Module 14.4: The Keel Model[1]

Learning objectives

- Liabilities have significant role in stochastic behavior of net worth or surplus
- Surplus decomposition measures sensitivity to key drivers including liabilities
- Keel model is intuitive, extensible, and will serve as an aid to decision-makers
- Mini cases: bank, defined benefit retirement system, and high net worth family

Executive summary

Performance attribution is extended to an enterprise level based on the keel model. The keel model, introduced here, is applied to the problem of attributing enterprise value changes to various risk factors and can also be used by investment professionals for portfolio surplus management purposes.

The keel model is general in form and can incorporate many factors. Changes in enterprise value are decomposed into four macro components. The horizon component captures change attributable to time. The spot rate component captures movement in some reference term structure of interest rates. The spread component captures change attributable to spread movement over this spot rate curve. There is an interaction component.

The spot rate and spread components are further decomposed into components attributable to duration, convexity, and cross-convexity. Each of these components are further decomposed based on level, slope, and curvature. Case studies of this attribution approach are provided.

Central finance concepts

Financial professionals seek to understand the sources of return as well as risk when making a variety of financial decisions. The goal here is attributing actual performance rather than issues related to general or partial equilibrium. The model presented here, termed the keel model, is designed consistent with the level, slope, and curvature model (denoted the LSC model) of Nelson and Siegel (1987) with a general extension along the lines of Svensson (1995). Steeley (2008) tests a variety of term structure estimation methods and finds this method to be extremely well specified.

The keel model is applied to the problem of attributing enterprise value changes to various risk factors and can also be used by investment professionals for portfolio surplus management purposes. The keel model is general in form and can incorporate as many factors as desired. Further, the degree of granularity is at the discretion of the finance professional, rather than being limited by the quantitative tool. Upton and Brooks (2016) apply this model to bond holding period returns of mutual funds.

Christensen, Diebold, and Rudebusch (2011) introduce a class of affine arbitrage free models that maintain the popular three-factor Nelson and Siegel structure. They explore the empirical fit with US Treasury data. They conclude that "All told, the independent-factor AFNS (Arbitrage Free Nelson-Siegel) model fares well in out-of-sample prediction, consistently outperforming, for example, the canonical $A_o(3)$ (a 22 free parameter representation based on the three-factor Nelson-Siegel)."[2] Given our interest in flexibility and potentially numerous sources of risk, we extend the Nelson-Siegel framework only where necessary. Our focus is performance attribution, not equilibrium valuation.

The keel model developed here is given this name due to its intended use. A ship's keel serves as its foundation, providing stability in rough seas as well as converting crosswinds into forward progress. The keel model provides unique insights in building investments that are designed to provide positive performance when the enterprise is suffering negative performance. The goal is to transition the task of risk management away from a view-driven strategy of guessing what is over the horizon towards a needs-driven

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[1]This chapter is based on Robert Brooks, "An Enterprise Perspective of Performance Attribution: Introducing the Keel Model," *The Journal of Risk*, Vol. 20, No. 2, (December 2017), 53-84. DOI: 10.21314/JOR.2017.370.

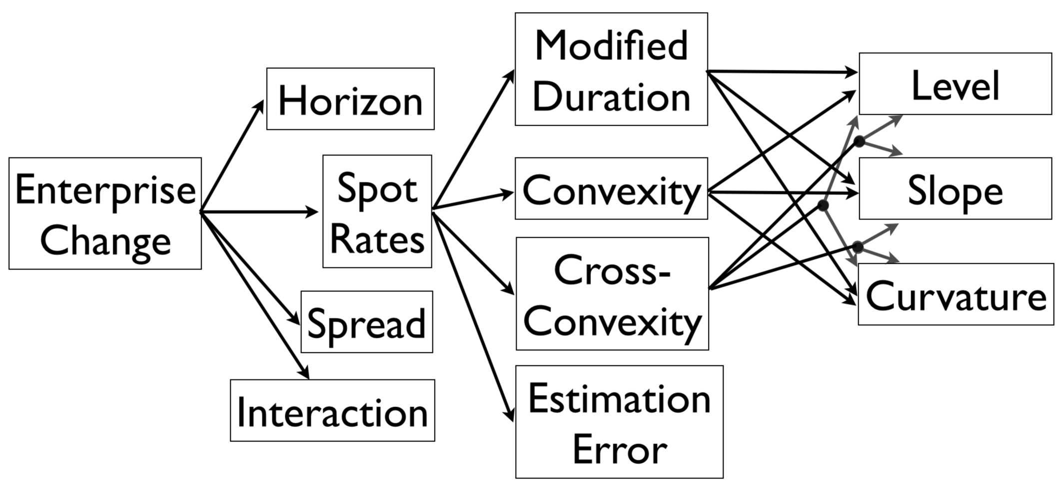
[2]Note US Treasury securities are exempt from state and local taxes and have numerous other unique attributes; therefore, requisite arbitrage trades are not feasible in practice.

risk management culture that serves to strengthen the financial durability of the enterprise. When implemented, the stakeholders have a clear perspective on the financial strength of the enterprise and what circumstances to avoid, much like navigating a ship.

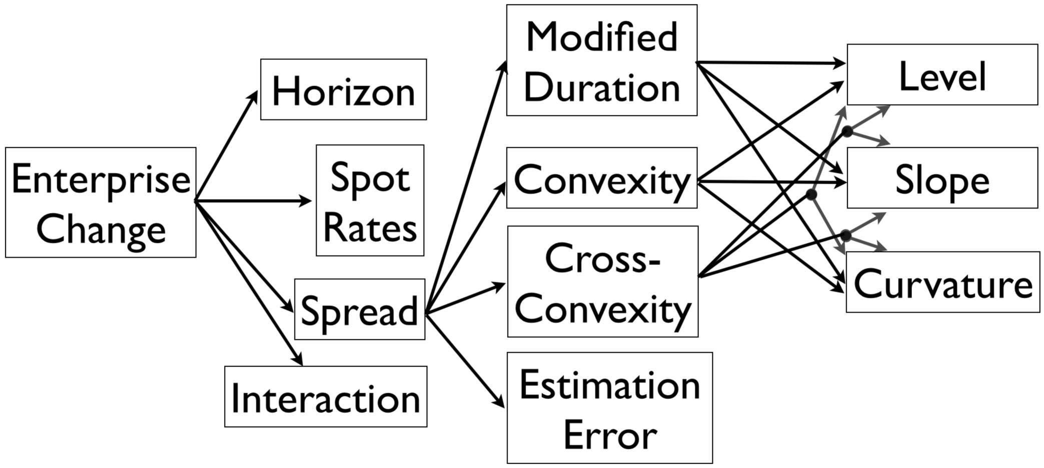
Enterprise value change is decomposed into four macro components: the non-random horizon component, the spot rate component, the spread component, and an interaction component. The horizon component captures the enterprise change attributable to the mere passage of time over the holding period horizon. The spot rate and spread components are estimated with the LSC model. The spot rate component captures movement in the fitted spot rate curve. The spread component captures the enterprise value change attributable to any change in the spread over the fitted spot rate curve. The interaction component contains the residual enterprise value change. Figure 14.4.1 illustrates the structure of the enterprise change decompositions provided by the keel model. The key insight of the keel model is that decomposition can be performed to whatever the desired level of granularity. As noted later, the model can easily incorporate expected cash flow growth rates, embedding growth rates as just another spread component.

The spot rate component of enterprise change can be further decomposed as needed into three components attributable to modified duration, convexity, and cross-convexity based on a Taylor series approximation. Thus, the approximation contains an error component. Each of these three spot rate components can be further decomposed, again, only as needed, into three subcomponents tied to movement in level, movement in slope, and movement in curvature. The keel model described in detail later is general enough to have multiple curvature components. The LSC model is also applied to the spread component, and a similar decomposition applied to the resultant spread curve is illustrated in Figure 14.4.1 b). For many applications, spot rate duration may be the only decomposition required, whereas in other cases understanding the spread level-slope cross-convexity is helpful. Multiple spreads can also easily be incorporated.

Figure 14.4.1. Illustration of Enterprise Change Attribution Provided by the Keel Model
a) Spot Rate Attribution



b) Spread Attribution



Given the highly technical nature of the keel model, we provide detailed explanations in the quantitative finance materials section.

Case studies

Three applications of the keel model are presented: a small bank, a defined benefit retirement system, and a high net worth family. With each case, we highlight different features of the keel model. The cases are provided for illustration purposes. Although based on anecdotal experience within these three settings, the case structure is purposefully designed to illustrate the keel model's ability to illuminate certain insights. Selected calculations are given in the Appendix D to further clarify how the keel model can be easily deployed.

Case study 1: Small bank

Suppose we have a small bank with total assets of $300 million and the market value balance sheet at year end as given in Table 14.4.1. We ignore the deep optionality contained in the balance sheets of most banks.[3] The data presented here is roughly like an actual small bank in the U.S.

Table 14.4.1. Small Bank Balance Sheet (in millions)

Assets		**Liabilities and Equity**	
Cash and Due	15	Noninterest Bearing Deposits	50
Securities	40	Interest Bearing Deposits	170
Loans and Leases	200	Borrowings	50
Premises	5	*Total Liabilities*	*270*
Other Assets	40	*Total Equity*	*30*
Total Assets	*300*	*Total Liabilities and Equity*	*300*

Table 14.4.2 provides selected information regarding each balance sheet category. We assume only Loans and Leases and Other Assets are exposed to spread changes.

[3] Banks are short numerous options, such as an early withdraw option on deposits and a prepayment option on loans. Analysis with the tools presented here of this optionality is the subject of another paper.

Table 14.4.2. Small Bank Duration and Convexity

Assets	Spot Rate Duration	Spot Rate Convexity	Spreads Duration	Spreads Convexity
Cash and Due	0.00	0.00		
Securities	3.55	14.06		
Loans and Leases	1.78	6.41	1.78	6.41
Premises	0.00	0.00		
Other Assets	2.29	6.63	2.29	6.63
Total Assets	*1.97*	*7.03*	*1.49*	*5.15*
Liabilities and Equity				
Nonint. Bearing Deposits	0.00	0.00		
Interest Bearing Deposits	3.47	16.00		
Borrowings	0.00	0.00		
Total Liabilities	*2.19*	*10.09*	*0.00*	*0.00*
Total Equity	***0.00***	*-20.30*	*14.84*	*51.27*
Total Liab. and Equity	*1.97*	*7.03*	*1.49*	*5.15*

Note that this bank's equity spot rate duration is zero, giving it the appearance of minimal exposure to interest rates. We assume assets and liabilities were structured with this in mind. Table 14.4.3 provides the LSC risk measures that paint a very different picture. The spot rate slope duration is 1.49 and the spread durations are material, exposing this bank to significant risk if spot rates or spreads rise.

