of the stock price is additive. The tree is recombining because the stock price is assumed to grow by addition such that $S_0 + u + d = S_0 + d + u$. Clearly, the order of addition does not matter. The additive approach presented here facilitates the convergence of the stock price to the normal distribution.

Figure 5.3.9 Two Period European-Style Binomial Model



Let us position ourselves in the time 1 up-state, where the underlying instrument price is $S + u$. At this point, we are now back in a one-period world. There are two outcomes possible outcomes in the next period, which is the expiration. It should be easy to see that the value of the call and put at this point would be:

$$c_u = \frac{\pi_{1,u} c_{2u} + \left(1 - \pi_{1,u}\right) c_{ud}}{1 + r} \text{ and } p_u = \frac{\pi_{1,u} p_{2u} + \left(1 - \pi_{1,u}\right) p_{ud}}{1 + r}, \tag{5.3.24}$$

where

$$\pi_{1,u} = \frac{S_u r - d}{u - d}. \tag{5.3.25}$$

Likewise, in the time 1 down-state, the option value would be

$$c_d = \frac{\pi_{1,d} c_{ud} + \left(1 - \pi_{1,d}\right) c_{2d}}{1 + r} \text{ and } p_d = \frac{\pi_{1,d} p_{ud} + \left(1 - \pi_{1,d}\right) p_{2d}}{1 + r}. \tag{5.3.26}$$

where we emphasize that these probabilities are observed at time 1 or

$$\pi_{1,d} = \frac{S_d r - d}{u - d}. \tag{5.3.27}$$

Note that $\pi_{1,u} \neq \pi_{1,d} \neq \pi_0$ because $S_u \neq S_d \neq S_0$. Stepping back to time 0, the value of the call and put options are again found with Equation (5.3.10), where the values of c_u and p_u are given in Equation (5.3.24) and c_d and p_d are given in Equation (5.3.26). Thus, to price options in the binomial framework in this multiperiod model, we start at the end—the exercise date—and work backwards to the present.

Although π is not constant, the special case for two-period options does lend itself to a simple formula that relates the initial option value to the value two periods later, essentially skipping over the first period.

$$c_0 = \frac{\pi_0 \pi_{1,u} c_{2u} + \left[\pi_0\left(1 - \pi_{1,u}\right) + \left(1 - \pi_0\right)\pi_{1,d}\right] c_{ud} + \left(1 - \pi_0\right)\left(1 - \pi_{1,d}\right) c_{2d}}{\left(1 + r\right)^2} \tag{5.3.28}$$

and

$$p_0 = \frac{\pi_0 \pi_{1,u} p_{2u} + \left[\pi_0\left(1-\pi_{1,u}\right)+\left(1-\pi_0\right)\pi_{1,d}\right] p_{ud} + \left(1-\pi_0\right)\left(1-\pi_{1,d}\right) p_{2d}}{\left(1+r\right)^2}. \tag{5.3.29}$$

Note that the three option payoffs two periods later are each weighted by the risk neutral probabilities, $\pi_0\pi_{1,u}$, $\left[\pi_0\left(1-\pi_{1,u}\right)+\left(1-\pi_0\right)\pi_{1,d}\right]$, and $\left(1-\pi_0\right)\left(1-\pi_{1,d}\right)$. These are the binomial probabilities for two trials, and they add up to 1.

ABM two period call and put option binomial model example

As with the GBM-BOVM examples, suppose the current stock price is \$99, the strike price is \$100, the annual, discretely compounded, risk free rate is 2%, the time to expiration is two years, $u = 24.75$, and $d = -19.80$. Now assume a two-period binomial model. Based on Equations (5.3.28) and (5.3.29), we can compute the call and put prices. First, we compute the terminal payoffs for both calls and puts as

$$\begin{aligned} c_{2u} &= \max\left(0, S+2u-X\right) = \max\left[0, 99+2\left(24.75\right)-100\right] = 48.50 \\ c_{ud} &= \max\left(0, S+u+d-X\right) = \max\left[0, 99+24.75-19.80-100\right] = 3.95 \text{ and} \\ c_{2d} &= \max\left(0, S+2d-X\right) = \max\left[0, 99-2\left(19.80\right)-100\right] = 0 \end{aligned} \tag{5.3.30}$$

$$\begin{aligned} p_{2u} &= \max\left(0, X-S-2u\right) = \max\left[0, 100-99-2\left(24.75\right)\right] = 0 \\ p_{ud} &= \max\left(0, X-S-u-d\right) = \max\left[0, 100-99-24.75+19.80\right] = 0. \\ p_{2d} &= \max\left(0, X-S+2d\right) = \max\left[0, 100-99-2\left(-19.80\right)\right] = 40.60 \end{aligned} \tag{5.3.31}$$

