Module 4.1: Valuation U.S. Treasuries

Learning objectives

- Review nuances of U.S. Treasury bond valuation, including accrued interest, yield to maturity reporting, modified business following, holidays, and bad dates
- Explore computing yield to maturity based on the traditional semi-annual bond equivalent basis as well as the continuously compounded basis
- Introduce constant maturity treasuries as a potential independent source for discounting
- Introduce U.S. Treasury bond valuation based on applying the LSC model to the constant maturity treasury rates

Executive summary

In this chapter, we introduce basic U.S. Treasury (UST) bond math calculation. We further explain several nuances with UST valuation calculations as well as issues related to computing yield to maturity. With this foundation, we extend our work to illustrate valuing UST bonds from a somewhat independent source, the CMT curve. Several interesting properties of USTs were explored with this new perspective.

Central finance concepts

Four finance concepts are covered in this module. First, the traditional approach to valuing U.S. Treasury (UST) instruments is reviewed with particular focus on technical details and quotation conventions. Second, the computation and interpretation of yield to maturity is address both from a traditional semi-annual bond equivalent basis as well as a continuously compounded basis. Third, constant maturity treasuries (CMT) are introduced as a potential independent source for discounting UST avoiding the tautology inherent in the traditional UST valuation approach. Finally, the LSC model is applied to CMT data to form the basis for an independent discount rate that can be applied to all UST instruments.

Traditional U.S. Treasury valuation

The United States Treasury market is very large, comprising approximately 15 percent of the global bond market and approximately 37 percent of the U.S. bond market. At the end of Quarter 1, 2019, the U.S. Treasury instruments outstanding totaled around $16 trillion according to SIFMA, an association representing the financial securities industry.

The standard approach to valuing U.S. Treasury notes and bonds (UST hereafter) is simply the present value of future cash flows. USTs pay semi-annual coupons and are quoted without including accrued interest. The yield to maturity is typically solved implicitly. Thus, the valuation equation above, by definition, will give the market bond value by the nature of how the input yield to maturity is acquired.

The semi-annual coupon amount is computed based on a 30/360 day count basis. Thus, every semi-annual period has 180 days based on a 360 day year. Hence, the semi-annual coupon amount will always be the stated annual amount divided by two.

There are several details related to bond calculations that are typically ignored in introductory materials. We now turn to address several of these minutiae related to USTs.

Payment dates

Legally, how does one handle the case when a coupon date falls on a week-end or is a non-existent date. For example, a UST bond that matures at the end of the month in August (8/31) will have a corresponding coupon that is paid on either February 28 or 29; depending on whether or not it is a leap year.

Often, coupon payment dates fall on week-ends or Federal Reserve System holidays. The formal rule states, "If any principal or interest payment date is a Saturday, Sunday, or other day on which the Federal Reserve System is not open for business, we will make the payment (without additional interest) on the next business day."[1] This payment date approach is known as Modified Business Following or MBF. Thus, to be technically accurate, one must discount coupons paid on week-ends and holidays as if they are paid on the next business day. Further, quality software will account for these technicalities.

[1]See www.gpo.gov/fdsys/granule/CFR-2011-title31-vol2/CFR-2011-title31-vol2-sec356-30.

Day count
As previously mentioned, UST day count convention for dollar coupon amount is 30/360 and paid semi-annually. Thus, the semi-annual dollar coupon amount will simply be one half the stated annual coupon amount. As we will cover next, the fraction of the period is computed on an ACT/ACT basis.
Accrued interest
UST bonds are quoted without accrued interest, whereas stocks are quoted with accrued dividends. For almost all bonds, the dollar amount of the next coupon payment is known as well as the date it will be paid. For stocks on the other hand, corporate executives can change dividend policy on very short notice. Thus, in the bond market, the accrued interest is handled separately from the quoted bond price. The day after a bond coupon payment, the quoted price does not materially change. The comparable ex-dividend date for a stock, the quoted stock price typically drops by about the amount of the dividend payment.

It is important to note that financial decisions should be based on the bond value not the quoted price.