Table 14.4.3. LSC Model Risk Measures

Risk Measures	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	0.00	1.49	0.16	14.84	4.31	3.65
Convexity	–20.30	1.54	–1.50	51.27	4.05	3.04
Cross-Convexity	–2.66	–5.53	–0.61	14.19	12.46	3.48

Note: Level / Cross-Convexity denotes cross-convexity between level and slope, Slope / Cross-Convexity denotes cross-convexity between slope and curvature, and Curvature / Cross-Convexity denotes cross-convexity between level and curvature.

Table 14.4.4 provides the initial and horizon spot rates and spreads for a particular scenario of concern—curves rising and flattening. Figure 14.4.2 illustrates these rates graphically.

Table 14.4.4. Initial and Horizon LSC Parameters (Horizon = 0.25 Years)

LSC Parameters	**Base-Rate**			**Spread**		
	Initial	**Horizon**	**Difference**	**Initial**	**Horizon**	**Difference**
Level	2.5%	2.5%	0.0%	0.5%	0.5%	0.0%
Slope	–2.4%	0.0%	2.4%	–0.4%	0.0%	0.4%
Curvature	–2.0%	0.0%	2.0%	–0.2%	0.0%	0.2%
Scalar	2.0	2.0		1.0	1.0	

Figure 14.4.2a. Spot Rate Initial and Horizon Spot Rates (Horizon = 0.25 Years)

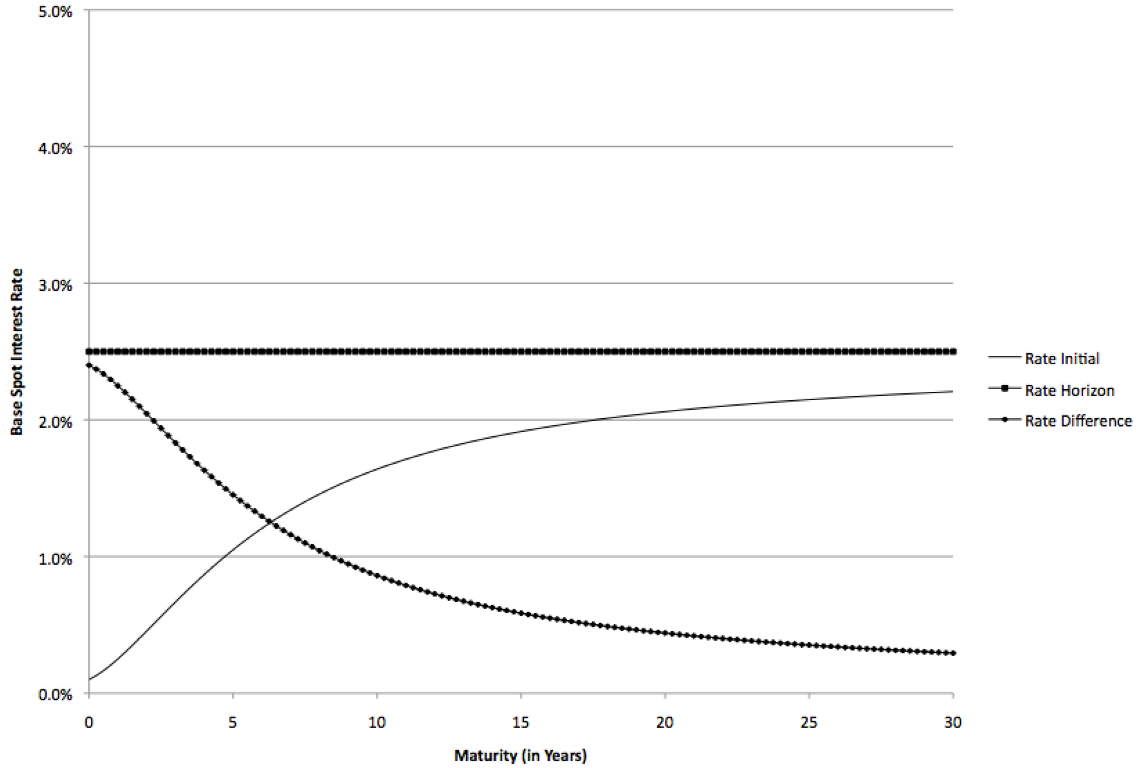


Figure 14.4.2b. Spread Initial and Horizon Rates (Horizon = 0.25 Years)

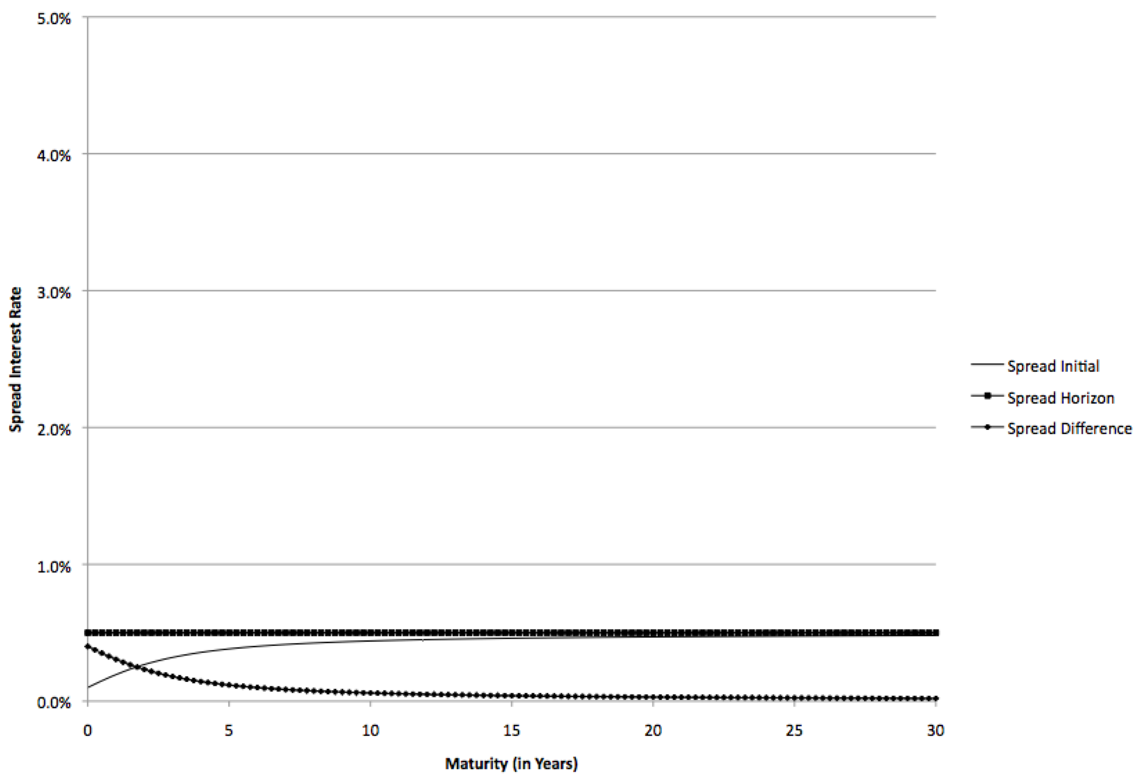


Figure 14.4.2c. All-in Initial and Horizon Spot Rates (Horizon = 0.25 Years)

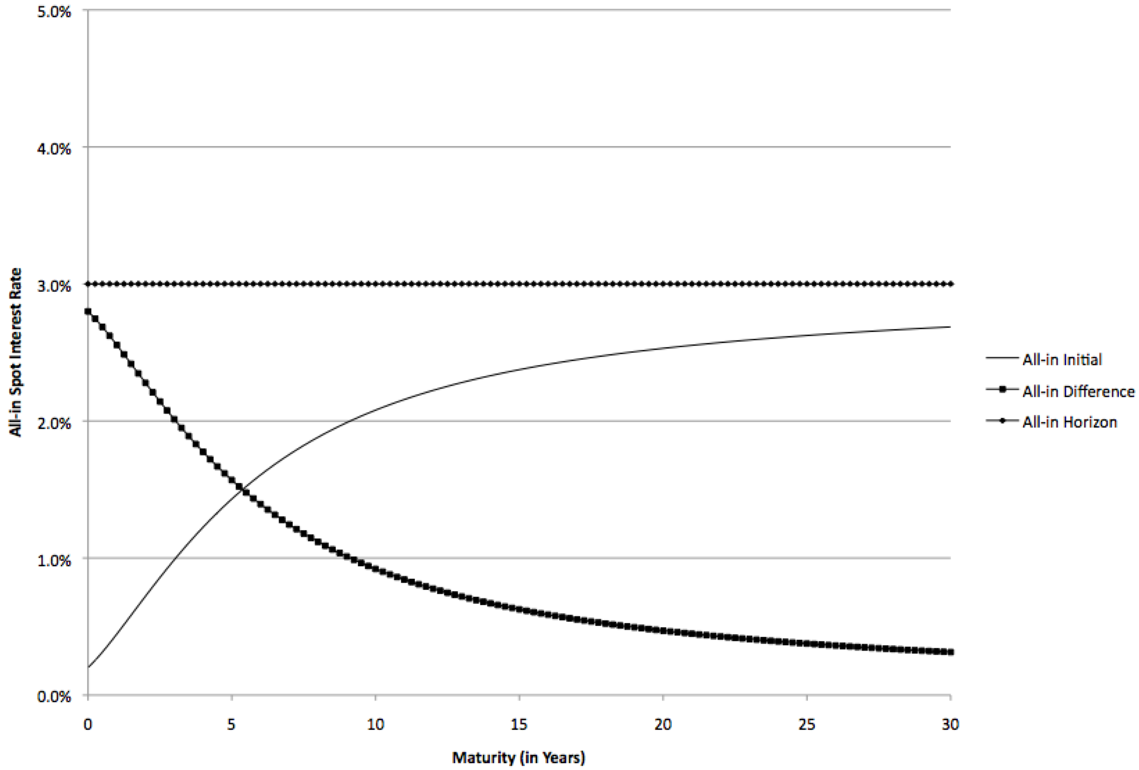


Table 14.4.5 presents the factor sensitivities and surplus decomposition. On an HPR basis, the horizon HPR accounts for a positive 82 basis points and the interaction HPR accounts for a positive 14 basis points. The spot rate HPR of a 390 basis point loss is comprised almost entirely of spot rate factor duration sensitivity (FDS) even though the measured spot rate duration is zero. The spot rate HPR is attributable to slope and curvature duration. The spread HPR of a 245 basis point loss is again attributable to factor duration sensitivities.

