Module 4.1

Valuation US Treasuries

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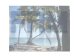
Overview

- Review technical details of US Treasuries (USTs)
- Understand weaknesses of traditional valuation approach
- Apply LSC model to constant maturity treasuries (CMT) for the purpose of relative UST valuation



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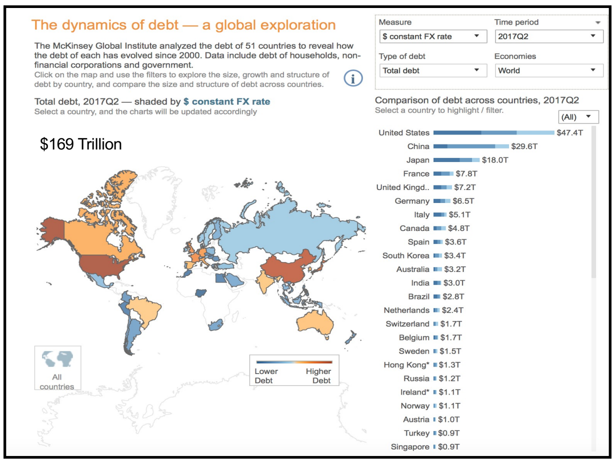

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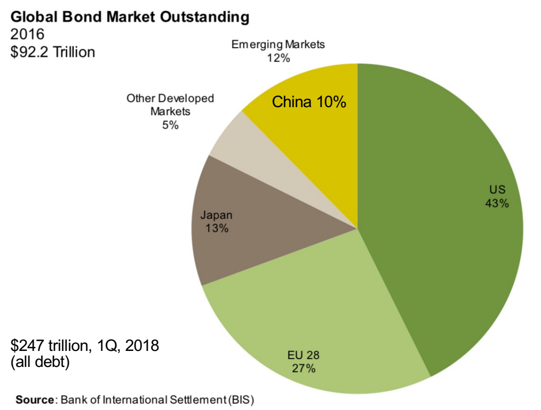

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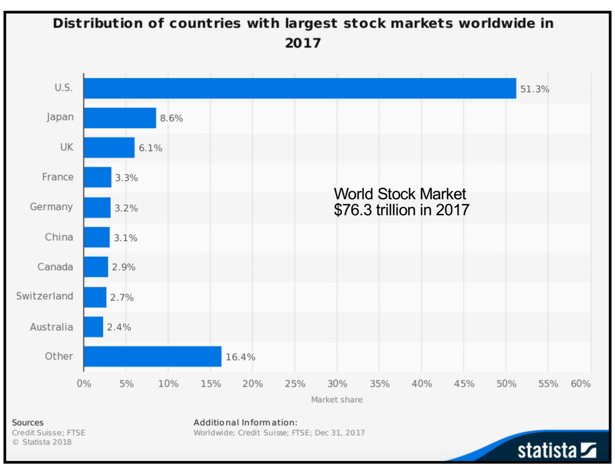

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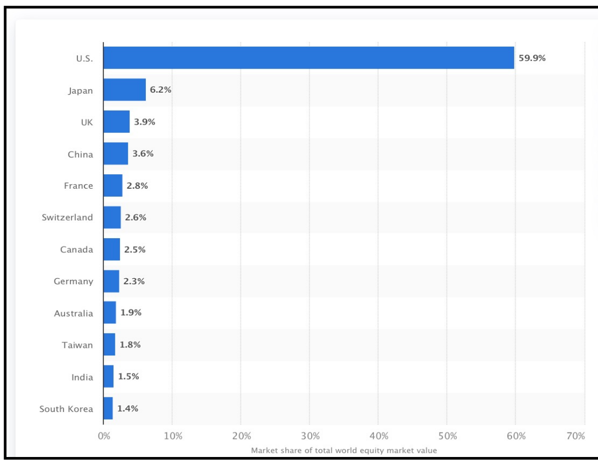

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US Treasury Market

- 15% of global bond market
- 37% of US bond market
- $16 trillion (Q1, 2019)
- UST denotes both notes and bonds
  - Semi-annual pay
  - Quoted without accrued interest
  - Quoted as % of Par