Quoted bond price in the market
UST bonds trade in percentage points plus fractions of 32nds of par value. Thus, 103-083 is 103.26171875% of par. Specifically, the digits 08 denote 8/32nds and the digit 3 denotes 3/8ths of a 32nd. Thus, the quoted bond price is 103 + (8 + 3/8)/32 percent of par. The UST bond price of 98-13+ is shorthand for 98 + (13 + 4/8)/32 percent of par. The plus (+) sign simply denotes 1/2 or more precisely 4/8ths.

The data used in this program was taken from *The Wall Street Journal*. Table 4.1.1 illustrates selected portions of the file.

Table 4.1.1 Reported Closing Data for the UST Market on September 17, 2019

MATURITY	COUPON	BID	ASKED	CHG	ASKED YIE
9/30/19	1	99.31	99.314	unch.	1.518
9/30/19	1.375	99.31	99.314	0.004	1.891
9/30/19	1.75	99.314	100	unch.	1.743
10/15/19	1	99.292	99.296	0.006	1.991
10/31/19	1.25	99.29	99.294	0.006	1.934
10/31/19	1.5	99.296	99.302	0.002	1.976
11/15/19	1	99.264	99.27	unch.	2.012
11/15/19	3.375	100.062	100.066	-0.002	1.993
11/30/19	1	99.25	99.254	unch.	2.037
11/30/19	1.5	99.282	99.286	0.002	2.015
11/30/19	1.75	99.294	99.3	0.004	2.065
12/15/19	1.375	99.274	99.28	0.004	1.901
12/31/19	1.125	99.242	99.246	0.008	1.938
12/31/19	1.625	99.282	99.286	0.002	1.987
12/31/19	1.875	99.306	99.312	0.004	1.946

…

11/15/26	2	101.236	101.242	0.034	1.738
11/15/26	6.5	132.026	132.032	0.732	1.713
2/15/27	2.25	103.166	103.172	0.046	1.739
2/15/27	6.625	133.284	133.29	0.052	1.727
5/15/27	2.375	104.162	104.166	0.042	1.741
8/15/27	2.25	103.206	103.212	0.046	1.752
8/15/27	6.375	134.042	134.046	0.736	1.733
11/15/27	2.25	103.222	103.226	0.048	1.759
11/15/27	6.125	133.064	133.07	0.058	1.738
2/15/28	2.75	107.206	107.212	0.05	1.765
5/15/28	2.875	108.254	108.26	0.068	1.772
8/15/28	2.875	108.316	109.002	0.072	1.777
8/15/28	5.5	130.214	130.22	0.058	1.761
11/15/28	3.125	111.08	111.084	0.064	1.786
11/15/28	5.25	129.096	129.102	0.062	1.767
2/15/29	2.625	107.074	107.08	0.76	1.784
2/15/29	5.25	130.012	130.016	0.746	1.767
5/15/29	2.375	105.064	105.07	0.764	1.784
8/15/29	1.625	98.184	98.19	0.084	1.781
8/15/29	6.125	139.094	139.104	0.092	1.777
5/15/30	6.25	143.03	143.04	0.78	1.787
2/15/31	5.375	136.2	136.21	0.116	1.804
2/15/36	4.5	136.01	136.02	0.86	1.926

Source: *The Wall Street Journal*.

Notice that for many maturities there are several different coupon rates. As we will learn in Chapter 8, the lower the coupon rate the higher the duration which, in this case, implies a higher yield.

Yield to maturity

The traditional yield to maturity is simply a function of the parameters related to a particular UST except in this case, we assume with know the bond price and are seeking to determine the yield to maturity. There is no simple expression where one can solve for the yield to maturity directly.

Yield to maturity is a mathematical result based on the particular bond valuation expression selected. Traditionally, yield to maturity for bonds is reported on a semi-annual discrete compounding basis because most bonds pay coupons semi-annually. In many quantitative applications, continuous compounding makes the analysis more easily tractable. Thus, there are several different bond valuation equations that could be deployed each resulting in different reported yield to maturities.

Regardless of the particular valuation equation selected, given the market price of the bond, we can compute the yield to maturity. Conversely, given the appropriate yield to maturity and using the correct valuation formula, then we could compute the quoted price of the bond. This valuation framework is tautological and hence is rather vacuous.