Table 14.4.5. Factor Sensitivities and Surplus Decomposition Without Accounting Hedges

Factor Sensitivities	**Dollar Change**	**Percentage**	**Surplus Decomposition**	**Dollar Change**	**Return**
Total FS(r)	*–$1.18*	*–3.9%*	*Actual*	*–$1.63*	*–5.44%*
FDS(r)	–$1.18	–3.9%	**Horizon**	$0.25	0.82%
FCS(r)	$0.0	0.0%	**Base-Rate**	–$1.18	–3.90%
FCCS(r)	–$0.01	0.0%	**Spread**	–$0.74	–2.45%
Total FS(sp)	*–$0.74*	*–2.5%*	**Interaction**	$0.04	0.14%
FDS(sp)	–$0.74	–2.5%			
FCS(sp)	$0.0	0.0%			
FCCS(sp)	$0.0	0.0%			

FS(r) denotes the factor sensitivity related to the base curve whereas FS(sp) denotes the factor sensitivity to the spread. FDS denotes factor duration sensitivity, FCS denotes factor convexity sensitivity, and FCCS denotes factor cross-convexity sensitivity.

Table 14.4.6 illustrates further decomposition of the factor sensitivities. The HPRs are primarily attributable to both spot rate and spread slope duration. Recall in this scenario that the level spot rate and level spread did not change. If this slope risk was deemed significant, then risk management strategies could be pursued to lower the slope durations. In this case, convexity and cross-convexity are not a concern. The key insight is that the keel model provides clarity on how to increase the stability of the enterprise through custom designed risk management solutions that are based on the entire enterprise.

Table 14.4.6. Factor Sensitivity Decomposition without Accounting Hedges

Factor Sensitivity	Base-Rate			Spread		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	0.00%	–3.59%	–0.32%	0.00%	–1.73%	–0.73%
Convexity	0.00%	0.04%	–0.03%	0.00%	0.01%	0.00%
Cross-Convexity	0.00%	0.00%	–0.03%	0.00%	0.00%	0.00%

Note: Level / Cross-Convexity denotes cross-convexity between level and slope, Slope / Cross-Convexity denotes cross-convexity between slope and curvature, and Curvature / Cross-Convexity denotes cross-convexity between level and curvature.

We now consider two popular "risk management" strategies related to managing interest rate risk. Although clearly not risk reducing based on the keel model, these two strategies are often promulgated by investment banking firms seeking transaction business.

Strategy #1: Suppose the small bank enters a $40 million receive floating, pay fixed interest rate swap as "hedge" against their floating rate borrowings. With this strategy, market value changes are recorded in Other Comprehensive Income (OCI), not earnings. [See Abdel-Khalik (2014).] This strategy will provide rates-up protection against Held-To-Maturity designated securities and will limit future problems with the balance sheet, such as underwater securities. This strategy is only hedging one financial instrument; it is not taking an enterprise perspective. Clearly, this "hedge" may be enterprise risk increasing.

By lengthening the duration of the borrowings, Table 14.4.7 illustrates the negative spot rate duration, thus the small bank is now exposed to losses if rates fall. One interesting insight is the effect on the horizon HPR that falls from 82 basis points (Table 14.4.5) to 26 basis points (Table 14.4.7). The synthetic fixed rate borrowing introduced by the swap results in the fixed payments riding down the spot rate curve and increasing the borrowing liability, thus resulting in a lower horizon HPR. This effect will have a mild influence on all the risk measures. The synthetic fixed borrowing will fall in value when rates rise, hence the actual HPR is now 299 basis points rather than a loss of 544 basis points.

Table 14.4.7. Risk Measures, Factor Sensitivities, and Surplus Decomposition with Borrowings-based Swap

Risk Measures	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	–6.18	–0.89	–1.62	14.93	4.33	3.67
Convexity	–49.55	–2.74	–3.94	51.56	4.07	3.06
Cross-Convexity	–13.84	–13.97	–3.84	14.27	12.53	3.50
Factor Sensitivity	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	0.00%	2.12%	3.24%	0.00%	–1.73%	–0.73%
Convexity	0.00%	–0.08%	–0.08%	0.00%	0.01%	0.00%
Cross-Convexity	0.00%	0.00%	–0.18%	0.00%	0.00%	0.00%

Factor Sensitivities	**Dollar Change**	**Percentage**	**Surplus Decomposition**	**Dollar Change**	**Return**
Total FS(r)	*\$1.52*	*5.1%*	*Actual*	*\$0.90*	*2.99%*
FDS(r)	\$1.62	5.4%	**Horizon**	\$0.08	0.26%
FCS(r)	–\$0.05	–0.2%	**Base-Rate**	\$1.52	5.05%
FCCS(r)	–\$0.06	–0.2%	**Spread**	–\$0.74	–2.46%
Total FS(sp)	*–\$0.74*	*–2.5%*	**Interaction**	\$0.04	0.14%
FDS(sp)	–\$0.74	–2.5%			
FCS(sp)	\$0.00	0.0%			
FCCS(sp)	\$0.00	0.0%			

Strategy #2: As an alternative, suppose the small bank pursued a $100 million receive fixed, pay floating interest rate swap as hedge against floating rate loans based on view that interest rates will not increase for an extended period. There will be an immediate net interest margin accretion. The justification is that it will allow for aggressive origination of floating rate loans at funding levels. There is the opportunity for an attractive roll down on the spot rate curve. Also, the value of swap may increase as time passes, helping to offset potential securities losses with future rate increases. Although accounting rules allow for this strategy to be deemed hedging, it is clearly view driven and not enterprise hedging.

Table 14.4.8 highlights some results using the keel model. The spot rate level duration is now almost 15, exposing the bank to significant losses if rates increase. The scenario considered is a large increase in rates, hence the small bank is seen suffering significant losses. Thus, although this strategy is currently allowed as an accounting hedge, the small bank loses 27.85 percent of its equity value under this scenario.

Table 14.4.8. Risk Measures, Factor Sensitivities, and Surplus Decomposition with Loans-based Swap

Risk Measures	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	14.98	7.25	4.48	29.58	7.42	6.61
Convexity	50.42	11.92	4.40	120.80	7.12	5.84
Cross-Convexity	24.42	14.90	7.21	28.77	26.40	6.41
Factor Sensitivity	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	0.00%	–17.73%	–9.12%	0.00%	–3.03%	–1.35%
Convexity	0.00%	0.35%	0.09%	0.00%	0.01%	0.00%
Cross-Convexity	0.00%	0.00%	0.35%	0.00%	0.00%	0.01%

Factor Sensitivities	**Dollar Change**	**Percentage**	**Surplus Decomposition**	**Dollar Change**	**Return**
Total FS(r)	*–$7.87*	–26.2%	*Actual*	*–$8.36*	–27.85%
FDS(r)	–$8.11	–27.0%	**Horizon**	$0.76	2.52%
FCS(r)	$0.13	0.4%	**Base-Rate**	–$7.88	–25.61%
FCCS(r)	$0.11	0.4%	**Spread**	–$1.32	–4.28%
Total FS(sp)	*–$1.31*	–4.4%	**Interaction**	$0.08	0.26%
FDS(sp)	–$1.32	–4.4%			
FCS(sp)	$0.00	0.0%			
FCCS(sp)	$0.00	0.0%			

Therefore, the performance attribution approach presented here can easily capture the salient interest rate and credit risk issues facing a small bank. Note that for this particular case, just the duration measures appear to adequately capture the important exposures, and thus convexity and cross convexity exposures do not have to be managed.

We turn now to apply the keel model to a defined benefit retirement system and wade into the controversial issue of valuing defined benefit liabilities.

Case Study 2: Defined Benefit Retirement System

Suppose we have a defined benefit system with $30 billion in assets, present value of defined benefits, valued based solely on the spot rate, is $39.9 billion. The data presented here is roughly like an actual defined benefit plan in the U.S. The market value balance sheet at year-end is given in Table 14.4.9.

Table 14.4.9. Defined Benefit Retirement System Balance Sheet (in billions)

Assets		**Liabilities and Surplus**	
Cash and Receivables	0.9	Payables	0.1
Stocks	18	Present Value of Defined Benefits	39.9
Bonds	10	*Total Liabilities*	*40*
Real Estate	1	*Surplus*	*–10*
Property	0.1		
Total Assets	*30*	*Total Liabilities and Surplus*	*30*

Table 14.4.10 provides selected information regarding each balance sheet category. We assume only Stocks, Bonds, and Real Estate Assets are exposed to spread changes. Stocks are modeled as essentially very long duration instruments—consistent with the constant growth dividend discount model. Spot rate and spread risk measures are the same as they exert the same influence on valuation. Note that because Surplus is negative, spread surplus duration and convexity measures are negative.

Table 14.4.10. Defined Benefit Retirement System Duration and Convexity

Assets	**Spot Rate Duration**	**Spot Rate Convexity**	**Spreads Duration**	**Spreads Convexity**
Cash and Receivables	0.00	0.00	0.00	0.00
Stocks	12.75	162.56	12.75	162.56
Bonds	2.83	8.62	2.83	8.62
Real Estate	12.75	162.56	12.75	162.56
Property	0.00	0.00	0.00	0.00
Total Assets	*9.03*	*106.06*	*9.03*	*106.06*
Liabilities and Equity				
Payables	0.00	0.00	0.00	0.00
PV(Defined Benefits)	9.81	155.49	0.00	0.00
Total Liabilities	*9.79*	*155.11*	*0.00*	*0.00*
Total Surplus	***12.17***	*309.39*	*-28.41*	*-333.59*
Total Liab. and Equity	*9.03*	*106.06*	*9.03*	*106.06*

Note that this retirement system's spot rate duration is significantly positive, and the surplus is negative, indicating significant exposure to *falling* interest rates. Table 14.4.11 provides the initial and horizon spot rates and spreads for a particular scenario of concern—spot rate curve is falling and flattening whereas the spread curve is rising and flattening.

