Module 3.7

Numerical Integration and the Lognormal Distribution

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Overview

- Explore numerical integration
  - Lognormal distribution
  - Normal distribution
  - Deep dive into properties as applied in finance
- Numerous comparisons and contrasts
- Bivariate distributions

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Univariate Distributions

- Cumulative distribution function (CDF)

$$F_X(x) \equiv \Pr(X \le x) \quad \text{(generic)}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(j)\,dj \quad \text{(continuous)}$$

- Probability density function (continuous)

$$f_X(j)$$

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Lognormal Distribution

- Parameters: μ (Mean), σ (Std. Dev.)
- Limits
  - Mean: $-\infty < \mu < +\infty$
  - Standard deviation: $\sigma > 0$
  - Range: $0 < x < +\infty$
- Note: x cannot equal zero
- If $y = \ln(x)$ is normally distributed (ND), then x is said to have a lognormal distribution (LD)

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ND and LD CDF and PDF

- CDF: $\Lambda(\mu,\sigma) = \Lambda(d) = \int_{-\infty}^{d} \frac{e^{-\frac{[\ln(x)-\mu]^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}}dx$ (Lognormal CDF) and

$$N(\mu,\sigma) = N(d) = \int_{-\infty}^{d} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}dx \text{ (Normal CDF).}$$

- PDF:

$$\lambda(\mu,\sigma) = \lambda(x) = \frac{e^{-\frac{[\ln(x)-\mu]^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} \text{ (Lognormal PDF) and}$$

$$n(\mu,\sigma) = n(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \text{ (Normal PDF).}$$

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Median, Mode, Mean

- Median: $\hat{x}$ such that $\int_{0}^{\hat{x}} f(x)dx = \int_{\hat{x}}^{+\infty} f(x)dx = \frac{1}{2}$.

$Median_\lambda = e^{\mu}$ and
$Median_n = \mu$.

- Mode: $f'(x) = 0$ and $f''(x) < 0$ $\quad Mode_\lambda = e^{\mu-\sigma^2}$ and
$Mode_n = \mu$.
- Mean: $Mean_\lambda = e^{\mu+\frac{\sigma^2}{2}}$ and
$Mean_n = \mu$.

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Variance and Skewness

- Variance: $\mu_{2\lambda} = Variance_\lambda = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$ and

  $\mu_{2n} = Variance_n = \sigma^2$.
- Skewness: $\mu_{3\lambda} = e^{3\mu+\frac{3\sigma^2}{2}}\left(e^{\sigma^2}-1\right)^2\left(e^{\sigma^2}+2\right)$ and

  $\mu_{3n} = 0$.
- Normalized Skewness: $\gamma_{S\lambda} = \frac{\mu_{3\lambda}}{\mu_{2\lambda}^{3/2}} = \sqrt{e^{\sigma^2}-1}\left(e^{\sigma^2}+2\right)$ and

  $\gamma_{Sn} = \frac{\mu_{3n}}{\mu_{2n}^{3/2}} = 0$.

Kurtosis

- Kurtosis:

  $\mu_{4\lambda} = e^{4\mu+2\sigma^2}\left(e^{\sigma^2}-1\right)^2\left(e^{4\sigma^2}+2e^{3\sigma^2}+3e^{2\sigma^2}-3\right)$ and

  $\mu_{4n} = 3\sigma^4$.
- Excess kurtosis:

  $\gamma_{K\lambda} = \frac{\mu_{4\lambda}}{\mu_{2\lambda}^2} - 3 = e^{4\sigma^2}+2e^{3\sigma^2}+3e^{2\sigma^2}-6$ and

  $\gamma_{Kn} = \frac{\mu_{4n}}{\mu_{2n}^2} - 3 = 0$.

Asset Price Distributions

- If $x \sim N\left(\mu_g,\sigma_g\right)$ (g denotes generic) and $y = e^x$, then $y \sim \Lambda\left(\mu_g,\sigma_g\right)$
- Rates of return: $S_T = S_t e^{R(T-t)}$
- Distribution of returns:

  $R \sim N\left[\mu(T-t),\sigma\sqrt{T-t}\right]$
- Distribution of prices:

  $S_T \sim \Lambda\left[\ln(S_0)+\mu(T-t),\sigma\sqrt{T-t}\right]$

Asset Price Distribution

- Terminal stock prices:

  $E\left(S_T\right) = S_0 e^{\left(\mu+\frac{\sigma^2}{2}\right)(T-t)}$ $\quad Var\left(S_T\right) = S_0^2\left(e^{2\left(\mu+\sigma^2\right)(T-t)} - e^{\left(2\mu+\sigma^2\right)(T-t)}\right)$
- Returns:

  $\mu = \ln\left[\frac{E\left(S_t\right)}{S_0}\right] \Big/ (T-t)$ $\quad \sigma^2 = \ln\left[\frac{Var\left(S_t\right)}{E\left(S_t\right)^2}+1\right] \Big/ (T-t)$

Asset Price Distributions

- Suppose S = 100, T = 1, $\hat{\mu} = 12$, σ = 30%
- Intend: $E\left(S_T\right) = S_0 e^{\hat{\mu}(T-t)} = 100e^{0.12(1)} = 112.749685$
- Two versions: $E\left(S_T\right) = S_0 e^{\hat{\mu}(T-t)}$ and

  $E\left(S_T\right) = S_0 e^{\left(\mu+\frac{\sigma^2}{2}\right)(T-t)}$.

$$E\left(S_T\right) = S_0 e^{\left(\mu+\frac{\sigma^2}{2}\right)(T-t)} = S_0 e^{\hat{\mu}(T-t)} = 100e^{0.12(1)} = 112.749685$$

$$\mu = \hat{\mu} - \frac{\sigma^2}{2} = 0.12 - \frac{0.3^2}{2} = 0.075.$$

Asset Price Distributions

$$S_T \sim \Lambda\left[\ln(S_0)+\left(\hat{\mu}-\frac{\sigma^2}{2}\right)(T-t),\sigma\sqrt{T-t}\right]$$
$$= \Lambda\left[\ln(S_0)+\mu(T-t),\sigma\sqrt{T-t}\right]$$
$$= \Lambda\left[\ln(100)+0.075(1),0.30\sqrt{1}\right] = \Lambda(4.680170,0.30)$$

$$Median = e^{\ln(S_0)+\left(\hat{\mu}-\frac{\sigma^2}{2}\right)(T-t)} = e^{4.680170} = 107.788415,$$

$$Mode = e^{\ln(S_0)+\left(\hat{\mu}-\frac{\sigma^2}{2}\right)(T-t)-\sigma^2(T-t)} = e^{4.680170-0.09(1)}, \text{ and}$$
$$= e^{4.680170-0.09(1)} = 98.511194$$

$$Variance = e^{2[4.680170]+0.09^2(1)}\left(e^{0.09^2(1)}-1\right) = 1{,}197.189781.$$

standard deviation is 34.600430.

Role of Standard Deviation

- Increasing return standard deviation (volatility) implies terminal price has
  - Increasing mean
  - Constant median
  - Decreasing mode
  - Increasing skewness
  - Increasing likelihood of very low values

Figure 3.7.1A. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 30%
Probability Density Function

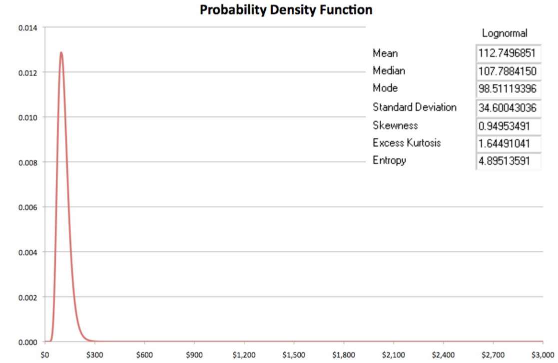

Figure 3.7.1B. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 30%
Cumulative Distribution Function

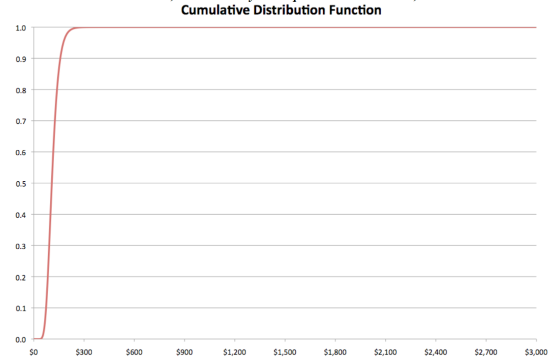

Figure 3.7.2A. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 80%
Probability Density Function