We seek a valuation framework whose inputs do not directly depend on the reported outputs. We illustrate an alternative valuation approach where the inputs are based on market data not directly derived from the individual bond's market value. But first we need to introduce constant maturity treasuries.

Constant Maturity Treasuries

Constant maturity treasury (CMT) data is freely available in the H.15 file produced by the Board of Governors of the Federal Reserve System.[2] The CMT yields are based on on-the-run (OTR) UST securities. CMT quotes are illustrated in Table 4.1.2.

[2]See https://www.federalreserve.gov/releases/h15/.

Table 4.1.2 Screen Shot of CMT Portion of H.15 File

Instruments	2019 Sep 11	2019 Sep 12	2019 Sep 13	2019 Sep 16	2019 Sep 17
U.S. government securities					
Treasury bills (secondary market) 3 4					
4-week	1.98	1.95	1.95	2.05	2.06
3-month	1.92	1.91	1.92	1.95	1.95
6-month	1.83	1.85	1.87	1.88	1.88
1-year	1.74	1.77	1.82	1.81	1.82
Treasury constant maturities					
Nominal 9					
1-month	2.01	1.99	1.99	2.08	2.10
3-month	1.96	1.95	1.96	1.99	1.99
6-month	1.88	1.90	1.92	1.93	1.93
1-year	1.79	1.82	1.88	1.86	1.87
2-year	1.68	1.72	1.79	1.74	1.72
3-year	1.62	1.67	1.76	1.71	1.68
5-year	1.60	1.65	1.75	1.69	1.66
7-year	1.68	1.72	1.83	1.77	1.75
10-year	1.75	1.79	1.90	1.84	1.81
20-year	2.02	2.06	2.17	2.11	2.08
30-year	2.22	2.22	2.37	2.31	2.27

Source: https://www.federalreserve.gov/releases/H15/. This screen shot captured on September 19, 2019 at 7:16am ET.

According to Jordan and Mansi (2000), "Constant-maturity yields represent yields on Treasury securities at (fixed or constant) maturities of from three months to thirty years that are interpolated by the Department of the Treasury from the daily yield curve. This interpolation is based on the closing market bid yields of the actively traded Treasury securities in the over-the-counter market and calculated from the composites of quotation obtained by the Federal Reserve Bank of New York. Fixed or constant maturities, in this context, mean that this interpolation method provides a yield for a particular maturity even if no outstanding security has exactly that fixed maturity. Constant-maturity yields are not identical to market yields because of the smoothing process and the aging of OTR bonds (Page 2)".

The particular smoothing method applied by the Department of the Treasury is not clear. Again, according to Jordan and Mansi (2000), "At the close of the trading day, composite closing market bids (yields) on outstanding Treasury securities are reported by five U.S. government securities dealers to the Federal Reserve Bank of New York and plotted by the Treasury on a graph. The horizontal axis shows the maturity date of each reported security, and the vertical axis measures the yield; a continuous curve is fitted either using a statistical model or by hand thorough the plotted points (Footnote 7, page 10)".

Applying the LSC Model to CMTs

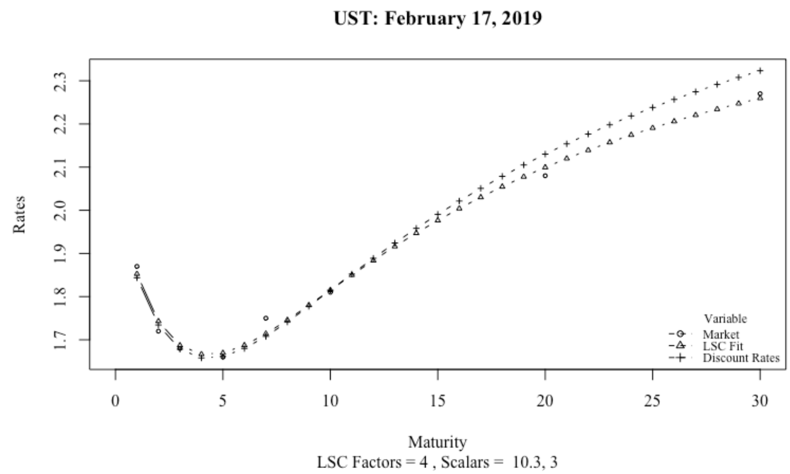
From Module 3.5, we learned the mechanics of the LSC model. In the Additional quantitative materials section below, we illustrate how to value UST bonds based on a fitted constant maturity treasuries (CMT) curve based on the LSC model.