Central Finance Concepts

- Traditional U.S. Treasury valuation
- Quoted bond price in the market
- Yield to maturity
- Constant Maturity Treasuries (CMT)

UST Quotation Conventions

- Quoted as percent of par
  - 101-16 = 101 + 16/32 = 101.5% of Par
  - 101-165 = 101 + (16 + 5/8)/32
    = 101 + 16.625/32 = 101.51953125
  - 101-16+ = 101 + (16 + 4/8)/32
    = 101 + 16.5/32 = 101.515625
- Par = $100 (and multiples of $100)

2/17/19
WSJ

MATURITY	COUPON	BID	ASKED	CHG	ASKED YIE
9/30/19	1	99.31	99.314	unch.	1.518
9/30/19	1.375	99.31	99.314	0.004	1.891
9/30/19	1.75	99.314	100	unch.	1.743
10/15/19	1	99.292	99.296	0.006	1.991
10/31/19	1.25	99.29	99.294	0.006	1.934
10/31/19	1.5	99.296	99.302	0.002	1.976
11/15/19	1	99.264	99.27	unch.	2.012
11/15/19	3.375	100.062	100.066	-0.002	1.993
11/30/19	1	99.25	99.254	unch.	2.037
11/30/19	1.5	99.282	99.286	0.002	2.015
11/30/19	1.75	99.294	99.3	0.004	2.065
12/15/19	1.375	99.274	99.28	0.004	1.901
12/31/19	1.125	99.242	99.246	0.008	1.938
12/31/19	1.625	99.282	99.286	0.002	1.987
12/31/19	1.875	99.306	99.312	0.004	1.946
		...			
11/15/26	2	101.236	101.242	0.034	1.738
11/15/26	6.5	132.026	132.032	0.732	1.713
2/15/27	2.25	103.166	103.172	0.046	1.739
2/15/27	6.625	133.284	133.29	0.052	1.727
5/15/27	2.375	104.162	104.166	0.042	1.741
8/15/27	2.25	103.206	103.212	0.046	1.752
8/15/27	6.375	134.042	134.046	0.736	1.733
11/15/27	2.25	103.222	103.226	0.048	1.759
11/15/27	6.125	133.064	133.07	0.058	1.738

Higher Coupons, Larger Change

UST Technical Details

- MBF – modified business following
- Paid on business day, if not, following business day without corresponding interest
- 30/360 day count basis (1/2 stated coupon)
- All coupons will be the same

Yield to Maturity

- Simply a function of the UST market price
- Mathematical result based on a particular bond valuation expression
  - Discretely compounded valuation equation
  - Continuously compounded valuation equation
- Simply a tautology without much insight

Constant Maturity Treasuries

- CMT data is freely available
- Based on on-the-run (OTR) UST securities
- Provides a yield estimate even when no bond is trading at that maturity
- Illustrated on next slide

Instruments	2019 Sep 11	2019 Sep 12	2019 Sep 13	2019 Sep 16	2019 Sep 17
U.S. government securities					
Treasury bills (secondary market) 3 4					
4-week	1.98	1.95	1.95	2.05	2.06
3-month	1.92	1.91	1.92	1.95	1.95
6-month	1.83	1.85	1.87	1.88	1.88
1-year	1.74	1.77	1.82	1.81	1.82
Treasury constant maturities					
Nominal 9					
1-month	2.01	1.99	1.99	2.08	2.10
3-month	1.96	1.95	1.96	1.99	1.99
6-month	1.88	1.90	1.92	1.93	1.93
1-year	1.79	1.82	1.88	1.86	1.87
2-year	1.68	1.72	1.79	1.74	1.72
3-year	1.62	1.67	1.76	1.71	1.68
5-year	1.60	1.65	1.75	1.69	1.66
7-year	1.68	1.72	1.83	1.77	1.75
10-year	1.75	1.79	1.90	1.84	1.81
20-year	2.02	2.06	2.17	2.11	2.08
30-year	2.22	2.22	2.37	2.31	2.27

Constant Maturity Treasuries

- CMT properties
  - 3 months to 30 years
  - Interpolated by the Department of the Treasury from the daily yield curve (not public)
  - Based on the closing market bid yields of the actively traded Treasury securities
  - Available in H.15 file produced by the Board of Governors of the Federal Reserve System