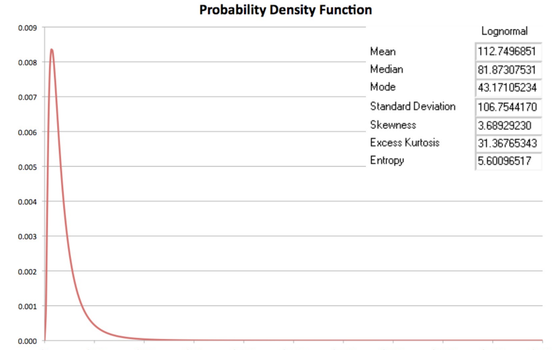

Figure 3.7.2B. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 80%
Cumulative Distribution Function

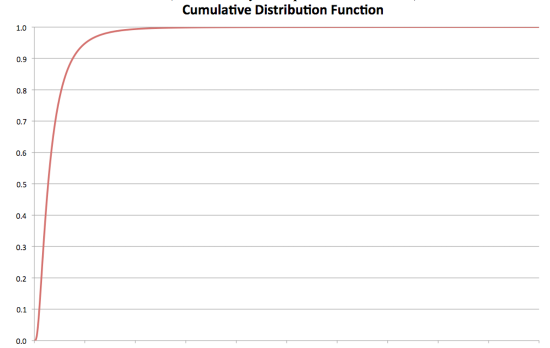

Figure 3.7.3A. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 130%
Probability Density Function

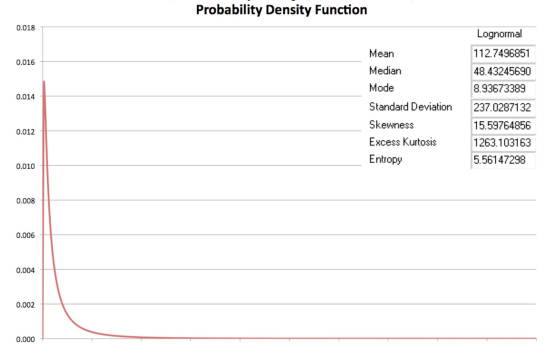

Figure 3.7.3B. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 130%

Cumulative Distribution Function

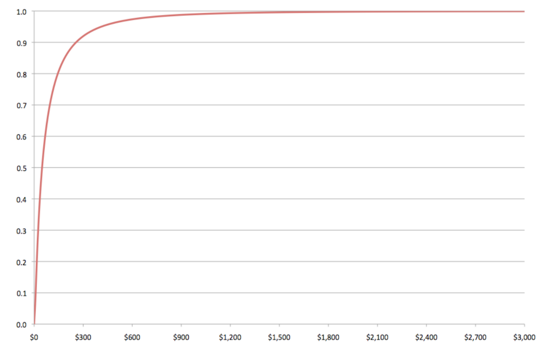

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Figure 3.7.4A. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 180%

Probability Density Function

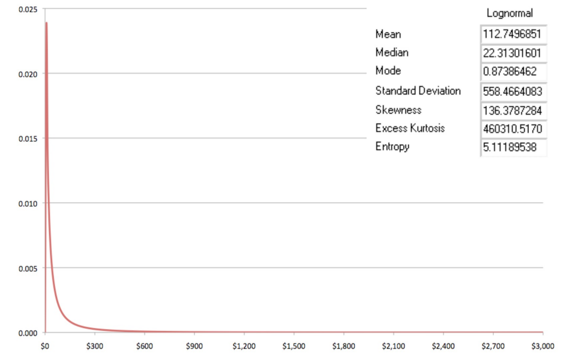

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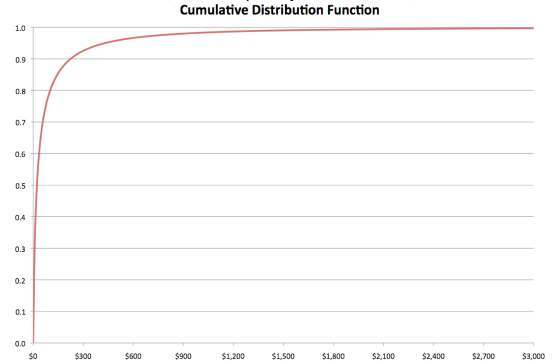
Figure 3.7.4B. Stock Price = $100, Horizon = 1 year Expected Return = 12%, Standard Deviation = 180%

Cumulative Distribution Function



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Probability Call In-The-Money