The CMT yields will form the basis for estimating a base spot rate curve. We first fit an LSC model to the CMT yields. Once we obtain the LSC spot yield factors, we can estimate the entire CMT curve. With this complete dataset, we can easily compute the annualized, continuously compounded discount rates for this base curve. These discount rates form the basis for valuing any UST bond.

This is a powerful approach to valuing UST bonds as the discount rates are independent of a particular bond. Thus, we can evaluate all UST bonds and appraise which bonds are over- or under-priced relative to the fitted CMT curve. Hence, we have an independent valuation technique that is no longer tautological.

With a fitted CMT curve, we can easily compute the CMT yields on any maturity as well as compute the corresponding discount rate curve. The R code illustrating this effort is provided. These curves along with the original data are illustrated in Figure 4.1.1.

Figure 4.1.1 CMT Yields With Fitted CMT Yields and Fitted Discount Factors



With these functions, it is straightforward to analyze all USTs as illustrated in Figures 4.1.2. The relative bond valuation error is the fitted price as a percentage of the actual bond price express as a percentage. For maturities less than a year as well as long maturities are very sensitive to errors in the fit. The bond valuation model based solely on the LSC model fit to the CMT yields is

$$V_{LSC,B} = \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right) Par}{\left(1+\frac{r_{LSC,i}}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{r_{LSC,N}}{m}\right)^{N-f}}. \quad (4.0.1)$$

Figure 4.1.2 Relative Bond Valuation Errors

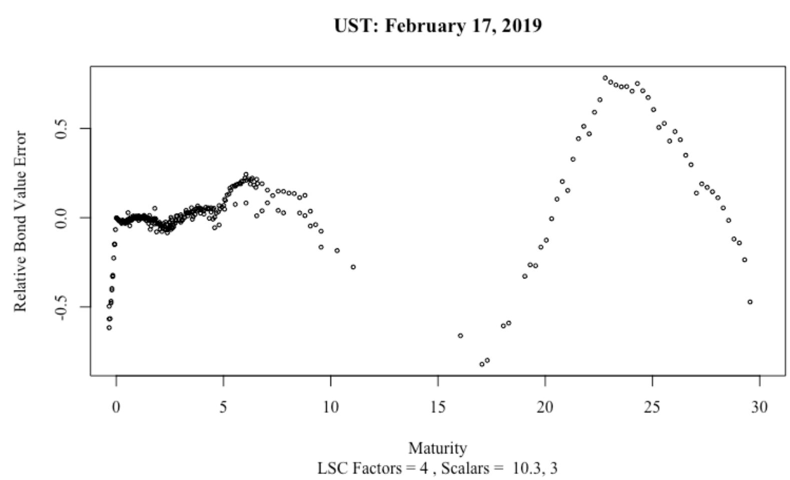


Figure 4.1.3 presents the absolute bond valuation error is the fitted price less the actual bond price express as a percentage of Par. For maturities less than a year as well as long maturities are very sensitive to errors in

the fit. Although a bit hard to see, the errors are less near some CMT rates. Overall, the errors are very small between years 1 to 7.