Applying LSC Model to CMTs

- CMT yields form the basis for estimating a base spot rate curve
- Fitted model used to estimate any maturity CMT proxy
- Derived discount rates form basis for valuing any UST
- Provides independent valuation approach

Quantitative Finance Materials

- Review technical details of UST valuation
- Computing yield to maturity
  - Discrete compounding
  - Continuous compounding
- Applying the LSC model to CMTs

Selected Notation

- Coupon – annual dollar coupon
- m – coupons per year
- Par – notional amount (principal)
- f – fraction of payment period elapsed since last coupon (NAD/NTD)
- N – number of remaining cash flows
- CF_i – ith cash flow, i = n: CF_n = (Coupon/m) + Par, otherwise CF_i = (Coupon/m)

Accrued Interest

- Accrued interest is legally defined:

$$AI_B = f\frac{CR}{m}Par$$

- Fraction of period already elapsed

$$f = \frac{NAD}{NTD}$$

- Quoted price

$$QP_B = V_B - AI_B = \sum_{i=1}^{N}\frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}} - \left(\frac{NAD}{NTD}\right)\frac{CR}{m}Par$$

Bond Valuation (BV)

- Discrete compounding, bond equivalent yield

$$V_B = \sum_{i=1}^{N}\frac{\left(\frac{Coupon}{m}\right)Par}{\left(1+\frac{y_d}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y_d}{m}\right)^{N-f}} = \sum_{i=1}^{N}\frac{CF_i}{\left(1+\frac{y_d}{m}\right)^{i-f}}$$

- Continuous compounding

$$V_B = \sum_{i=1}^{N}\left(\frac{Coupon}{m}\right)Par\left(e^{-y_c\tau_i}\right) + Par\left(e^{-y_c\tau_i}\right) = \sum_{i=1}^{N}CF_i\left(e^{-y_c\tau_i}\right) = \sum_{i=1}^{N}CF_i\left[e^{-y_c\left(\frac{i-f}{m}\right)}\right]$$

- Note that the bond value does not change, but the yields are different

Bond Valuation Example

- Settlement date: 6/4/2021
- Maturity date: 5/15/2041
- Coupon rate: 2.25%
- Quoted bond price: 100-13
- Reported yield to maturity: 2.224632%
- Thus, semi-annual pay implies 40 remaining coupons

Intermediate Calculations

- *NAD* = 20 (16 in May and 4 in June)
- *NTD* = 184 (16+30+31+31+30+31+15)
- *f* = *NAD*/*NTD* = 20/184 = 0.1086957
- $AI_B = f(CR/m)Par$ = 0.1086957(0.0225/2)100 = 0.1222826
- Thus, accrued interest is $1,222.83 per $1,000,000 par

Bond Valuation

- Based on the data above, we have

$$V_B = \sum_{i=1}^{N}\frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}} + \frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}}$$

$$= \sum_{i=1}^{40}\frac{\left(\frac{0.0225}{2}\right)100}{\left(1+\frac{0.0224632}{2}\right)^{i-0.1086957}} + \frac{100}{\left(1+\frac{0.0224632}{2}\right)^{40-0.1086957}}$$

$$= 100.5285$$

- Note: 100-13 implies 100.40625
- Adding AI, market value of 100.528533 (close enough due to y rounding)

Bond Yield to Maturity (*y*)

- Derived from simple bond math for

$$y = f\left(QP_B, CR, m, Par, NAD, NTD\right)$$

- The bond function can alternatively be based on continuously compounded formula

$$QP_B = V_B - AI_B$$

$$= \sum_{i=1}^{N}e^{-y_c\tau_i}\left(\frac{CR}{m}\right)Par + e^{-y_c\tau_N}Par - \left(\frac{NAD}{NTD}\right)\frac{CR}{m}Par$$

$$y_c = f_c\left(QP_B, CR, m, Par, NAD, NTD\right)$$

Solving for *y*

- Setting the bond valuation equation to zero

$$\sum_{i=1}^{N}\frac{\left(\frac{CR}{m}\right)Par}{\left(1+\frac{y}{m}\right)^{i-f}}+\frac{Par}{\left(1+\frac{y}{m}\right)^{N-f}}-\left(\frac{NAD}{NTD}\right)\frac{CR}{m}Par-QP_B=0$$