The binomial probability of an up move at time 0 is π_0 = [99(0.02) – (–19.80)]/[24.75 – (–19.80)] = 48.89%, at time 1 assuming up is $\pi_{u,1}$ = [123.75(0.02) – (–19.80)]/[24.75 – (–19.80)] = 50%, and at time 1 assuming down is $\pi_{d,1}$ = [79.20(0.02) – (–19.80)]/[24.75 – (–19.80)] = 48%. Therefore, based on Equation (5.3.28), we find

$$\begin{aligned} c_0 &= \frac{\pi_0 \pi_{1,u} c_{2u} + \left[\pi_0\left(1-\pi_{1,u}\right)+\left(1-\pi_0\right)\pi_{1,d}\right] c_{ud} + \left(1-\pi_0\right)\left(1-\pi_{1,d}\right) c_{2d}}{\left(1+r\right)^2} \\ &= \frac{0.4889\left(0.50\right)20 + \left[0.4889\left(1-0.50\right)+\left(1-0.4889\right)0.48\right]0 + \left(1-0.4889\right)\left(1-0.48\right)0}{\left(1+0.02\right)^2} \\ &= 13.255 \end{aligned} \tag{5.3.32}$$

and, based on Equation (5.3.29), we have

$$\begin{aligned} p_0 &= \frac{\pi_0 \pi_{1,u} p_{2u} + \left[\pi_0\left(1-\pi_{1,u}\right)+\left(1-\pi_0\right)\pi_{1,d}\right] p_{ud} + \left(1-\pi_0\right)\left(1-\pi_{1,d}\right) p_{2d}}{\left(1+r\right)^2} \\ &= \frac{0.4889\left(0.50\right)0 + \left[0.4889\left(1-0.50\right)+\left(1-0.4889\right)0.48\right]0 + \left(1-0.4889\right)\left(1-0.48\right)40.60}{\left(1+0.02\right)^2}. \\ &= 10.37 \end{aligned} \tag{5.3.33}$$

Alternatively, the two period binomial model can be viewed as three one period binomial models and the no arbitrage model applied. The call results are illustrated in Figure 5.3.10. Note that at node (1,0) both the call value and hedge ratio are zero because it is not possible that this option will end up in-the-money at time 2. Node (2,0) is out-of-the-money and node (2,1) is at-the-money. At node (1,1), the call value is

$$c_u = \frac{\pi_{1,u} c_{2u} + \left(1 - \pi_{1,u}\right) c_{ud}}{1+r} = \frac{0.50\left(48.50\right) + \left(1 - 0.50\right)3.95}{1 + 0.02} = 25.71 . \tag{5.3.34}$$

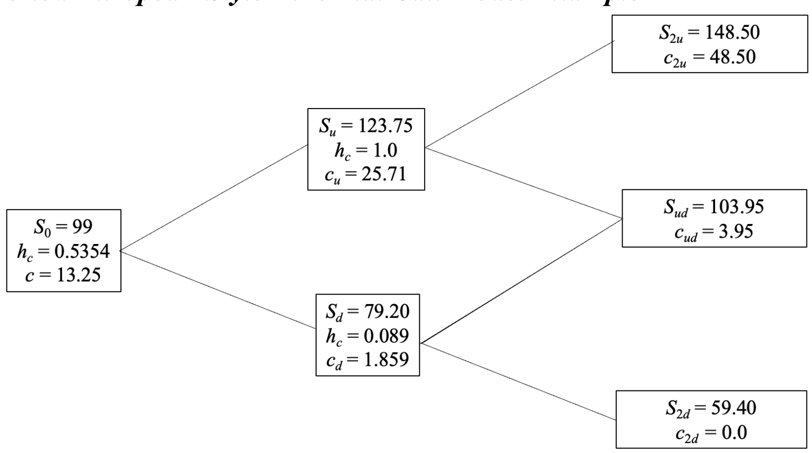
At time 0, the call hedge ratio is

$$h_{c,0} = \frac{c_u - c_d}{u - d} = \frac{25.71 - 1.8588}{24.75 - \left(-19.80\right)} = 0.5354 . \tag{5.3.35}$$

At time 0, the call value is

$$c_0 = \frac{\pi_0 c_u + \left(1 - \pi_0\right) c_d}{1+r} = \frac{0.4889\left(25.71\right) + \left(1 - 0.4889\right)1.8588}{1 + 0.02} = 13.255 . \tag{5.3.36}$$

Figure 5.3.10 Two period European-Style Binomial Call Model Example



The put results are illustrated in Figure 5.3.11. Note that at node (1,1) both the put value and hedge ratio are zero because it is not possible that this option will end up in-the-money at time 2. Node (2,2) is out-of-the-money and node (2,1) is at-the-money. At node (1,0), the put value is

$$p_d = \frac{\pi_{1,d} p_{ud} + \left(1 - \pi_{1,d}\right) p_{2d}}{1+r} = \frac{0.48\left(0\right) + \left(1 - 0.48\right)40.60}{1 + 0.02} = 20.698 . \tag{5.3.37}$$