Figure 4.1.3 Absolute Bond Valuation Errors

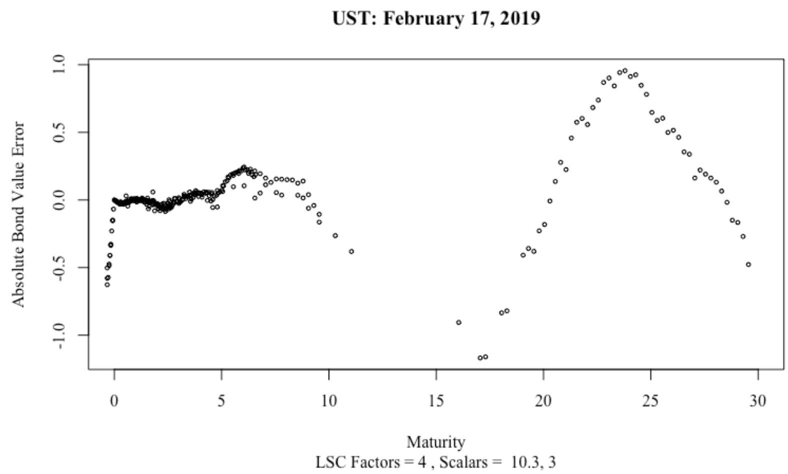
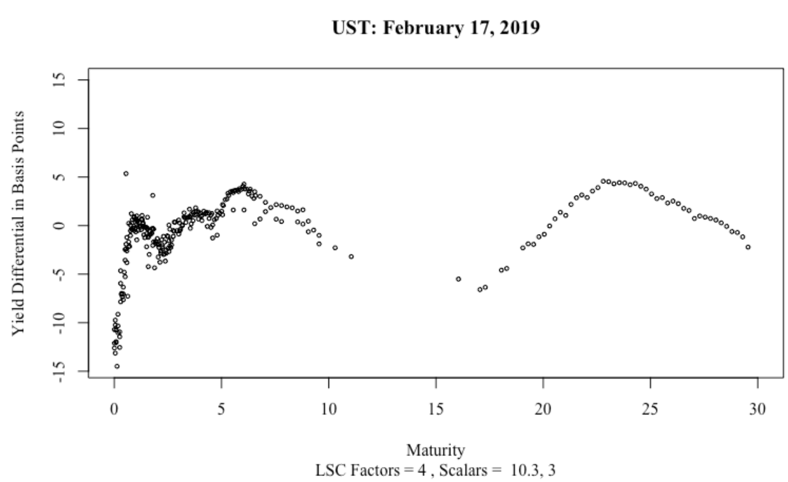


Figure 4.1.4 presents the yield to maturity differential in basis points is the fitted yield to maturity (y_{LSC}) less the market yield to maturity (y) expressed in basis points. The fitted yield to maturity is based on solving for the yield to maturity based on the fitted bond value from the LSC model or

$$V_{LSC,B} = \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y_{LSC}}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y_{LSC}}{m}\right)^{N-f}}. \tag{4.0.2}$$

For longer term bonds, small changes in yield to maturity result in large changes in price so it is no surprise that the error diminishes. One interesting insight gleaned is the role of coupon. We will see later that higher coupon bonds are treated by investors as essentially being shorter maturity bonds, hence they will have lower yield to maturities. We will more thoroughly examine this phenomenon when covering bond duration in later chapters.

Figure 4.1.4 Yield Differential in Basis Points



In summary, we reviewed the traditional discretely compounded UST bond valuation approach with particular focus on the technical nuances such as semi-annual compounding and adjusting for the fraction of the period elapsed since the last coupon date. After reviewing market quotation conventions and providing a conceptual overview of calculating yield to maturity, we introduced the independent CMT data as a means for comparing relative valuation and illustrated some results from the R code. We now explore these topics with a much more quantitative approach.

Quantitative finance materials

In this section, we will analyze in detail the quantitative specifics related to UST valuation and related issues. Specifically, we will provide the precise equations used to compute various values related to USTs, including valuation, yield to maturity, and accrued interest. Finally, we will introduce the LSC model as a means to independently value particular USTs without the tautological reference to its own yield to maturity.

Traditional U.S. Treasury valuation

The UST bond's value today, V_B, depends on the number of payments per year, $m = 2$, the par value, *Par*, fraction of payment period elapsed already, f, the number of remaining coupon payments, N, the stated annual dollar coupon rate in decimal, CR, and the traditional discretely compounded yield to maturity in decimal, y.[3] Mathematically, the bond value can be expressed as

$$V_B = \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}}. \text{ (\textbf{Discretely Compounded Valuation})} \quad (4.0.3)$$

We illustrate this calculation with two actual UST bonds of the same maturity, a newly issued roughly par bond and an outstanding premium bond.