- Example with prior data

$$\sum_{i=1}^{N}\frac{\left(\frac{0.0225}{2}\right)100}{\left(1+\frac{y}{2}\right)^{i-0.1086957}}+\frac{100}{\left(1+\frac{y}{2}\right)^{N-0.1086957}}-\left(\frac{20}{184}\right)\frac{0.0225}{2}100-100.8125=0$$

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Weakness of Yield to Maturity

- Overlapping yields
  - 5 year compared to 10 year
  - Marginal information of the next year
- Sharply declining volatility of yields for longer maturities
- Coupon effects

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LSC Application

- OLS regression

$$y_{CMT,i}=\sum_{j=0}^{N}x_{i,j}f_j$$

- Coefficients

$$x_{i,0}=1$$

$$x_{i,1}=\frac{s_1}{\tau_i}\left(1-e^{-\tau_i/s_1}\right)$$

$$x_{i,j}=\frac{s_j}{\tau_i}\left(1-e^{-\tau_i/s_j}\right)-e^{-\tau_i/s_j};j>1$$

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LSC applied to UST

- UST valuation independent of individual UST

$$V_B=\sum_{i=1}^{N_t}CF_iDF_i=\sum_{i=1}^{N_t}CF_ie^{-y_i\tau_i}$$

$$=\sum_{i=1}^{N_t}CF_ie^{-\left(r_i^{LSC}+\varepsilon_i\right)\tau_i}=\sum_{i=1}^{N_t}CF_ie^{-\left(\sum_{j=0}^{N^r}x_{i,j}f_j^r+\varepsilon_i\right)\tau_i}$$

- Spot rates and LSC estimated spot rates

$$y_i=\sum_{j=0}^{N^r}x_{i,j}f_j^r+\varepsilon_i \qquad r_i^{LSC}\equiv\sum_{j=0}^{N^r}x_{i,j}f_j^r$$

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Relative UST Valuation

- Assuming no error in LSC estimate

$$V_B\cong V_B^{LSC}=\sum_{i=1}^{N_t}CF_iDF_i^{LSC}=\sum_{i=1}^{N_t}CF_ie^{-r_i^{LSC}\tau_i}$$

- Each discount factor based on LSC model

$$DF_i^{LSC}\equiv e^{-r_i^{LSC}\tau_i}$$

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R Code Comments

- Valuation UST Test.R (Traditional)
  - UST Functions.R
- UST Book Spreads Over CMT Test.R (LSC)
  - UST Book Inputs.R
  - USTYYYYMMDD.xlsx
  - CMTYYYYMMDD.xlsx

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Traditional Analysis Key Functions

- CouponsRemaining(B): Number of remaining payments, semi-annual frequency
- Elapsed(B): Fraction of coupon period elapsed, last, next, & current date
- FractionElapsed(B): Only fraction of period elapsed
- AccruedInterest(B): Dollar accrued interest based on par amount

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Traditional Analysis Key Functions

- BondValue(B): Dollar bond value including accrued interest
- TimeToMaturity(B): Years to maturity of bond
- PriceDifference(YTM, B): Difference between market price and model value
- YieldToMaturitySolver(B): Estimates yield to maturity using optimize

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Input Structure in Data Frame

```
#
# UST functions (semi-annual only)
#
source("UST Functions.R")
BONDInputData <- list(inputFrequency, inputCouponRate, inputPar,
  inputYieldToMaturity, inputYtMType, inputBondPrice,
  SettlementDateMonth, SettlementDateDay, SettlementDateYear,
  MaturityDateMonth, MaturityDateDay, MaturityDateYear)
names(BONDInputData) <- c("Frequency", "CouponRate", "Par",
  "YieldToMaturity", "YtMType", "BondPrice",
  "SettlementDateMonth", "SettlementDateDay", "SettlementDateYear",
  "MaturityDateMonth", "MaturityDateDay", "MaturityDateYear")
# Data frame easier to manage later
BONDInputData <- as.data.frame(BONDInputData)
```