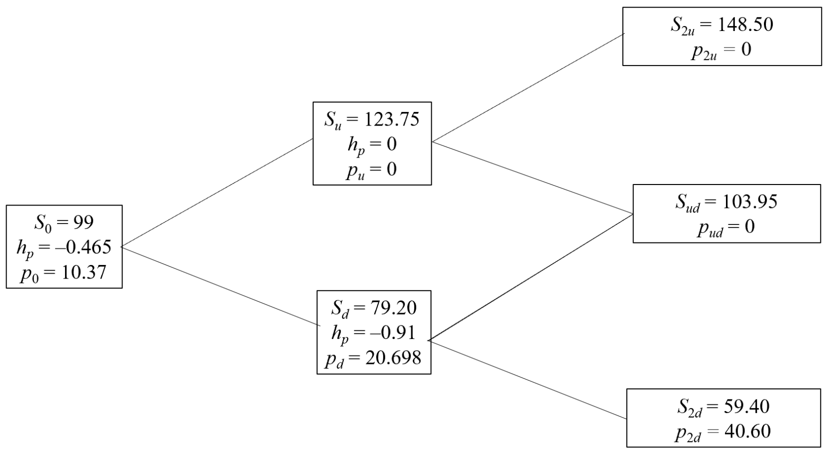
At time 0, the put hedge ratio is

$$h_{p,0} = \frac{p_u - p_d}{u - d} = \frac{0 - 20.698}{24.75 - \left(-19.80\right)} = -0.4646 . \tag{5.3.38}$$

At time 0, the put value is

$$\begin{aligned} p_0 &= \frac{\pi_0 p_u + (1-\pi_0) p_d}{1+r} \\ &= \frac{0.4889(0) + (1-0.4889)20.698}{1+0.02} = 10.37 \end{aligned} . \quad (5.3.39)$$

Figure 5.3.11 Two Period European-Style Binomial Put Model Example



We turn now to address American-style options where early exercise may enhance the worth of an option.

ABM American-style option two period model[4]

Recall if the options are American-style, they can be exercised early. Cash payments, such as dividends, will influence the early exercise decision. Thus, we first examine this influence.

American-stye options and dividends

It is well known that American call options will not be exercised early unless there is some cash or cash-equivalent amount paid by the underlying instrument, in which case early exercise could be justified immediately before the cash is paid. An example of a non-cash benefit is ski lift tickets given to stockholders of a ski company. The typical assumption is that any benefits of this nature are immediately sold for cash and this cash amount is included in any holding period return calculations. Obviously, one could go skiing but the financial analysis assumes that it is sold. Note that cash dividend on the stock result in less equity per share remaining with the company and hence, the stock price should decline by the dividend amount. This stock price decline is detrimental to call holders.

There are two primary methods for handling the underlying instrument paying out something of value, the yield method, and the escrow method. We focus here on cash dividends on a stock. The yield method assumes the dividend is a constant rate of the value of the stock. This approach, however, would imply a very small dividend at every time step. Options on stock indexes come close to a continuous yield and can be approximated by a yield.

The escrow method assumes the present value of the dividends to be paid out over the life of the option is placed in a bankruptcy proof escrow account denoted *PVD*. The escrow account is then used to make the future dividend payments. Thus, the remaining stock value is simply based on subtracting the escrow amount from the current value of the underlying. The stock price minus the present value of dividends, $S' = S - PVD$, is modeled with the binomial tree according to the factors u and d. At a given node at which the dividend is paid, we decide if the option is worth exercising just before the stock goes ex-dividend. If so, the exercise value replaces the value obtained using the formula.

[4]Much of this section repeats from the prior model. We keep this material here as some readers may skip around rather than working sequentially.

For example, suppose at a point in the tree, we have a value of the stock price minus the present value of all remaining dividends over the life of the option of \$42. Suppose that using the binomial formula, we compute the value of the call at that point as \$2.25. Assume there is a \$3 dividend being paid at this time point. Then the stock price with the dividend is \$45. If the exercise price is \$42, we could exercise it and collect a value of \$3, which is more than its unexercised value of \$2.25. Thus, we would replace \$2.25 with \$3. This early exercise check would be done at all points in the tree in which the option is in-the-money.

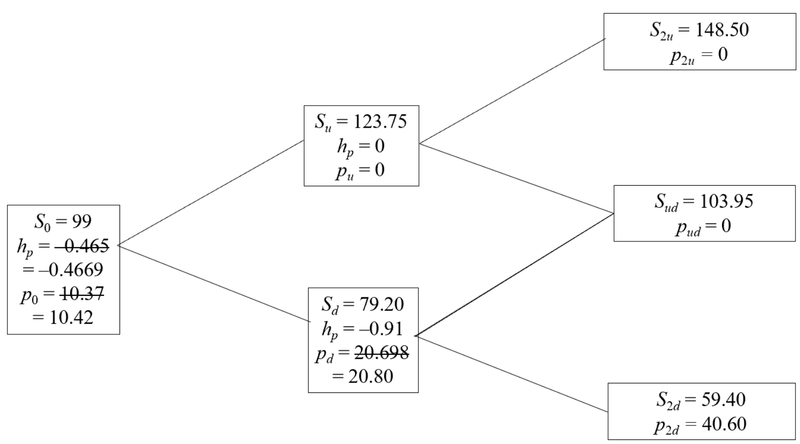
It is known that early exercise could occur regardless of a dividend for put options. At every in-the-money point in the binomial tree, we examine whether the put is worth more exercised or not. If it is worth more to early exercise, the exercise value is used at that point into the tree as the option value. If it is not worth more to early exercise, we simply continue to use the computed value obtained by the single period binomial formula. Dividends will reduce the frequency of early exercise since dividends drive the stock price down, which makes puts worth more. Exercising early negates this benefit. Early exercise generally occurs just after a dividend when the stock price falls.