Payment dates, day count, and accrued interest

Recall USTs are based on MBF. Thus, to be technically accurate, one must discount coupons paid on week-ends and holidays as if they are paid on the next business day. Recall, UST day count convention for dollar coupon amount is 30/360 and paid semi-annually. A particular UST's accrued interest is based on a legal agreement and not mathematics, where

$$AI_B = f\frac{CR}{m}Par. \quad (4.0.4)$$

The fraction of the elapsed period since the last coupon payment date is based on the actual number of accrued days divided by the actual number of total days

The fraction of the payment period that has elapsed already, f, is computed based on an ACT/ACT basis. That is, first you compute the number of accrued days between the last coupon date and the current date or NAD. More precisely, the NAD is the actual number of days since the last stated coupon payment date and the current settlement date. Next, you compute the number of total days between coupon payment dates or NTD. More precisely, NTD is the actual number of days between the last stated coupon payment date and the next stated coupon payment date. Consequently,

$$f = \frac{NAD}{NTD}. \quad (4.0.5)$$

Hence, the quoted bond price is more completely expressed as

[3]Note the traditional method assumes semi-annual compounding frequency.

$$QP_B = V_B - AI_B$$
$$= \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}} - \left(\frac{NAD}{NTD}\right)\frac{CR}{m}Par \quad (4.0.6)$$

It is important to note that financial decisions should be based on the bond value not the quoted price.

Numerical illustration: Newly issued UST

We first examine a newly issued UST bond. Specifically, the data was based on a June 4, 2021 settlement date for a 2.25% UST issued on 5/15/2021 and maturing on 5/15/2041. The observed quoted bond price was 100-13 and the reported yield to maturity was 2.224632%. Based solely on this information, we can extract several results.

First, the number of accrued days is 20 [NAD = 16 (May) + 4 (June)] and the number of total days in the semi-annual period is 184 [NTD =16 (May) + 30 (June) + 31 (July) + 31 (August) + 30 (September) + 31 (October) + 15 (November)]. Thus, we have

$$f = \frac{NAD}{NTD} = \frac{20}{184} = 0.1086957.$$

The accrued interest amount assuming \$100 par and coupon rate of 2.25% is

$$AI_B = f\frac{CR}{m}Par$$
$$= 0.1086957\frac{0.0225}{2}100.$$
$$= 0.1222826$$

The accrued interest amount is consistent reported accrued interest of \$1,222.83 for \$1,000,000 par.

Third, the number of remaining coupons is 40 because this 20 year bond was recently issued. Thus, the market value of the bond therefore is

$$V_B = \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}}$$
$$= \sum_{i=1}^{40} \frac{\left(\frac{0.0225}{2}\right)100}{\left(1+\frac{0.0224632}{2}\right)^{i-0.1086957}} + \frac{100}{\left(1+\frac{0.0224632}{2}\right)^{40-0.1086957}}.$$
$$= 100.5285$$

The quoted price of 100-13 is equal to \$100.40625. Adding the accrued interest results in a bond value of \$100.528533, consistent with our valuation. Note that there are 19.94565 years remaining [= 20 – 0.1086957(2)].

Numerical illustration: Previously issued UST

We now examine a UST bond that has been outstanding for a while. Specifically, the data was based on a June 4, 2021 settlement date for a 4.375% UST issued on 5/16/2011 and maturing on 5/15/2041. The observed quoted bond price was 136-05 and the reported yield to maturity was 2.138633%. Again based solely on this information, we can extract several results. Given the same maturity as the newly issued UST, the number of accrued days is 20 and the number of total days in the semi-annual period is 184; hence, the fraction of the period elapsed is 0.1086957 as before. The accrued interest in this case is 0.2377717 for \$100 par consistent with market observations.