Table 14.4.11. Initial and Horizon LSC Parameters (Horizon = 0.25 Years)

LSC Parameters	**Base-Rate**			**Spread**		
	Initial	**Horizon**	**Difference**	**Initial**	**Horizon**	**Difference**
Level	2.5%	1.5%	–1.0%	5.0%	6.0%	1.0%
Slope	–2.4%	–1.4%	1.0%	–4.0%	–3.0%	1.0%
Curvature	–2.0%	–2.0%	0.0%	–3.0%	–2.0%	1.0%
Scalar	2.0	2.0		1.0	1.0	

Table 14.4.12 provides the market value balance sheet at the initial as well as horizon dates based on the LSC parameters above. The column denoted $I(i)$ is an indicator function where 0 denotes spread is not incorporated and 1 denotes spread is incorporated. Note that though the surplus value fell, the measured HPR is positive because the denominator is negative. Importantly, a positive surplus duration indicates exposure to falling rates.

Table 14.4.12. Retirement System Market Value Balance Sheet

Assets	*I*(*i*)	Value (*r*,*sp*) *t*	Value (*r*+,*sp*+) *t*+Δ	Actual Change	Actual Return
Cash and Receivables	0	$0.90	$0.90	$0.00	0.03%
Stocks	1	$18.00	$17.62	–$0.38	–2.11%
Bonds	1	$10.00	$9.83	–$0.17	–1.75%
Real Estate	1	$1.00	$0.98	–$0.02	–2.11%
Property	0	$0.10	$0.10	$0.00	0.03%
Total Assets		$30.00	$29.43	–$0.57	–1.92%
Liabilities and Equity					
Payables	0	$0.10	$0.10	$0.00	0.03%
Present Value of Defined Benefits	0	$39.90	$43.60	$3.70	9.27%
Total Liabilities		$40.00	$43.70	$3.70	9.25%
Total Equity		–$10.00	–$14.28	–$4.27	42.75%*
Total Liabilities and Equity		$30.00	$29.43	–$0.57	–1.92%

* Note the positive number indicates a loss due to negative equity. (*r*+,*sp*+) denote after the rate and spread shock.

Table 14.4.13 provides the LSC risk measures, factor sensitivities, and surplus decomposition for this case. The significant factor sensitivities for this scenario are spot rate level and spread level durations, with spread contributing significantly more to the losses. In this scenario, the initial funding ratio of 74.94 percent falls to 67.26 percent—a significant loss to the system. The funding ratio here is defined as the total assets less payables divided by the present value of defined benefits.

Table 14.4.13. Retirement System Risk Measures, Factor Sensitivities, and Surplus Decomposition

Risk Measures	Base-Rate			Spread		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	12.17	1.51	1.26	–28.41	–2.96	–2.77
Convexity	309.39	2.72	2.05	–333.59	–2.88	–2.59
Cross-Convexity	23.77	22.62	2.35	–28.22	–27.76	–2.72
Factor Sensitivity	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	–$1.21	$0.15	$0.00	–$2.71	–$0.28	–$0.26
Convexity	–$0.15	$0.00	$0.00	$0.32	$0.00	$0.00
Cross-Convexity	$0.02	$0.00	$0.00	$0.03	$0.03	$0.00

Factor Sensitivities	Dollar Change	Percentage	Surplus Decomposition	Dollar Change	Return
Total FS(r)	*–$1.16*	*11.6%*	*Actual*	*–$4.27*	*42.75%*
FDS(r)	–$1.03	10.3%	**Horizon**	$0.30	–3.05%
FCS(r)	–$0.15	1.5%	**Base-Rate**	–$1.17	12.10%
FCCS(r)	$0.02	–0.2%	**Spread**	–$3.10	31.96%
Total FS(sp)	*–$2.92*	*29.2%*	**Interaction**	–$0.31	3.18%
FDS(sp)	–$3.31	33.1%	**Funding Ratio:**		
FCS(sp)	$0.33	–3.3%	**Initial:** 74.94%	**Horizon:**	67.26%
FCCS(sp)	$0.06	–0.6%			

Table 14.4.14 provides the market value balance sheet at the initial date, as well as the horizon date, based on the LSC parameters above assuming the present value of defined benefits are based on both the spot rate and the spread. Note the column denoted *I*(*i*) for Present Value of Defined Benefits is 1. By including a spread in the valuation of benefits, the surplus is a positive $2.49 billion. As is widely used in practice, one clearly observes from this analysis the perverse incentive to raise the spread applied to the benefits to improve the system. See, for example, Minahan (2014) and Ambachtsheer (2015).

Table 14.4.14. Retirement System Balance Sheet Applying Spread to Benefits

Assets	***I*(*i*)**	**Value (*r*,*sp*) *t***	**Value (*r*+,*sp*+) *t*+Δ**	**Actual Change**	**Actual Return**
Cash and Receivables	0	$0.90	$0.90	$0.00	0.03%
Stocks	1	$18.00	$17.62	–$0.38	–2.11%
Bonds	1	$10.00	$9.83	–$0.17	–1.75%
Real Estate	1	$1.00	$0.98	–$0.02	–2.11%
Property	0	$0.10	$0.10	$0.00	0.03%
Total Assets		$30.00	$29.43	–$0.57	–1.92%
Liabilities and Equity					
Payables	0	$0.10	$0.10	$0.00	0.03%
Present Value of Defined Benefits	1	$27.41	$26.94	–$0.48	–1.74%
Total Liabilities		$27.51	$27.04	–$0.48	–1.73%
Total Equity		$2.49	$2.39	–$0.10	–3.97%
Total Liabilities and Equity		$30.00	$29.43	–$0.57	–1.92%

[*] Note the positive number indicates a loss due to negative equity. (*r*+,*sp*+) denote after the rate and spread shock.

Table 14.4.15 provides the LSC risk measures, factor sensitivities, and surplus decomposition applying spread to defined benefit liabilities. From a risk management perspective, the keel model illustrates the flaw in the current practice of applying a risk-adjusted discount rate to the present value of defined benefits. The incentive would be to make the system less credit worthy to be able to discount the benefits based on a higher rate. This approach clearly violates the fundamental principles of enterprise risk management.

Table 14.4.15. Retirement System Risk Measures, Factor Sensitivities, and Surplus Decomposition Applying Spread to Benefits

Risk Measures	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	27.52	3.64	4.15	27.52	1.61	1.93
Convexity	216.89	8.09	9.33	216.89	1.87	1.99
Cross-Convexity	56.36	58.33	8.77	27.84	27.96	1.98
Factor Sensitivity	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	\$0.69	–\$0.09	\$0.00	–\$0.69	–\$0.04	–\$0.05
Convexity	\$0.03	\$0.00	\$0.00	\$0.05	\$0.00	\$0.00
Cross-Convexity	–\$0.01	\$0.00	\$0.00	\$0.01	\$0.01	\$0.00

Factor Sensitivities	**Dollar Change**	**Percentage**	**Surplus Decomposition**	**Dollar Change**	**Return**
Total FS(r)	*\$0.63*	*25.3%*	*Actual*	*–\$0.10*	*–3.97%*
FDS(r)	\$0.61	24.7%	**Horizon**	\$0.08	3.28%
FCS(r)	\$0.03	1.2%	**Base-Rate**	\$0.63	24.39%
FCCS(r)	–\$0.01	–29.2%	**Spread**	–\$0.76	–29.38%
Total FS(sp)	*–\$0.73*	*–29.2%*	**Interaction**	–\$0.05	–2.03%
FDS(sp)	–\$0.80	–32.1%	**Funding Ratio:**		
FCS(sp)	\$0.06	2.3%	**Initial:** 109.08%	**Horizon:**	108.87%
FCCS(sp)	\$0.01	0.6%			

The analysis presented here supports Minahan (2014). From a risk management perspective, the correct approach is to value the benefits based on the economic cost to defease the liability; that is, the correct discount rate is just the spot rate derived from risk-free instruments.

We now consider two simple, but flawed, strategies.

Strategy #1: Pursue the "less risky" strategy by investing in cash (short-term instruments) rather than stocks, bonds and real estate.

Table 14.4.16 provides the market value balance sheet at the initial and horizon dates based on the LSC parameters above, assuming the asset categories are moved to cash. Note the column denoted $I(i)$ for Present Value of Defined Benefits is reset to 0. While moving assets to cash does mitigate the asset losses—compare Total Assets of \$29.43 billion in Table 14.4.12 with Total Assets of \$30.01 billion in Table 14.4.16—Total Equity (Surplus) still suffers a \$3.69 billion loss. Thus, a purely asset view is inappropriate.

Table 14.4.16. Retirement System Balance Sheet When Asset Categories Moved to Cash

Assets	*I*(*i*)	Value (*r*,*sp*) *t*	Value (*r*+,*sp*+) *t*+Δ	Actual Change	Actual Return
Cash and Receivables	0	\$0.90	\$0.90	\$0.00	0.03%
Stocks	0	\$18.00	\$18.01	\$0.01	0.03%
Bonds	0	\$10.00	\$10.00	\$0.00	0.03%
Real Estate	0	\$1.00	\$1.00	\$0.00	0.03%
Property	0	\$0.10	\$0.10	\$0.00	0.03%
Total Assets		\$30.00	\$30.01	\$0.01	0.03%
Liabilities and Equity					
Payables	0	\$0.10	\$0.10	\$0.00	0.03%
Present Value of Defined Benefits	0	\$39.90	\$43.60	\$3.70	9.27%
Total Liabilities		\$40.00	\$43.70	\$3.70	9.25%
Total Equity		–\$10.00	–\$13.69	\$3.69	36.90%
Total Liabilities and Equity		\$30.00	\$30.01	\$0.01	0.03%

[*] Note the positive number indicates a loss due to negative equity. (*r*+,*sp*+) denote after the rate and spread shock.