ABM two period American-style put option binomial model example

Recall the data related to Figure 5.3.11. If this put option was American-style, we would exercise the put at node (1,0). Figure 5.3.12 illustrates this adjustment. Therefore at time 0, the put value is

$$\begin{aligned} p_0 &= \frac{\pi_0 p_u + \left(1-\pi_0\right) p_d}{1+r} \\ &= \frac{0.4889\left(0\right)+\left(1-0.4889\right)20.80}{1+0.02} = 10.42 \end{aligned}. \tag{5.3.40}$$

Figure 5.3.12 Two period American-Style Binomial Put Model Example



We now explore coherence conditions when there are no dividends.

ABM coherence conditions

In this section, we will follow closely the framework used with the GBM-based binomial model. The difference here is the variance is based on dollar changes in the underlying and not percentage changes.

No dividend coherence conditions

We seek to build an option valuation model with certain assumption known as the coherent conditions. We assume a time step of Δt. The coherent conditions comprise four assumptions:[5]

1) $d < S_0\left(e^{r\Delta t}-1\right) < u$ (no arbitrage boundary condition).
2) $0 << \pi << 1$ (probability condition, distribution independent, not "close" to 0 or 1).

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[5]Based, in part, on Don Chance, "A Synthesis of Binomial Option Pricing Models for Lognormally Distributed Assets," *Journal of Applied Finance* (Spring/Summer 2008).

3) $\pi = \dfrac{S_0\left(e^{r\Delta t}-1\right)-d}{u-d}$ (no arbitrage distribution independent condition).

4) $Var_\pi\left(\Delta S_T\right)=\left(u-d\right)^2\pi\left(1-\pi\right)$ (variance condition of the price changes, distribution independent).

We briefly comment on each coherent condition. Arithmetic Brownian motion converges in the limit to the normal distribution at every future point in time; hence, zero or negative instrument values are certainly possible. Recall limited liability required that d is greater than 0, but limited liability does not imply the financial instrument cannot be valued at zero at some future point in time. Limited liability can be incorporated with ABM using zero strike put options. In this case, $S_{\Delta t}$ = 0 is certainly possible.

Investing in a "risk-free" financial instrument should change by more than the down change (less negative) if the down event, d, occurs; otherwise, no one would buy the "risk-free" instrument ($d < S_0\left(e^{r\Delta t}-1\right)$). Specifically, if $d \geq S_0\left(e^{r\Delta t}-1\right)$, then buy the risky financial instrument with borrowed money. Thus, you have at least a positive probability of future positive cash flow, with no initial investment. Similarly, investing in the risky instrument should earn more than the "risk-free" instrument at some future point (u occurs); otherwise, no one would buy the risky instrument ($S_0\left(e^{r\Delta t}-1\right)<u$). Specifically, if $u \leq S_0\left(e^{r\Delta t}-1\right)$, then buy the "risk-free" financial instrument and short sell the risky financial instrument. Thus, you have at least a positive probability of future positive cash flow, with no initial investment.

The equivalent martingale measure (risk-neutral probability) of the up event cannot be "too close" to zero or one. If π is "too close" to 0, then u will tend to positive infinity. If π is "too close" to 1, then d will tend to negative infinity. In both cases, one will encounter stability problems with numerical implementations.

The no arbitrage condition is the result of potential arbitrage trading activities forcing specific relationship between the option and underlying stock. Under the equivalent martingale measure, the present value of an option (O) is

$$O_0 = PV\left[E_\pi\left(O_{\Delta t}\right)\right]=e^{-r\Delta t}\left[\pi O_u+\left(1-\pi\right)O_d\right]. \tag{5.3.41}$$

Thus, for an underlying instrument, S, we observe

$$S_0 = PV\left[E_\pi\left(S_{\Delta t}\right)\right]=e^{-r\Delta t}\left[\pi\left(S_0+u\right)+\left(1-\pi\right)\left(S_0+d\right)\right], \text{ and} \tag{5.3.42}$$

$$S_0\left(e^{r\Delta t}-1\right)=\pi u+\left(1-\pi\right)d=E_\pi\left(\Delta S_{\Delta t}\right). \tag{5.3.43}$$

Note that these equations hold only if no arbitrage condition above is true.