Again, assuming \$100 par value with coupon rate is 0.04375 and the number of remaining coupons is 40 with semi-annual pay ($m = 2$), then the bond market value is

$$\begin{aligned}V_B &= \sum_{i=1}^{N} \frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}} \\ &= \sum_{i=1}^{40} \frac{\left(\frac{0.04375}{2}\right)100}{\left(1+\frac{0.02138633}{2}\right)^{i-0.1086957}} + \frac{100}{\left(1+\frac{0.02138633}{2}\right)^{40-0.1086957}}. \\ &= 136.394\end{aligned}$$

The quoted price of 136-05 is equal to \$136.15625. Adding the accrued interest results in a bond value of \$136.156228, consistent with our valuation.

Yield to maturity

Recall the traditional yield to maturity is simply a function of the parameters related to a particular UST where QP_B denotes the quoted market price of the bond. That is,

$$y = f\left(QP_B, CR, m, Par, NAD, NTD\right). \tag{4.0.7}$$

There is no simple expression where one can solve for y directly. Fortunately, there are several search routines that will rapidly find the solution for y that makes Equation (4.0.3) hold.

Frequently, it is convenient to use annualized continuously compounded yield to maturity, denoted y_c. The bond valuation equation is expressed as

$$\begin{aligned}QP_B &= V_B - AI_B \\ &= \sum_{i=1}^{N} e^{-y_c\tau_i}\left(\frac{CR}{m}\right)Par + e^{-y_c\tau_N}Par - \left(\frac{NAD}{NTD}\right)\frac{CR}{m}Par\end{aligned}. \ \textbf{(Continuously Compounded Valuation)} \tag{4.0.8}$$

where $\tau_i = (i - f)/2$ and $\tau_N = (N - f)/2$. Thus the annualized continuously compounded yield to maturity is found based on Equation (4.0.8) as

$$y_c = f_c\left(QP_B, CR, m, Par, NAD, NTD\right). \tag{4.0.9}$$

Yield to maturity requires an iterative search algorithm of some form. Typically, these solution tecnologies require reframing the initial problem so the equation equals zero when the correct yield to maturity is found. Recall, the bond model value (V_B) less accrued interest (AI_B) should equal to the bond quoted price (QP_B) or solving for zero we have

$$V_B - AI_B - QP_B = 0. \tag{4.0.10}$$

Substituting for bond model value and accrued interest we have

$$\sum_{i=1}^{N}\frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}}+\frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}}-\left(\frac{NAD}{NTD}\right)\frac{CR}{m}Par-QP_B=0. \tag{4.0.11}$$

Based on the newly issued bond information provided above, we have

$$\sum_{i=1}^{N}\frac{\left(\frac{0.0225}{2}\right)100}{\left(1+\frac{y}{2}\right)^{i-0.1086957}}+\frac{100}{\left(1+\frac{y}{2}\right)^{N-0.1086957}}-\left(\frac{20}{184}\right)\frac{0.0225}{2}100-100.8125=0. \tag{4.0.12}$$

Thus, we solve for the implied yield to maturity of 2.224632%. According to the observed market data, the reported yield to maturity of 2.224664% easily within rounding error.

Applying the same procedure with the premium bond, we have

$$\sum_{i=1}^{N}\frac{\left(\frac{0.04375}{2}\right)100}{\left(1+\frac{y}{2}\right)^{i-0.1086957}}+\frac{100}{\left(1+\frac{y}{2}\right)^{N-0.1086957}}-\left(\frac{20}{184}\right)\frac{0.04375}{2}100-136.15625=0. \tag{4.0.13}$$

Again, we solve for the implied yield to maturity of 2.138643%. Again, according to the observed market data, the reported yield to maturity of 2.138633% easily within rounding error.

We now provide further insights on applying the LSC model to CMT yields.

Applying the LSC model to CMTs

Recall that yield to maturity is a simple mathematical artifact of applying quoted UST bond prices and other inputs to the bond valuation equations. Similarly, the bond value is a simple mathematical artifact of applying quoted yield to maturity and other inputs to the iterative search routine for finding yield to maturity. We have worked our way through the tautology. Surely we can enhance our UST valuation methods so that it will not be simple tautologies. We illustrate one method of many that allow us to escape the tautological trap.