Table 14.4.17 provides the LSC risk measures, factor sensitivities, and surplus decomposition when asset categories are moved to cash. Note that all exposure to spread has been eliminated, yet the funding ratio still declines to 68.60 percent—only a slight improvement over the 67.26 percent seen in Table 14.4.13.

Table 14.4.17. Retirement System Risk Measures, Factor Sensitivities, and Surplus Decomposition When Asset Categories Moved to Cash

Risk Measures	Base-Rate			Spread		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	38.64	6.70	5.37	0.00	0.00	0.00
Convexity	612.25	12.38	10.10	0.00	0.00	0.00
Cross-Convexity	75.34	71.91	10.98	0.00	0.00	0.00
Factor Sensitivity	**Base-Rate**			**Spread**		
	Level	**Slope**	**Curvature**	**Level**	**Slope**	**Curvature**
Duration	–\$3.92	\$0.68	\$0.00	\$0.00	\$0.00	\$0.00
Convexity	–\$0.31	–\$0.01	\$0.00	\$0.00	\$0.00	\$0.00
Cross-Convexity	\$0.08	\$0.00	\$0.00	\$0.00	\$0.00	\$0.00

Factor Sensitivities	Dollar Change	Percentage	Surplus Decomposition	Dollar Change	Return
Total FS(r)	*–\$3.49*	*34.9%*	*Actual*	*–\$3.69*	*36.90%*
FDS(r)	–\$3.25	32.5%	**Horizon**	–\$0.18	1.82%
FCS(r)	–\$0.32	3.2%	**Base-Rate**	–\$3.51	34.46%
FCCS(r)	\$0.08	–0.8%	**Spread**	\$0.00	0.00%
Total FS(sp)	*\$0.00*	*0.0%*	**Interaction**	\$0.00	0.00%
FDS(sp)	\$0.00	0.0%	**Funding Ratio:**		
FCS(sp)	\$0.00	0.0%	**Initial:** 74.94%	**Horizon:**	68.60%
FCCS(sp)	\$0.00	0.0%			

Strategy #2: The system managers pursue a high-risk strategy, as it is assumed the associated risk premium will pay off. This doubling down strategy could end disastrously as seen in the tables below. The key here is that management is addressing the wrong issue—they should focus on surplus as clarified by the keel model.

Table 14.4.18 provides the market value balance sheet at the initial as well as horizon dates based on the LSC parameters above assuming the categories Stock, Bonds, and Real Estate were exposed to another constant spread. One way to model this exposure is a simple credit default swap where the system is paid if this spread narrows, and the system pays if the spread widens. From a risk management viewpoint, we examine the consequences of a 300 basis point widening of the spread. Note that the losses more than double to the base case in Table 14.4.12. The Total Equity (Surplus) declines by $10.98 billion rather than $4.27 billion.

Table 14.4.18. Retirement System Balance Sheet With Additional Spread Exposure

Assets	**$I(i)$**	**Value (r,sp) t**	**Value (r+,sp+) $t+\Delta$**	**Actual Change**	**Actual Return**
Cash and Receivables	0	$0.90	$0.90	$0.00	0.03%
Stocks	1	$18.00	$12.02	–$5.98	–33.22%
Bonds	1	$10.00	$9.03	–$0.97	–9.70%
Real Estate	1	$1.00	$0.67	–$0.33	–33.22%
Property	0	$0.10	$0.10	$0.00	0.03%
Total Assets		$30.00	$22.72	–$7.28	–24.27%
Liabilities and Equity					
Payables	0	$0.10	$0.10	$0.00	0.03%
Present Value of Defined Benefits	0	$39.90	$43.60	$3.70	9.27%
Total Liabilities		$40.00	$43.70	$3.70	9.25%
Total Equity		–$10.00	–$20.98	–$10.98	109.82%
Total Liabilities and Equity		$30.00	$22.72	–$7.28	–24.27%

* Note the positive number indicates a loss due to negative equity. (r+,sp+) denote after the rate and spread shock.

Table 14.4.19 provides the surplus decomposition with additional spread exposure. Although in this case a spread decline of 4.168 percent would lead to a fully funded system, the keel model clearly identifies the folly of the doubling down strategy.

Table 14.4.19. Retirement System Surplus Decomposition With Additional Spread Exposure

Surplus Decomposition		**Dollar Change**	**Return**
Actual		–$10.98	109.82%
Horizon		$0.30	–3.05%
Base-Rate		–$1.17	12.10%
Spread		–$9.19	94.80%
Interaction		–$0.92	9.52%
Funding Ratio:			
Initial:	74.94%	**Horizon:**	51.88%

We conclude this module by applying this approach to a high net worth family.

Case Study 3: High Net Worth Family

As an investment advisor, suppose we have a high net worth family with a market value balance sheet as shown in Table 14.4.20. We assume a 30 year fixed rate mortgage valued without the spread. The Other Obligations category is a five year commitment to fund a particular charity; again, it is valued without the spread. The family firm is assumed to be exposed both to the spot rate and to the spread.

Table 14.4.20. Family Market Value Balance Sheet (in thousands)

Assets		Liabilities and Equity	
Cash and Other Liquid	300	Payables	10
Home and Related	1,000	Mortgage	700
Bonds	500	Other Obligations	3,000
Stocks	2,000	*Total Liabilities*	*3,710*
Family Firm	15,000	*Equity*	*15,090*
Total Assets	*18,800*	*Total Liabilities and Equity*	*18,800*

Table 14.4.21 provides selected information regarding each family balance sheet category. We assume only Stocks, Bonds, and Family Firm Assets are exposed to spread changes. Note that based on the keel model, we quickly see that this family has significant exposure to both spot rates as well as the spread, even with traditional risk measures.

Table 14.4.21. Family Duration and Convexity

	Spot Rate		Spreads	
Assets	**Duration**	**Convexity**	**Duration**	**Convexity**
Cash and Other Liquid	0.00	0.00	0.00	0.00
Home and Related	0.00	0.00	0.00	0.00
Bonds	2.83	8.62	2.83	8.62
Stocks	12.75	162.56	12.75	162.56
Family Firm	29.75	885.06	29.75	885.06
Total Assets	*25.20*	*724.71*	*25.20*	*724.71*
Liabilities and Equity				
Payables	0.00	0.00	0.00	0.00
Mortgage	4.73	30.66	0.00	0.00
Other Obligations	2.35	7.61	0.00	0.00
Total Liabilities	*2.80*	*11.95*	*0.00*	*0.00*
Total Equity	***30.61***	*896.82*	*31.29*	*899.70*
Total Liab. and Equity	*25.20*	*724.71*	*25.20*	*724.71*

Table 14.4.22 provides the initial and horizon spot rates and spreads for a particular scenario of concern—spot rate curve is rising and flattening, and the spread curve is rising and flattening.

Table 14.4.22. Initial and Horizon LSC Parameters (Horizon = 0.25 Years)

LSC Parameters	**Base-Rate**			**Spread**		
	Initial	**Horizon**	**Difference**	**Initial**	**Horizon**	**Difference**
Level	2.5%	3.5%	1.0%	5.0%	5.0%	0.0%
Slope	–2.4%	0.0%	2.4%	–4.0%	0.0%	4.0%
Curvature	–2.0%	0.0%	2.0%	–3.0%	0.0%	3.0%
Scalar	2.0	2.0		1.0	1.0	

Table 14.4.23 provides the market value balance sheet at the initial and horizon dates based on the LSC parameters above. The column denoted $I(i)$ is an indicator function where 0 denotes spread is not incorporated and 1 denotes spread is incorporated. Based on the scenario modeled, the family suffers a \$5,583.84 loss to equity. Clearly, the significant exposure to spot rate and spread changes within the family firm should influence the construction of the stock and bond portfolio. The keel model provides clear direction to investment managers on the design and implementation of investment portfolios. Specifically,

the investment portfolio will provide the counterbalance required during the rough financial times as well as providing a forward push during choppy financial times much like a sailboat's keel.

Table 14.4.23. Family's Balance Sheet

Assets	**I(i)**	**Value (r,sp) t**	**Value (r+,sp+) t+Δ**	**Actual Change**	**Actual Return**
Cash and Receivables	0	\$300.00	\$300.10	\$0.10	0.03%
Home	0	\$1,000.00	\$1,000.32	\$0.32	0.03%
Bonds	1	\$500.00	\$441.34	–\$58.66	–11.73%
Stocks	1	\$2,000.00	\$1,532.53	–\$467.47	–23.37%
Family firm	1	\$15,000.00	\$9,692.00	–\$5,308.00	–35.39%
Total Assets		\$18,800.00	\$12,966.29	–\$5,833.71	–31.03%
Liabilities and Equity					
Payables	0	\$10.00	\$10.00	\$0.00	0.03%
Mortgage	0	\$700.00	\$631.72	–\$68.28	–9.75%
Other Obligations	0	\$3,000.00	\$2,818.40	–\$181.60	–6.05%
Total Liabilities		\$3,710.00	\$3,460.12	–\$249.88	–6.74%
Total Equity		\$15,090.00	\$9,506.16	–\$5,583.84	–37.00%
Total Liabilities and Equity		\$18,800.00	\$12,966.29	–\$5,833.71	–31.03%

* Note the positive number indicates a loss due to negative equity. (r+,sp+) denote after the rate and spread shock.

Table 14.4.24 provides the LSC risk measures, factor sensitivities, and surplus decomposition for this case. The significant factor sensitivities for this scenario are spot rate durations with level contributing significantly more than slope and curvature. Given this significant risk exposure, the investment advisor would consider alternative investments that provide an offset to risk exposures present in the other family accounts.