The variance condition is required to converge to the ABMOVM (discussed in Module 5.5) as well as be consistent at each node. The variance of the dollar change in price is ($S_0 > 0$). Therefore,

$$Var_\pi\left(\Delta S_{\Delta t}\right)\equiv E\left\{\left[\Delta S_{\Delta t}-E\left(\Delta S_{\Delta t}\right)\right]^2\right\}. \tag{5.3.44}$$

Substituting for $S_{\Delta t}$ and cancelling S_0,

$$Var_\pi\left(\Delta S_{\Delta t}\right)=\left[u-E\left(\Delta S_{\Delta t}\right)\right]^2\pi+\left[d-E\left(\Delta S_{\Delta t}\right)\right]^2\left(1-\pi\right). \tag{5.3.45}$$

Substituting for the mean and rearranging, we have the results in the variance condition above,

$$\begin{aligned} Var_\pi\left(\Delta S_{\Delta t}\right)&=\left\{u-\left[\pi u+\left(1-\pi\right)d\right]\right\}^2\pi+\left\{d-\left[\pi u+\left(1-\pi\right)d\right]\right\}^2\left(1-\pi\right) \\ &=\left(1-\pi\right)^2\left(u-d\right)^2\pi+\pi^2\left(u-d\right)^2\left(1-\pi\right)=\left(u-d\right)^2\pi\left(1-\pi\right) \end{aligned}. \tag{5.3.46}$$

As with the GBM-based model, with these four coherence conditions, we can demonstrate the functional form for *u* and *d*.

No dividend u and d conditions

With these coherence conditions, we can establish the following expressions for *u* and *d*:

$$u = S_0\left(e^{r\Delta t} - 1\right) + \left(1 - \pi\right)\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1 - \pi\right)}}, \tag{5.3.47}$$

and

$$d = S_0\left(e^{r\Delta t} - 1\right) - \pi\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1 - \pi\right)}}. \tag{5.3.48}$$

We now prove these two conditions. Isolating *u* based on the no arbitrage condition, we have

$$u = \frac{S_0\left(e^{r\Delta t} - 1\right) - d}{\pi} + d = \frac{S_0\left(e^{r\Delta t} - 1\right) - d\left(1 - \pi\right)}{\pi}. \tag{5.3.49}$$

Substituting this result into the variance condition of absolute volatility, we have

$$\begin{aligned}\left(u - d\right)^2 \pi\left(1 - \pi\right) &= \left[\frac{S_0\left(e^{r\Delta t} - 1\right) - d\left(1 - \pi\right)}{\pi} - d\right]^2 \pi\left(1 - \pi\right) \\ &= \left[\frac{S_0\left(e^{r\Delta t} - 1\right) - d}{\pi}\right]^2 \pi\left(1 - \pi\right) = \sigma^2 \Delta t\end{aligned}. \tag{5.3.50}$$

Solving for d,

$$\begin{aligned}\left[\frac{S_0\left(e^{r\Delta t} - 1\right) - d}{\pi}\right]\sqrt{\pi\left(1 - \pi\right)} &= \sigma\sqrt{\Delta t} \\ \frac{S_0\left(e^{r\Delta t} - 1\right) - d}{\pi} &= \frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1 - \pi\right)}} \\ S_0\left(e^{r\Delta t} - 1\right) - d &= \pi\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1 - \pi\right)}}\end{aligned}. \tag{5.3.51}$$

Thus the normal coherent binomial down move for the single period model is

$$d = S_0\left(e^{r\Delta t} - 1\right) - \pi\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1 - \pi\right)}}. \tag{5.3.52}$$

Solving for the up move

$$u = \frac{S_0\left(e^{r\Delta t}-1\right)-d\left(1-\pi\right)}{\pi} = \frac{S_0\left(e^{r\Delta t}-1\right)-\left[S_0\left(e^{r\Delta t}-1\right)-\pi\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}\right]\left(1-\pi\right)}{\pi} \\ = \frac{\pi S_0\left(e^{r\Delta t}-1\right)+\pi\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}\left(1-\pi\right)}{\pi}. \tag{5.3.53}$$

The normal coherent binomial up move for a single period is

$$u = S_0\left(e^{r\Delta t}-1\right)+\left(1-\pi\right)\frac{\sigma\sqrt{\Delta t}}{\sqrt{\pi\left(1-\pi\right)}}. \tag{5.3.54}$$

Let us consider the range of π. As π tends to 1 from below, note that u tends to $S_0\left(e^{r\Delta t}-1\right)$ and d tends to 0. As π tends to 0 from above, note that u tends to positive infinity and d tends to $S_0\left(e^{r\Delta t}-1\right)$. Thus, π cannot be too 'close' to either 0 or 1. So long as π is in a reasonable range, then numerically $\pi = \frac{S_0\left(e^{r\Delta t}-1\right)-d}{u-d}$ exactly.

Table 5.3.1 illustrates the relationship between u, d, and π. The first column is selected values for the equivalent martingale probability of up move. The values for u and d are computed based on Equations (5.3.54) and (5.3.48), respectively. Finally, the fourth column (Prob Check) recomputes the equivalent martingale probability of up move based on the coherence condition 3 (no arbitrage distribution independent condition) as well as the computed values for u and d.