From Module 3.5, we learned the mechanics of the LSC model. We now illustrate a powerful application. By applying the LSC model to CMT yields, we have

$$y_{CMT,i}=\sum_{j=0}^{N}x_{i,j}f_j, \tag{4.0.14}$$

where $y_{CMT,i}$ denotes some input maturity time variables such as an interest rate for some maturity corresponding to i, $x_{i,j}$ denotes input LSC coefficients based on some maturity and some factor, and f_j denotes the output factors. The LSC model in general form assumes

$$x_{i,0}=1,\; x_{i,1}=\frac{s_1}{\tau_i}\left(1-e^{-\tau_i/s_1}\right), \text{ and } x_{i,j}=\frac{s_j}{\tau_i}\left(1-e^{-\tau_i/s_j}\right)-e^{-\tau_i/s_j}; j>1. \tag{4.0.15}$$

Recall the input scalars, s_j, are defined where $s_1 = s_2$. Again s_j denotes scalars that applies various weights to different locations on the term structure, $x_{i,j}$ denotes LSC maturity coefficients, and f_j denotes the output LSC factor, a parameter that is typically found using ordinary least squares regression applied to maturity time CMT yields.

Thus, the LSC model is applied to CMT yields to form the basis for estimating a base spot rate curve. Here, we first fit an LSC model to the CMT rates. Once we obtain the LSC spot rate factors, we can estimate the entire CMT curve. With this complete dataset, as before we can easily compute the annualized, continuously compounded discount rates for this base curve. We term this component of the analysis the base rate function.

For our purposes, we now decompose the UST value into the major component, the base rate function, and solve for any remaining error over or under the base rate function. Thus, we can decompose the spot yield, y_i, into LSC spot rates, r_i^{LSC} and an error term ε_i. Note that with this approach any error in estimating LSC models is completely captured in the error term. Therefore, the current bond value can be expressed as

$$\begin{aligned} V_B &= \sum_{i=1}^{N_t} CF_i DF_i = \sum_{i=1}^{N_t} CF_i e^{-y_i \tau_i} \\ &= \sum_{i=1}^{N_t} CF_i e^{-\left(r_i^{LSC} + \varepsilon_i\right)\tau_i} = \sum_{i=1}^{N_t} CF_i e^{-\left(\sum_{j=0}^{N^r} x_{i,j} f_j^r + \varepsilon_i\right)\tau_i} \end{aligned}. \tag{4.0.16}$$

As calendar time will be an important component, we add the subscript t to the LSC model notation.

From Module 3.5, we apply the LSC model to UST yields, we have

$$y_i = \sum_{j=0}^{N^r} x_{i,j} f_j^r + \varepsilon_i . \tag{4.0.17}$$

We define the portion of the yield to maturity attributable to the base rate as

$$r_i^{LSC} \equiv \sum_{j=0}^{N^r} x_{i,j} f_j^r , \tag{4.0.18}$$

where the LSC parameters are given in Equation (4.0.15).

Thus, the approximate value of the bond, based on the LSC model, is expressed as

$$V_B \cong V_B^{LSC} = \sum_{i=1}^{N_t} CF_i DF_i^{LSC} = \sum_{i=1}^{N_t} CF_i e^{-r_i^{LSC} \tau_i} . \tag{4.0.19}$$

With enough LSC factors, the residual error will be negligible. Thus, we define the discount factor based solely on the LSC model applied to the base rates as

$$DF_i^{LSC} \equiv e^{-r_i^{LSC} \tau_i} . \tag{4.0.20}$$

Thus, we have a framework for appraising the relative value of UST bonds. Clearly, if a particular bond has a significantly positive error term, then that implies the potential mispricing where the UST bond is underpriced. Alterntatively, if a particular bond has a significantly negative error term, then that implies the potential mispricing where the UST bond is overpriced.

Summary

In this chapter, we introduced basic U.S. Treasury (UST) bond math calculation. We further explained several nuances with UST valuation calculations as well as issues related to computing yield to maturity. To illustrate being creative, we demonstrate valuing UST bonds from a somewhat independent source, the CMT curve. The LSC model was applied. Several interesting properties of USTs were explored with this new perspective.

References

See https://en.wikipedia.org/wiki/Bond_valuation and references contained therein.