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Intermediate Building Blocks

```
#
# Calendar manipulations
#
N = CouponsRemaining(BONDInputData)
# ElapsedOutput contains fraction, JLastDate, JNextDate, and JCurrentDate
ElapsedOutput = Elapsed(BONDInputData)
# Number of Total Days
NTD <- ElapsedOutput$NextDate - ElapsedOutput$LastDate
# Number of Accrued Days since last semi-annual coupon
NAD <- ElapsedOutput$Fraction * NTD
# Fraction of coupon period that has elapsed already
f <- ElapsedOutput$Fraction
# Bond maturity, in years
Mat <- TimeToMaturity(BONDInputData)
NAD; NTD; f; N; Mat
```


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Example Bond Function

```
#
# BondValue: Dollar value of bond including accrued interest (Traditional method)
#
BondValue = function(B){
  with(B,{
    PV = 0.0
    RemainingCoupons = CouponsRemaining(B)
    ElapsedTime = FractionElapsed(B)
    if(YtMType == 'BEY'){
      for(i in 1:RemainingCoupons){
        PV = PV + ( (CouponRate/(Frequency*100.0))*Par ) /
          ((1.0 + (YieldToMaturity/(Frequency*100.0)))^(i - ElapsedTime))
      }
      PV = PV+Par/((1.0 +
        (YieldToMaturity/(Frequency*100.0)))^(RemainingCoupons - ElapsedTime))
    } else if(YtMType == 'CC'){
      for(i in 1:RemainingCoupons){
        PV = PV + ( (CouponRate/(Frequency*100.0))*Par ) *
          exp(-(YieldToMaturity/100)*((i - ElapsedTime)/Frequency))
      }
      PV = PV + Par *
        exp(-(YieldToMaturity/100)*((RemainingCoupons - ElapsedTime)/Frequency))
    } else PV = -99
    return( PV )
  })
}
```

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Deploying the LSC Model

- UST Book Spreads Over CMT Test.R
- Input files
  - UST parameters for portfolio
  - CMT parameters
- Outputs
  - Various data derived from valuing UST bonds based on CMT curve

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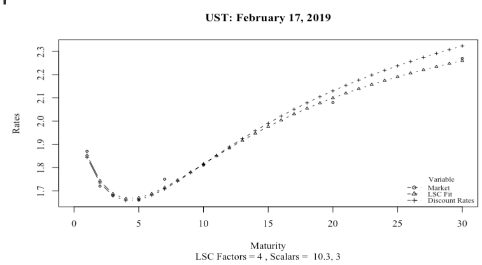
LSC CMT Example (4 Factors)



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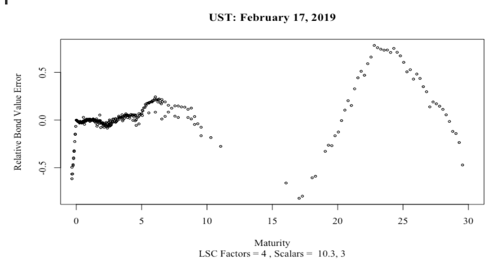
Relative Error in LSC Fit



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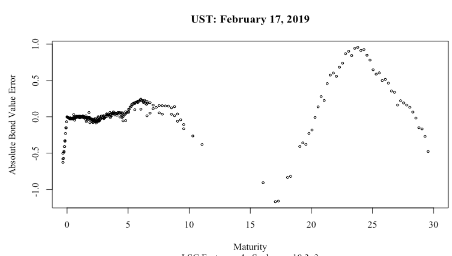
Absolute Error (% of Par)



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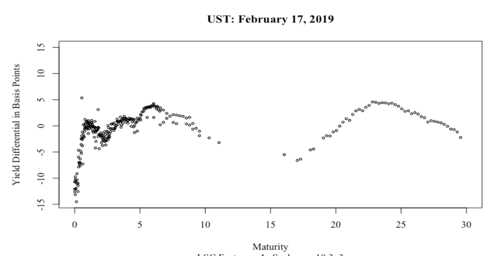
Yield Differential (in bps)



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Summary

- Reviewed technical details of US Treasuries (USTs)
- Weaknesses of traditional valuation approach (overlapping yields)
- Apply LSC model to constant maturity treasuries (CMT) for the purpose of relative UST valuation

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