Table 5.3.1. Relationship between u, d, and π

Probability	u	d	Prob Check
0	#DIV/0!	#DIV/0!	#DIV/0!
0.00000001	300000.049	0.0470125	0.00000001
0.0000001	94868.3751	0.04052567	0.0000001
0.000001	30000.035	0.02001249	0.000001
0.00001	9486.83556	-0.0448563	0.00001
0.0001	2999.90001	-0.2500025	0.0001
0.001	948.25885	-0.89914549	0.001
0.01	298.546244	-2.96510094	0.01
0.1	90.0500125	-9.9499875	0.1
0.2	60.0500125	-14.9499875	0.2
0.3	45.8757695	-19.5895976	0.3
0.4	36.7923586	-24.4448849	0.4
0.5	30.0500125	-29.9499875	0.5
0.6	24.5449099	-36.6923336	0.6
0.7	19.6896226	-45.7757444	0.7
0.8	15.0500125	-59.9499875	0.8
0.9	10.0500125	-89.9499875	0.9
0.99	3.06512595	-298.446219	0.99
0.999	0.9991705	-948.158825	0.999
0.9999	0.3500275	-2999.79998	0.9999
0.99999	0.14488131	-9486.73553	0.99999
0.999999	0.08001252	-29999.935	0.999999
0.9999999	0.05949934	-94868.275	0.9999999
0.99999999	0.0530125	-299999.948	0.99999999
1	#DIV/0!	#DIV/0!	#DIV/0!

Because π is arbitrary, the coherence conditions comprise a family of binomial option valuation models. These models converge to the arithmetic Brownian motion option valuation model in the limit as the number of time steps tends to infinity (or the step size tends to zero). Specifically, based on the use of u and d above, the coherent normal binomial model converges to the arithmetic Brownian option valuation model presented in Module 5.5.

Before addressing dividends, we present the ABM-BOVM.

ABM-based binomial option valuation model: No dividends, European-style

The GBM European-style multiperiod option model results in a recombining tree in both outcomes as well as probabilities. Thus, the well-known result for a call option can be expressed as

$$\begin{aligned} O_0 &= PV_r\left[E_0\left(O_T\right)\right] \\ &= PV_r\left[\sum_{j=0}^{n} \Pr\left(n,j\right) Payoff\left(\iota,n,j\right)\right] \\ &= PV_r\left\{\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)\pi^j\left(1-\pi\right)^{n-j}\max\left[0,\iota\left(S_0u^jd^{n-j}-X\right)\right]\right\} \end{aligned}, \qquad (5.3.55)$$

where O_0 denotes the current call or put value, ι denotes an indicator function that equals +1 if call and –1 if put, and PV_r is simply a present value factor.

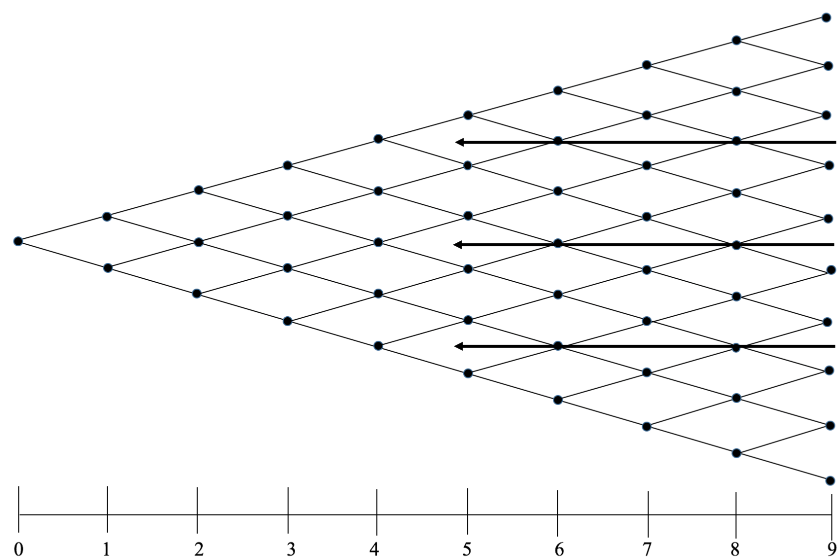
Unfortunately, for ABM the probabilities are path dependent due to the geometric growth rate assumed for the underlying instrument. That is,

$$
\begin{aligned}
O_0 &= PV_r\left[E_0\left(O_T\right)\right] \\
&= PV_r\left[\sum_{j=0}^{n} \Pr\left(n,j\right) Payoff\left(\iota,n,j\right)\right] \\
&= PV_r\left[\sum_{j=0}^{n} \Pr\left(n,j\right) \max\left\{0, \iota_u\left[S_0 + ju + \left(n-j\right)d - X\right]\right\}\right]
\end{aligned} \quad . \tag{5.3.56}
$$

There are several ways to handle the computation of Pr(*n*,*j*). We focus here on the approach that is easiest to build a computer solution, but not the fastest solution.