- Terminal stock price: $\tilde{S}_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\tilde{\varepsilon}}$
- Probability: $\Pr\left(S_T > X\right) = \Pr\left[S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\tilde{\varepsilon}} > X\right]$

$$\Pr\left(S_T > X\right) = \Pr\left[\ln\left(\frac{S_0}{X}\right) + \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\tilde{\varepsilon} > 0\right]$$

$$= \Pr\left[\tilde{\varepsilon} > \frac{-\ln\left(\frac{S_0}{X}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] = 1 - \text{Prob}\left[\tilde{\varepsilon} < -\frac{\ln\left(\frac{S_0}{X}\right) + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right]$$

$$= 1 - N\left(-d_2\right) = N\left(d_2\right)$$

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Multivariate Distributions

- Finance suffers from "curse of dimensionality"
- S&P 500 index
  - Technically a 500 dimensional problem (each stock)
  - Each stock has multiple factors or dimensions
  - Quants aggressively seek to reduce dimensionality
  - In practice, one rarely gets the dimensions down to only one. Hence, we need the capacity to model multiple dimensional problems.

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Multivariate Normal

- PDF: $$n_N\left(\mathbf{x} \middle| \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) = \frac{e^{\left[-\frac{\left(\mathbf{x}-\boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)}{2}\right]}}{\sqrt{\left(2\pi\right)^N \left|\boldsymbol{\Sigma}\right|}}$$
- If $\mathbf{x} \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ and $\mathbf{y} = e^{\mathbf{x}}$, then

$$E\left(y_i\right) = e^{\mu_i + \frac{\Sigma_{ii}}{2}} \text{ and}$$

$$\operatorname{cov}\left(y_i, y_j\right) = e^{\mu_i + \mu_j + \frac{1}{2}\left(\Sigma_{ii} + \Sigma_{jj}\right)}\left(e^{\Sigma ij} - 1\right).$$

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Bivariate PDFs and CDFs

- Normal:

$$n_2(x_1,x_2;\rho)\equiv\frac{e^{-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

$$N_2(a,b;\rho)\equiv\int_{-\infty}^{a}\int_{-\infty}^{b}n_2(x_1,x_2;\rho)\,dx_1\,dx_2.$$

- Lognormal:

$$\lambda_2(x_1,x_2;\rho)\equiv\frac{e^{-\frac{\left[\frac{\ln(x_1)-\mu_1}{\sigma_1}\right]^2-2\rho\left[\frac{\ln(x_1)-\mu_1}{\sigma_1}\right]\left[\frac{\ln(x_2)-\mu_2}{\sigma_2}\right]+\left[\frac{\ln(x_2)-\mu_2}{\sigma_2}\right]^2}{2(1-\rho^2)}}}{2\pi x_1x_2\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

$$\Lambda_2(a,b;\rho)\equiv\int_{-\infty}^{a}\int_{-\infty}^{b}\lambda_2(x_1,x_2;\rho)\,dx_1\,dx_2.$$

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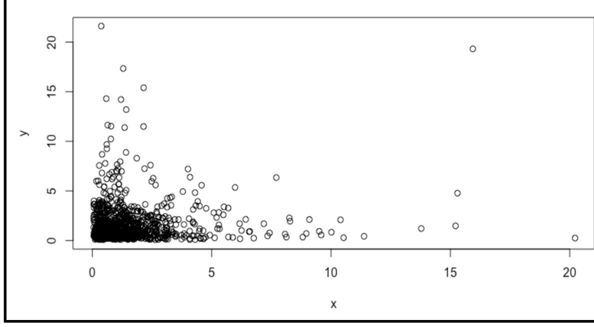
Bivariate Lognormal Simulation

Figure 3.7.5 Bivariate Lognormal Simulation



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Conditional Normal Distribution

- Note

$$n(x_2\mid x_1)=\frac{e^{-\frac{\left\{x_2-\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right]\right\}^2}{2\sigma_2^2(1-\rho^2)}}}{\sigma_2\sqrt{2\pi(1-\rho^2)}}$$

- Thus, $n(x_2\mid x_1)\sim N\left(\mu_{2|1},\sigma_{2|1}^2\right)$

$$\mu_{2|1}=\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\text{ and}$$

$$\sigma_{2|1}^2=\sigma_2^2(1-\rho^2).$$

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Contrasting ND and LD

- Next several graphs generated in R illustrate role of volatility on
  - PDFs
  - CDFs
  - Mean, Median, and Mode
  - Simulations

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Figure 3.7.6 Univariate Normal PDFs (30%, 80%, 130%, 180%)