Table 14.4.24. Family's Risk Measures, Factor Sensitivities, and Surplus Decomposition

Risk Measures	Base-Rate			Spread		
	Level	Slope	Curvature	Level	Slope	Curvature
Duration	30.61	1.99	2.09	31.29	1.15	1.15
Convexity	896.82	4.10	4.30	899.70	1.15	1.14
Cross-Convexity	61.43	61.67	4.23	31.28	31.27	1.15
Factor Sensitivity	Base-Rate			Spread		
	Level	Slope	Curvature	Level	Slope	Curvature
Duration	–$4,627.87	–$720.08	–$631.23	$0.00	–$698.39	–$521.08
Convexity	$678.05	$17.83	$13.00	$0.00	$27.88	$15.56
Cross-Convexity	$222.89	$186.49	$30.69	$0.00	$0.00	$20.81

Factor Sensitivities	Dollar Change	Percentage	Surplus Decomposition	Dollar Change	Return
Total FS(r)	–$4,925.55	–32.6%	*Actual*	–$5,583.84	–37.00%
FDS(r)	–$6,096.46	–40.4%	**Horizon**	$318.29	2.11%
FCS(r)	$722.38	4.8%	**Base-Rate**	–$5,061.53	–32.85%
FCCS(r)	$448.53	3.0%	**Spread**	–$1,200.05	–7.79%
Total FS(sp)	–$1,176.93	–7.8%	**Interaction**	$359.45	2.33%
FDS(sp)	–$1,242.39	–8.2%			
FCS(sp)	$44.25	0.3%			
FCCS(sp)	$21.20	0.1%			

To conclude this case, we briefly comment on two popular, but flawed, approaches.

Strategy #1: Build the investment portfolio based on personal risk preferences identified based on simple games of chance. This approach completely ignores what is happening within the family and the resultant risk exposures. Every family brings a different set of exposures to the financial market that will often result in different optimal portfolios. The key insight is that the keel model places the focus on the family equity value rather than on preference for a particular mutual fund based on the advisor's potential faulty view of the future.

Strategy #2: Many families are advised to follow a bucketing strategy; that is, identifying several areas of importance, such as a college fund and a retirement fund. Once the buckets are identified, then the investment advisor manages each bucket based on the risk level for that particular liability. While the risk level may be optimal for each bucket, it is likely suboptimal for the entire family. Again, the focus is at the micro level, rather than the enterprise level. Obviously, regulatory and institutional structures, such as various tax minimization strategies with minors, will influence the different accounts created. The investments in each account, however, should have an eye toward the family as a whole. Hence, the keel model provides this valuable perspective.

Quantitative finance materials

First, we introduce the keel model for measuring enterprise risk, focusing on changes in the surplus such as the difference between the market value of assets and liabilities. Second, the change in the surplus is decomposed into a non-stochastic horizon component, a stochastic spot rate component, a stochastic spread component, and a stochastic interaction term. Although only one spread is illustrated here, the keel model is easily extended to multiple spreads. Third, the spot rate and spread components are decomposed into duration, convexity, and cross-convexity sensitivities. Fourth, the spot rate and spread components are further decomposed based on the LSC model factors of level, slope, and curvatures. Finally, the keel model is illustrated with three cases: a small bank, a defined benefit retirement system, and a high net worth family.

Modeling financial risk of an enterprise

The keel model is focused on the change in an entity's equity value, or similarly, the change in a portfolio's surplus after incorporating liabilities. The keel model is based on the change in the surplus value as a function of an instrument's holding period return. Based on this approach, the discretely compounded holding period return for each instrument—assets and liabilities—are decomposed into a horizon component, a spot rate component, a spread component, and an interaction term. Based on the LSC model, each component can be aggregated to the enterprise level. Further, each component can be decomposed based on LSC-related risk measures greatly clarifying the task of enterprise risk management. The technical details are relegated to online appendices.

Recall that the goal here is performance attribution and not determinants of market prices. In these case studies, we make simplifying assumptions regarding the valuation of a variety of financial instruments. As described below, we employ a simple discounted cash flow approach due to its tractability. Alternatively, more sophisticated models, such as KMV-type models for credit issues and Christensen, Diebold, and Rudebusch (2011) for arbitrage free Nelson-Siegel (rate issues), are beyond the scope of the objectives here.[4]

Change in Surplus Value as Function of Instrument's Holding Period Return

The enterprise value can be expressed as the difference between the total assets market value and the total liabilities market value. In a portfolio context, the portfolio surplus is the difference between the assets held and the corresponding liabilities. The goal here is to illustrate the keel model for managing the inherent financial risks of either enterprises or portfolios. Due to computational needs later, we model liabilities as negative values. Thus, we assume

$$S_t = A_t + L_t \text{ (Enterprise value at time } t\text{)} \tag{14.4.1}$$

where S_t denotes the portfolio surplus or enterprise value at time t (denoted S because E is used for expectation), A_t denotes the enterprise's total asset portfolio value at time t, and L_t denotes the enterprise's total liability portfolio value at time t. All analysis is based on the market values of assets and liabilities and not the book values. Assets have positive values by definition, liabilities thus have negative values, and the surplus can be positive, negative, or even zero.

Focusing on assets, we assume

$$A_t = \sum_{l=1}^{M_A} A_{l,t} = \sum_{l=1}^{M_A} Q_{A_l,t} V_{A_l,t} > 0 \text{ (Enterprise asset value at time } t\text{)} \tag{14.4.2}$$

where M_A denotes the number of assets in the portfolio, $A_{l,t} > 0$ denotes the total value of asset l at time t held by the enterprise, $Q_{A_l,t} > 0$ denotes the quantity of asset l in portfolio at time t, and $V_{A_l,t} > 0$ denotes the value of one unit of asset l in portfolio at time t.

Focusing on liabilities, we assume

$$L_t = \sum_{l=1}^{M_L} L_{l,t} = \sum_{l=1}^{M_L} Q_{L_l,t} V_{L_l,t} < 0 \text{ (Enterprise liability value at time } t\text{)} \tag{14.4.3}$$

where M_L denotes the number of separate liabilities represented in the portfolio, $L_{l,t} < 0$ denotes the total value of enterprise's liability portfolio exposure to liability l at time t (negative reflects liability), $Q_{L_l,t} < 0$ denotes the quantity of liability l (negative quantity reflects liability) in portfolio at time t, and $V_{L_l,t} > 0$ denotes the value of one unit of liability l in portfolio at time t.[5]

Our focus is on measuring and managing the stochastic attributes of the enterprise or portfolio. The enterprise risk can be modeled based on the change in the enterprise value or the change in the portfolio

[4]See Vasicek (1999), Crosbie and Bohn (2002), and Kealhofer and Bohn (2001) and related literature for more sophisticated approaches to credit valuation.

[5]The advantage of modeling liabilities with negative quantities is the instrument valuations can be estimated in the same manner as assets as demonstrated below.

surplus. The goal is to develop the analytically tractable keel model for measuring and managing the inherent financial risks of either enterprises or portfolios. We model the change in the enterprise value between time t and time $t + \Delta$ as

$$\Delta\tilde{S} = \Delta\tilde{A} + \Delta\tilde{L} = \tilde{A}_{t+\Delta} - A_t + \tilde{L}_{t+\Delta} - L_t, \text{ (Change in enterprise value over period } \Delta\text{)} \tag{14.4.4}$$

where $\Delta\tilde{A} = \tilde{A}_{t+\Delta} - A_t$ denotes the change in total asset value and $\Delta\tilde{L} = \tilde{L}_{t+\Delta} - L_t$ denotes the change in total liability value, again expressed as negative. The ~ denotes the variable is not known at time t. Ignoring any rebalancing over period Δ and substituting, we have

$$\begin{aligned}\Delta\tilde{S} &= \sum_{l=1}^{M_A} Q_{A_l,t}\tilde{V}_{A_l,t+\Delta} - \sum_{l=1}^{M_A} Q_{A_l,t}V_{A_l,t} + \sum_{l=1}^{M_L} Q_{L_l,t}\tilde{V}_{L_l,t+\Delta} - \sum_{l=1}^{M_L} Q_{L_l,t}V_{L_l,t} \\ &= \sum_{l=1}^{M_A} Q_{A_l,t}\left(\tilde{V}_{A_l,t+\Delta} - V_{A_l,t}\right) + \sum_{l=1}^{M_L} Q_{L_l,t}\left(\tilde{V}_{L_l,t+\Delta} - V_{L_l,t}\right)\end{aligned}. \tag{14.4.5}$$

A model for valuing underlying instruments based on the LSC model is developed in Online Appendix A. Assuming discrete compounding of holding period returns (HPRs), the change in enterprise value can be expressed as

$$\Delta\tilde{S} = \sum_{l=1}^{M} Q_{l,t}V_{l,t}\tilde{R}_{l,\Delta}, \tag{14.4.6}$$

where M denotes the total number of ordered assets and liabilities ($M = M_A + M_L$) and

$$\tilde{R}_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta} - V_{l,t}}{V_{l,t}} \text{ (HPR for instrument } l\text{)} \tag{14.4.7}$$

$$\mathrm{V}_{l,t} = \sum_{i=0}^{\mathrm{N}_l} \mathrm{E}_t\left(\mathrm{C}\tilde{\mathrm{F}}_{l,i}\middle|\Im_t\right)e^{-k_{l,i,t}\tau_{l,i}}, \text{ and (Value of instrument } l \text{ at time } t\text{)} \tag{14.4.8}$$

$$\tilde{\mathrm{V}}_{l,t+\Delta} = \sum_{i=0}^{\mathrm{N}_l} \mathrm{E}_{t+\Delta}\left(\mathrm{C}\tilde{\mathrm{F}}_{l,i}\middle|\Im_{t+\Delta}\right)e^{-k_{l,i,t+\Delta}\left(\tau_{l,i}+\Delta\right)}. \text{ (Value of instrument } l \text{ at time } t+\Delta\text{)} \tag{14.4.9}$$

Note $E_t\left(C\tilde{F}_{l,i}\middle|\Im_t\right)$ denotes the expected i^{th} cash flow for instrument l, given the information at time t, k denotes the appropriate discount rate, and τ denotes the time to maturity.

Although our focus is primarily on capturing the interaction between assets and liabilities based on changes in the discount rates, the keel model is easily extended to capture interactions between assets and liabilities based on changes in cash flow growth rates as shown in the Online Appendix A. Thus, the keel model is in the spirit of Campbell and Shiller (1988a, 1988b, 2001), Ang and Liu (2001, 2004), and Chen and Zhao (2009). Our objective, however, is more in line with performance attribution.

Typically, the change in enterprise value is known at time t + Δ and the task addressed here is attributing this change to various risk factors. With the keel model, it is an easy transition to proactively measure and manage the extant enterprise risk exposure.