Recall the ABM binomial lattice is additive and recombining. Thus, the number of nodes grows only at a rate of one per additional node. Unfortunately, the number of path calculations for determining terminal probabilities is exploding. We pursue backward recursion as a valuation solution. Specifically, we assume the time to maturity is divided into n time steps of size Δt. Figure 5.3.13 illustrates the case with $n = 9$. For example, assume we are evaluating a 9 month option where each time step is 1 month. At expiration, $n = 9$ or $T = 9/12$, we know the terminal payoffs based on $Payoff\left(\iota,n,j\right)$. At point in time 8, we apply Equation (5.3.24) at each node. We then repeat the process at point in time 7 and so forth. Note that the number of nodes is declining by one as we recurse backward through the lattice.

Figure 5.3.13 Nine Period Binomial Model



ABM-based binomial option valuation model: No dividends, American-style

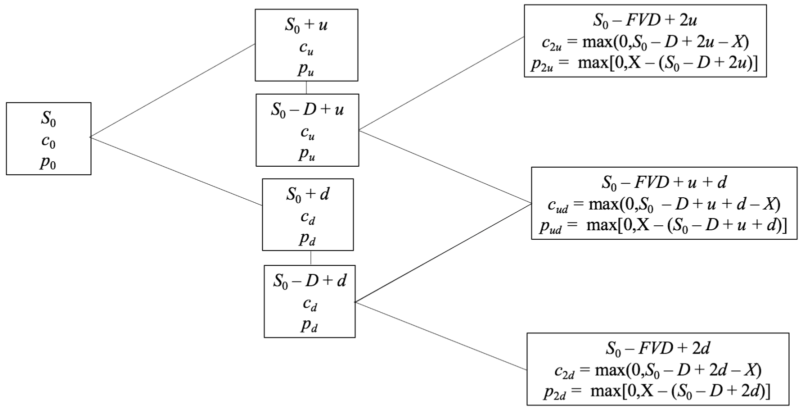
The process for valuing American-style options is similar to European-style options. The only difference is that at each step, except the point in time of expiration, we consider whether early exercise is more valuable than continuation. Also, we consider whether there is a violation of lower boundary conditions.

We now further explore the role of dividend on both European and American options.

Dividends

Dividends do not pose a problem with the additive binomial model. Figure 5.3.14 illustrates that we do not lose of the recombining property in the presence of a cash dividend at time 1. Recall that the stock price falls by the dollar dividend amount on the ex-dividend date. Optimal early exercise may occur either right before the ex-dividend date for calls, or right after the ex-dividend date for puts. Due to the additive nature of ABM, the binomial tree does recombine after a dollar dividend payment.

Figure 5.3.14 Two Period Binomial Model with Discrete Dividends



Note that mathematically,

$$S_0 - D + u + d = S_0 - D + d + u. \tag{5.3.57}$$

Thus, the presence of discrete dividends poses no computational problem with ABM. We turn now to deal with the challenges related to a multiperiod model.

Summary

ABM results in a normally distributed terminal distribution. In this module, we derived a binomial model that converges to the ABMOVM.

A lattice approach to valuing various options consistent with a normal terminal distribution (as opposed to the lognormal distribution in the last module) was presented in this module. The valuation approach was based on dynamic arbitrage. Dynamic arbitrage is based on the capacity to continuously rebalance a custom-designed portfolio.

In this module, we present the non-traditional binomial valuation model, ABM-BOVM. Again, the key weakness of the ABM-BOVM is the need to use backward recursion with European-style options. We argue that like tools in a toolbox for the quantitative analyst, the varied challenges analysts face will warrant the unique tool selected. Unorthodox tools often prove vital with particularly challenging tasks.

References

See references in Module 5.2.

Appendices for Module 5.3.

Several technical issues are covered in these appendices.

Appendix 5.3A. Derivation of Equation (5.3.10)

We start with Equation (5.3.7), re-stated here as[6]

$$\frac{h_c\left(S + u\right) - c_u}{1 + r} = h_c S - c. \tag{5.3.58}$$

Now substitute for *h*, using Equation (5.3.5),

[6] Note that we suppress the time subscript for ease of exposition.

$$\frac{\left(\frac{c_u - c_d}{u-d}\right)(S+u) - c_u}{1+r} = \left(\frac{c_u - c_d}{u-d}\right)S - c. \tag{5.3.59}$$

Rearranging, we have

$$\left(\frac{c_u - c_d}{u-d}\right)S - \frac{\left(\frac{c_u - c_d}{u-d}\right)(S+u) - c_u}{1+r} = c. \tag{5.3.60}$$

Then we multiply through by 1 + r,

$$\left(\frac{c_u - c_d}{u-d}\right)S(1+r) - \left(\frac{c_u - c_d}{u-d}\right)(S+u) + c_u = c(1+r). \tag{5.3.61}$$

Then we cancel S,

$$\left(\frac{c_u - c_d}{u-d}\right)Sr - \left(\frac{c_u - c_d}{u-d}\right)u + c_u = c(1+r). \tag{5.3.62}$$

Using the common denominator $u - d$, we obtain

$$\begin{gathered}\frac{c_u Sr - c_d Sr - c_u u + c_d u + c_u u - c_u d}{u-d} = c(1+r) \\ \frac{c_u(Sr-d) + c_d(u-Sr)}{u-d} = c(1+r)\end{gathered}. \tag{5.3.63}$$

Now, let us define π as in Equation (5.3.10),

$$\pi = \frac{Sr-d}{u-d}. \tag{5.3.64}$$

Then $1 - \pi$ is

$$1-\pi = \frac{u-Sr}{u-d}. \tag{5.3.65}$$

So the solution is

$$c = \frac{\pi c_u + (1-\pi)c_c}{1+r}. \tag{5.3.66}$$

which is Equation (5.3.10).