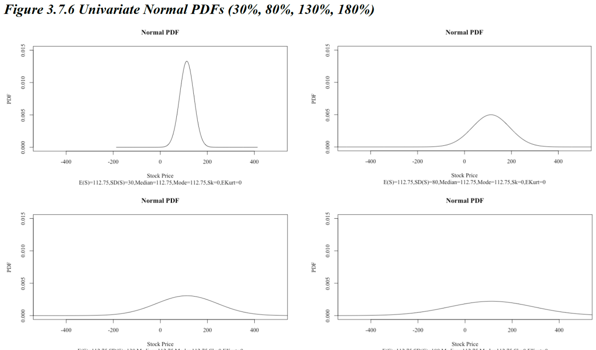

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Figure 3.7.7 Univariate Lognormal PDFs ($34, $106.75, $237, $558)

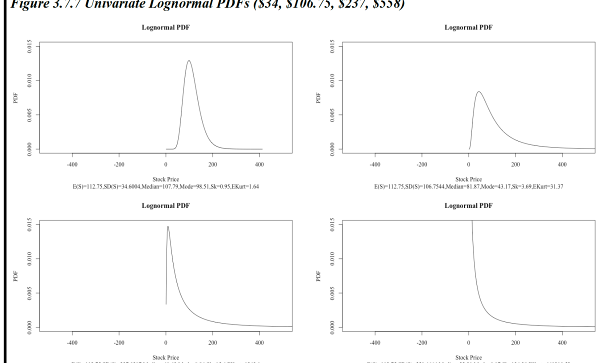

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Figure 3.7.8 Univariate Normal CDFs (30%, 80%, 130%, 180%)

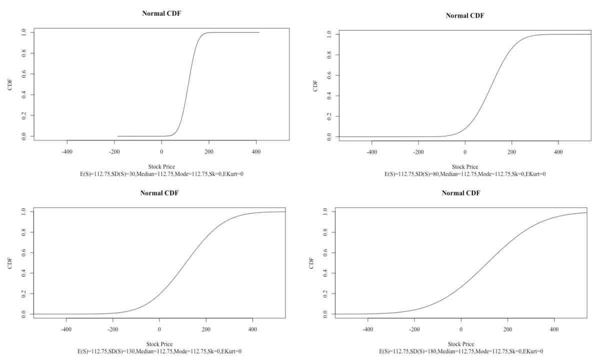

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Figure 3.7.9 Univariate Lognormal CDFs ($34, $106.75, $237, $558)

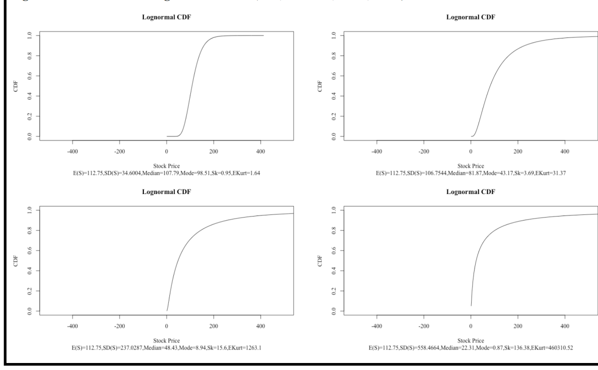

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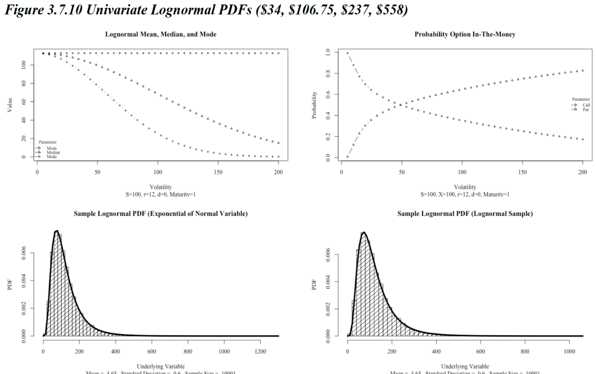
Figure 3.7.10 Univariate Lognormal PDFs ($34, $106.75, $237, $558)



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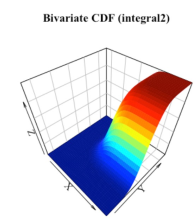
```
N2CDFV1[i, j] <- as.numeric(integral2(Normal2PDF, xmin = MinX1, xmax = MaxX1,
  ymin=MinX2, ymax = MaxX2, reltol = 1e-6, Mu1 = Mu1, Mu2 = Mu2,
  SD1 = SD1, SD2 = SD2, rho = rho)[1])
```

Figure 3.7.11 Bivariate Normal CDF based on double integral



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Summary

- Explored numerical integration
  - Lognormal distribution
  - Normal distribution
  - Deep dive into properties as applied in finance
- Numerous comparisons and contrasts
- Bivariate distributions

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