Discretely compounded return attribution

Discretely compounded returns can be expressed as a linear combination of consistent, parsimonious factors as shown in the Appendix B; therefore, the change in the enterprise value can be attributed to these factors. With this infrastructure, one can easily measure and manage enterprise risk. Specifically, we demonstrate the HPR for instrument l over horizon Δ can be expressed as

$$\tilde{R}_{l,\Delta} = R^h_{l,\Delta} + \left(1 + R^h_{l,\Delta}\right)\left(\tilde{R}^r_{l,\Delta} + \tilde{R}^{sp}_{l,\Delta} + \tilde{I}_{l,\Delta}\right) \tag{14.4.10}$$

where $R^h_{l,\Delta}$ denotes the HPR on instrument *l* attributable to the passage of time (horizon), Δ, $\tilde{R}^r_{l,\Delta}$ denotes the HPR on instrument *l* attributable to the overall change in the fitted spot rate, $\tilde{R}^{sp}_{l,\Delta}$ denotes the HPR on instrument *l* attributable to the overall change in the fitted spread, and $\tilde{I}_{l,\Delta}$ denotes the HPR on instrument *l* attributable to the interaction term. Substituting equation (5) into equation (4), we have

$$\begin{aligned}\Delta\tilde{S} &= \sum_{l=1}^{M} Q_{l,t}V_{l,t}\tilde{R}_{l,\Delta} = \sum_{l=1}^{M} Q_{l,t}V_{l,t}\left[R^h_{l,\Delta} + \left(1+R^h_{l,\Delta}\right)\left(\tilde{R}^r_{l,\Delta} + \tilde{R}^{sp}_{l,\Delta} + \tilde{I}_{l,\Delta}\right)\right] \\ &= \sum_{l=1}^{M} Q_{l,t}V_{l,t}R^h_{l,\Delta} + \sum_{l=1}^{M} Q_{l,t}V_{l,t}\left(1+R^h_{l,\Delta}\right)\tilde{R}^r_{l,\Delta} + \sum_{l=1}^{M} Q_{l,t}V_{l,t}\left(1+R^h_{l,\Delta}\right)\tilde{R}^{sp}_{l,\Delta} + \sum_{l=1}^{M} Q_{l,t}V_{l,t}\left(1+R^h_{l,\Delta}\right)\tilde{I}_{l,\Delta}\end{aligned} \tag{14.4.11}$$

or ΔS = Horizon + Spot Rate + Spread + Interaction. Thus, the surplus amount can be attributable to several HPRs, the passage of time or horizon, the fitted spot rate, the fitted spread, and an interaction term.

The horizon HPR can be expressed simply as

$$R^h_{1,\Delta} = \frac{V_{l,t+\Delta}(r,sp) - V_{l,t}(r,sp)}{V_{l,t}(r,sp)} \tag{14.4.12}$$

where (*r*,*sp*) emphasizes the dependence on the spot rate and spread. Note that neither value variable is stochastic in this expression. Clearly, there is no risk in the horizon HPR as we assume the spot rates and credit spreads are as estimated at time t.

As demonstrated in the Appendix B, the spot rate HPR can be further attributed to factor durations, factor convexities, and factor cross convexities. Specifically,

$$\begin{aligned}\tilde{R}^r_{1,\Delta} &= \frac{\tilde{V}_{l,t+\Delta}(\tilde{r}_\Delta, sp) - V_{l,t+\Delta}(r,sp)}{V_{l,t+\Delta}(r,sp)} \\ &= -\sum_{n=0}^{N_F^r} FD_{n,r,l}\Delta\tilde{b}_{n,r} + \frac{1}{2}\sum_{n=0}^{N_F^r} FC_{n,r,l}\Delta\tilde{b}^2_{n,r,l} + \sum_{n=0}^{N_F^r}\sum_{n'=n+1}^{N_F^r} FCC_{n,n',r,l}\Delta\tilde{b}_{n,r}\Delta\tilde{b}_{n',r} + \tilde{R}^\eta_{l,r} \\ &= FD\tilde{S}_r + FC\tilde{S}_r + FCC\tilde{S}_r + \tilde{R}^\eta_{l,r}\end{aligned} \tag{14.4.13}$$

where $\Delta\tilde{b}_{n,r}$ denotes the change in LSC factor *n* related to the spot rate and $\tilde{R}^\eta_{l,\Delta}$ denotes the estimation error. The factor sensitivities related to the spot rate are

$$FD\tilde{S}_r = -\sum_{n=0}^{N_F^r} FD_{n,r,l}\Delta\tilde{b}_{n,r}, \text{ (Duration)} \tag{14.4.14}$$

$$FC\tilde{S}_r = \frac{1}{2}\sum_{n=0}^{N_F^r} FC_{n,r,l}\Delta\tilde{b}^2_{n,r,l}, \text{ and (Convexity)} \tag{14.4.15}$$

$$FCC\tilde{S}_r = \sum_{n=0}^{N_F^r}\sum_{n'=n+1}^{N_F^r} FCC_{n,n',r,l}\Delta\tilde{b}_{n,r}\Delta\tilde{b}_{n',r}. \text{ (Cross-convexity)} \tag{14.4.16}$$

Note $FD_{n,r,l}$ denotes the factor duration, n^{th} factor associated with spot rates (*r*), instrument *l*, horizon Δ, $FC_{n,r,l}$ denotes the factor convexity, n^{th} factor associated with spot rates (*r*), instrument *l*, horizon Δ, and $FCC_{n,r,l}$ denotes the factor cross convexity, n^{th} factor associated with spot rates (*r*), instrument *l*, horizon Δ.

The denominator of Equation (14.4.13) is at the horizon based on the initial parameters. Thus, the spot rate HPR is calculated only after adjusting for the horizon effect.

The fitted spread HPR also can be further attributed to factor durations, factor convexities, and factor cross convexities. Specifically,

$$\tilde{R}^{sp}_{1,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)}$$

$$= -\sum_{n=N_F^r+1}^{N_F} FD_{n,sp,l}\Delta\tilde{b}_{n,sp} + \frac{1}{2}\sum_{n=N_F^r+1}^{N_F} FC_{n,sp,l}\Delta\tilde{b}^2_{n,sp,l} + \sum_{n=N_F^r+1}^{N_F}\sum_{n'=n+1}^{N_F} FCC_{n,n',sp,l}\Delta\tilde{b}_{n,sp}\Delta\tilde{b}_{n',sp} + \tilde{R}^{\eta}_{l,sp} \quad (14.4.17)$$

$$= FD\tilde{S}_{sp} + FC\tilde{S}_{sp} + FCC\tilde{S}_{sp} + \tilde{R}^{\eta}_{l,sp}$$

where again $\Delta\tilde{b}_{n,sp}$ denotes the change in LSC factor n related to the spread and $\tilde{R}^{\eta}_{l,sp}$ denotes the estimation error related to the spread. The factor sensitivities related to the spread are

$$FD\tilde{S}_{sp} = -\sum_{n=N_F^r+1}^{N_F} FD_{n,sp,l}\Delta\tilde{b}_{n,sp}, \text{ (Duration)} \quad (14.4.18)$$

$$FC\tilde{S}_{sp} = \frac{1}{2}\sum_{n=N_F^r+1}^{N_F} FC_{n,sp,l}\Delta\tilde{b}^2_{n,sp,l}, \text{ and (Convexity)} \quad (14.4.19)$$

$$FCC\tilde{S}_{sp} = \sum_{n=N_F^r+1}^{N_F}\sum_{n'=n+1}^{N_F} FCC_{n,n',sp,l}\Delta\tilde{b}_{n,sp}\Delta\tilde{b}_{n',sp}. \text{ (Cross-convexity)} \quad (14.4.20)$$

Note $FD_{n,sp,l}$ denotes the factor duration, n^{th} factor associated with spread (*sp*), instrument l, horizon Δ, $FC_{n,sp,l}$ denotes the factor convexity, n^{th} factor associated with spread (*sp*), instrument l, horizon Δ, and $FCC_{n,sp,l}$ denotes the factor cross convexity, n^{th} factor associated with spread (*sp*), instrument l, horizon Δ.
After accounting for spot rate related HPRs and spread-related HPRs, there is an interaction term. Specifically,

$$\tilde{I}_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - \tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - \left[\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)\right]}{V_{l,t+\Delta}\left(r, sp\right)}. \quad (14.4.21)$$

The interaction term is expected to be small in most cases.
LSC factor risk measures
Based on the LSC model, the LSC risk factor equations are derived in the Appendix C. For each cash flow i of underlying instrument l with maturity time $\tau - \Delta$ at calendar time $t + \Delta$ based on the information set at calendar time t, the present value weight is expressed as

$$w_{l,i,\tau_i-\Delta,t} = \frac{E_t\left(C\tilde{F}_{l,i}\middle|\Im_t\right)e^{-\left[\sum_{n=0}^{N_F}\hat{C}_{i,n}\left(\tau_i-\Delta;s_n\right)b_{n,t}+\varepsilon_{l,i,t}\right]\left(\tau_{l,i}-\Delta\right)}}{V_{l,t+\Delta}\left(r, sp\right)}, \quad (14.4.22)$$

where $\hat{C}_{i,n}\left(\tau_i; s_n\right) = \alpha_{l,n}C_{i,n}\left(\tau_i; s_n\right)$ denotes the alpha adjusted generic LSC coefficient and $b_{n,t}$ denotes the fitted LSC parameters for level, slope, and curvatures. Note that alpha provides flexibility if one should want to ascribe only a portion of a spread to a particular cash flow. For example, one may wish to assign only a 60% weight to a particular credit spread due to unique instrument properties. The approach here also easily handles multiple spreads. For example, a particular instrument may be valued based on a spot rate, spread over spot rate to a BBB credit spread, and one half of the spread between BB and BBB.

Based on the attribution approach here, the horizon effect is accounted elsewhere, so the risk measures focus solely on deviations from values at time $t + \Delta$, not time t. Also, the aggregate error term from the LSC model is assumed to be minimal and safely ignored. Again, the market values are often known, hence the aggregate error term can be known with certainty after ex-post analysis.