Appendix 5.3B. Arbitraging price discrepancies within a one period model

If the actual market price of the option differs from the model price, an arbitrage is possible. Consider the call option case. If the call can be sold for more than the formula value, Equation (5.3.8), the call is overpriced. Overpriced instruments should be sold. Simply selling the call, however, hardly qualifies as an arbitrage. If the call expires in-the-money, one could incur a significant loss, even though the call were underpriced. Instead, the arbitrage should be completed, and the risk eliminated by holding an offsetting number of units of the stock.

The arbitrageur would, thus, buy h_c units of the stock for each call sold and borrow B_c. It should be easy to see that the investment required would be less than what is received from the written call. Convergence of the option value to its exercise value is assured one period later, as the option is expiring and can clearly be worth only its exercise value. With less money invested and the same payoff as before, the rate of return clearly exceeds the risk-free rate. If the option trades at below the formula price, it would be purchased and h_c units of the stock would be sold, creating a net short position. The proceeds would be invested in risk-free bonds to earn the rate r. With the option purchased at a lower than fair price, the stock and option would finance the purchase of the risk-free instrument at a lower cost than it should if correctly priced, so the investor would earn an arbitrage profit.

Based on the information given in the past two examples, suppose we have the following market quotes, c_Q = \$11.43 and p_Q = \$10.37. Recall S_0 = \$99, X = \$100, $r = 0.02$, $\tau = 1$, u = \$24.75, and d = –\$19.8. In equilibrium, we found c_0 = \$11.38 and p_0 = \$10.42, thus the call price is too high and the put price is too low. Arbitrageurs typically prefer to receive positive cash flow today with no chance of any future liability.

Because the quoted call price is too high, the arbitrageur would sell it and buy the synthetic call option. Buying the synthetic call entails buying the stock with borrowed money. Table 5.3B.1 illustrates capturing the arbitrage profit available with the call option.

Table 5.3B.1. Cash Flow Table for Single Period ABM Model Applied to Call Options

Strategy		Today	Down Event at Expiration	Up Event at Expiration
Sell Call		$+c_{0,q} = +11.43$	$-\max(0, S_0 + d - X) = 0$	$-\max(0, S_0 + u - X) = -23.75$
Buy h_c Shares		$-h_c S_0 = -52.78$	$+h_c(S_0 + d) = +42.22$	$+h_c(S_0 + u) = +65.97$
Borrow		$+B_c = +41.39$	$-B_c(1 + r) = -42.22$	$-B_c(1 + r) = -42.22$
Net Cash Flow		+0.04	0	0

Thus, the arbitrageur receives \$0.04 today with no chance of a future liability. Within this simple one period binomial world, trading pressure will drive down the quoted call price and drive up the quoted stock price until the net cash flow is zero.

If the quoted put price, however, is too low, the arbitrageur would buy it and sell the synthetic put option. Selling the synthetic put entails buying the stock with borrowed money. Table 5.3B.2 illustrates capturing the arbitrage profit available with the put option.

Table 5.3B.2. Cash Flow Table for Single Period ABM Model Applied to Put Options

Strategy	Today	Down Event at Expiration	Up Event at Expiration
Buy Put	$-p_{0,q} = -10.37$	$+\max[0, X - (S_0 + d)] = +20.80$	$+\max[0, X - (S_0 + u)] = 0$
Buy h_p Shares	$-h_p S_0 = -46.22$	$+h_p(S_0 + d) = +36.98$	$+h_p(S_0 + u) = +57.78$
Borrow	$+B_c = +56.65$	$-B_c(1 + r) = -57.78$	$-B_c(1 + r) = -57.78$
Net Cash Flow	+0.06*	0	0

* Note the quoted price is \$10.37 and the model price is \$10.42, a difference of \$0.05. The table reports an arbitrage profit of \$0.06 The 0.01 discrepancy is simply rounding error.

Thus, the arbitrageur receives \$0.06 today with no chance of a future liability. Within this simple one period binomial world, trading pressure may simply drive up the quoted put price. Alternatively, buying shares may drive up the quoted stock price with some influence on the put price. Ultimately, the initial net cash flow must be zero. There is another arbitrage opportunity based on put call parity but we will not address it here.

Regardless of the direction of the mispricing, the ability to earn an arbitrage profit would force a price alignment until the option price conforms to the model price.