LSC risk factors can be expressed as functions of maturity time, LSC coefficients, and cash flow weights. Specifically,

$$FD_{n,l} = \sum_{i=0}^{N} \left(\tau_i - \Delta\right) C_{i,n}\left(\tau_i - \Delta; s_n\right) w_{l,i,\tau_i - \Delta,t}, \text{ (Factor durations)} \tag{14.4.23}$$

$$FC_{n,l} = \sum_{i=0}^{N} \left(\tau_i - \Delta\right)^2 C_{i,n}\left(\tau_i - \Delta; s_n\right)^2 w_{l,i,\tau_i - \Delta,t}, \text{ and (Factor convexities)} \tag{14.4.24}$$

$$FCC_{n,n',l} = \sum_{i=0}^{N} \left(\tau_i - \Delta\right)^2 C_{i,n}\left(\tau_i - \Delta; s_n\right) C_{i,n'}\left(\tau_i - \Delta; s_{n'}\right) w_{l,i,\tau_N - \Delta,t}. \text{ (Factor cross convexities)} \tag{14.4.25}$$

Therefore, the keel model expresses the change in enterprise value in terms of factor sensitivities based on Equation (14.4.11) and substituting in Equations (14.4.13) and (14.4.17), we have

$$\begin{aligned}\Delta\tilde{S} &= \sum_{l=1}^{M} Q_{l,t} V_{l,t} R_{l,\Delta}^{h} + \sum_{l=1}^{M} Q_{l,t} V_{l,t}\left(1 + R_{l,\Delta}^{h}\right)\left(FD\tilde{S}_r + FC\tilde{S}_r + FCC\tilde{S}_r + \tilde{R}_{l,r}^{\eta}\right) \\ &+ \sum_{l=1}^{M} Q_{l,t} V_{l,t}\left(1 + R_{l,\Delta}^{h}\right)\left(FD\tilde{S}_{sp} + FC\tilde{S}_{sp} + FCC\tilde{S}_{sp} + \tilde{R}_{l,sp}^{\eta}\right) + \sum_{l=1}^{M} Q_{l,t} V_{l,t}\left(1 + R_{l,\Delta}^{h}\right)\tilde{I}_{l,\Delta}\end{aligned}. \tag{14.4.26}$$

Thus, we see that the change in enterprise value depends on a horizon HPR, an interaction HPR, and two key returns based on movements in the spot rate and spread. The spot rate and spread HPRs can be further decomposed into factor sensitivities.

The keel model affords an easy decomposition of enterprise value changes. The enterprise value change can be attributable to the mere passage of time, the change in the spot rate curve, the changes in spread curves, and an interaction term. A further decomposition is available for both the spot rate curve and the spread curves with LSC risk factors. We now turn to three applications for this enterprise value decomposition based on the keel model, a small bank, a defined benefit retirement system, and a high net worth family. The goal is to demonstrate the widespread applicability of the keel model to various common financial challenges.

Summary

The keel model has been introduced and seen to provide valuable insights into the problem of attributing enterprise value changes to various risk factors. The keel model is based on the level, slope, and curvature model (denoted the LSC model) of Nelson and Siegel (1987) with a general extension along the lines of Svensson (1995). The keel model developed here is general in form and can have as many factors as required by the user, particularly multiple spread curves. Further, the degree of granularity is at the discretion of the finance professional.

Enterprise value change is decomposed here into four macro components: the non-random horizon component, the spot rate component, the spread component, and an interaction component. The horizon component captures the enterprise change attributable to the mere passage of time over the holding period horizon. The spot rate and spread components are estimated with the LSC model. The spot rate component captures movement in the fitted spot rate curve. The spread component captures the enterprise change attributable to any change in the spread over the fitted spot rate curve. The interaction component contains the residual enterprise change.

The spot rate component of enterprise change can be further decomposed as needed into three components attributable to modified duration, convexity, and cross-convexity based on a Taylor series approximation. Thus, the approximation contains an error component. Each of these three spot rate components can be further decomposed, again, only as needed, into three subcomponents tied to movement in level, movement in slope, and movement in curvature. The keel model is general enough to have multiple curvature components. The LSC model, upon which the keel model is based, is also applied to the spread component and a similar decomposition is applied to the resultant spread curve. For many applications, spot rate duration may be all the decomposition required, whereas in other cases understanding the spread level-slope cross-convexity is helpful. Multiple spreads can also easily be incorporated.

The keel model was illustrated with three cases: a small bank, a retirement system, and a high net worth family. For the cases and scenarios considered here, cross-convexity was not significant and often convexity was also not significant.

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Appendix A. Modeling Underlying Instruments

We focus here on a particular asset, let

$$V_{A_l,t} = \sum_{i=0}^{N_{A_l}} E_t\left(\tilde{CF}_{A_l,i}\middle|\Im_t\right)e^{-k_{A_l,i,t}\tau_{A_l,i,t}} > 0, \text{ (Value of asset } l \text{ at time } t\text{)} \quad (14.4.27)$$

where

N_{A_l} – Number of future cash flows for asset l (and counting any current cash flows with $i = 0$),

$\tilde{CF}_{A_l,i}$ – Cash flow i of asset l paid at maturity time $\tau_{A_l,i}$ periods, assumed random at time t,

$k_{A_l,i,t}$ – Continuously compounded discount rate applied to cash flow i of asset l at time t, deterministic across maturity time, but stochastic across calendar time,

$\tau_{A_l,i,t}$ – Time to maturity of cash flow i of asset l at time t expressed as a fraction of years,

$E_t(\ldots)$ – Standard expectations operator based on the information at time t, and

$\tilde{x}\middle|\Im_t$ – a random variable based on the information set available at time t.

Similarly, focusing on a particular liability expressed as a positive value, let

$$V_{L_l,t} = \sum_{i=0}^{N_{L_l}} E_t\left(\tilde{CF}_{L_l,i}\middle|\Im_t\right)e^{-k_{L_l,i,t}\tau_{L_l,i,t}} > 0, \text{ (Value of liability } l \text{ at time } t\text{)} \quad (14.4.28)$$

where

N_{L_l} – Number future cash flows for liability l (and counting any current cash flows with i = 0),

$\tilde{CF}_{L_l,i}$ – Cash flow i of liability l paid in $\tau_{L_l,i}$ periods, assumed random at time t,

$k_{L_l,i,t}$ – Continuously compounded discount rate applied to cash flow i of liability l at time t, deterministic across maturity time, but stochastic across calendar time, and

$\tau_{L_l,i,t}$ – Time to maturity of cash flow i of liability l at time t.

Suppressing *A* and *L* and recall the quantity of liabilities is represented as negatives, we have

$$S_t = \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_t\left(\tilde{CF}_{l,i}\middle|\Im_t\right)e^{-k_{l,i,t}\tau_{l,i,t}} \underset{<}{\geq} 0, \text{ (Enterprise value at time } t\text{)} \quad (14.4.29)$$

where assets are ordered first and then liabilities where (x denotes generic variable)

$$M = M_A + M_L, \text{ (Total number of assets and liabilities)} \quad (14.4.30)$$

$$x_l \in \{x_l : l = 1,\ldots,M_A\}, \text{ and (Asset items)} \quad (14.4.31)$$

$$x_l \in \{x_l : l = M_A + 1,\ldots,M\}. \text{ (Liability items)} \quad (14.4.32)$$

The enterprise value may be zero or negative. We suppress *A* and *L* where convenient and just use l to denote both. The generic asset or liability will be termed instrument. In general,

$$V_{l,t} = \sum_{i=0}^{N_l} E_t\left(\tilde{CF}_{l,i}\middle|\Im_t\right)e^{-k_{l,i,t}\tau_{l,i}} > 0. \text{ (Value of generic instrument l at time t)} \quad (14.4.33)$$

Although not our focus here, future expected cash flows could easily be modeled with similar maturity varying growth rates. That is, $E_t\left(\tilde{CF}_{l,i}\middle|\Im_t\right) = CF_{l,0}e^{g_{l,i,t}\tau_{l,i}}$, where g denotes the appropriate cash flow growth rate.

LSC model applied to discount rates

We now expand on the discount rate using the LSC model described below. For a generic instrument, assume

$$k_{l,i,t} = r^{LSC}_{\tau_i,t} + sp^{LSC}_{l,\tau_i,t} + \varepsilon_{l,t}, \text{ (Discount rate, generic instrument } l\text{, cash flow } i\text{, time } t\text{)} \quad (14.4.34)$$

where

$r^{LSC}_{\tau_i,t}$ – Base interest rate using the LSC model for maturity τ_i at time *t*.

$sp^{LSC}_{l,\tau_i,t} = \sum_{j=1}^{N_{sp}} \alpha_{l,j} sp^{LSC}_{j,\tau_i,t}$ – Total spread rate for instrument *l* using the LSC model for spread *j*, maturity τ_i at time *t*, where each instrument can have differing exposures ($\alpha_{l,j}$) to each spread (for example, AAA to Base Rate, BBB to AAA, and Equity to BBB). Total spread is additive across multiple spreads.

$\varepsilon_{l,t} = \varepsilon_l$ – Residual error from using the LSC model for the base interest rate and spread rate, note the error can typically only be measured at the instrument level, not the individual cash flow level.

LSC model

Motivated by the work of Nelson and Siegel (1987), Svensson (1995) develops a methodology employing estimates of level, slope, and multiple curvature terms. We call this approach the LSC model due to the estimates of level, slope, and multiple curvature terms. We use the general form that can be expressed in either rate or spread terms as

$$r^{LSC}_{\tau_i,t} = \sum_{n=0}^{N^r_F} C_{i,r,n}\left(\tau_i; s_{r,n}\right) b_{n,r,t} \text{ and} \quad (14.4.35)$$

$$sp^{LSC}_{j,\tau_i,t} = \sum_{n=0}^{N^{spj}_F} C_{i,sp_j,n}\left(\tau_i; s_{sp_j,n}\right) b_{n,sp_j,t}, \quad (14.4.36)$$

where

$$C_{i,r,0}\left(\tau_i; s_0\right) = C_{i,sp,0}\left(\tau_i; s_0\right) = 1, \text{ (Intercept coefficient)} \quad (14.4.37)$$

$$C_{i,r,1}\left(\tau_i; s_{1,r}\right) = \frac{s_{1,r}}{\tau_i}\left(1 - e^{-\tau_i/s_{1,r}}\right), \text{ (Base rate slope coefficient)} \quad (14.4.38)$$

$$C_{i,sp_j,1}\left(\tau_i; s_{1,sp_j}\right) = \frac{s_{1,sp_j}}{\tau_i}\left(1 - e^{-\tau_i/s_{1,spj}}\right), \text{ (}j^{\text{th}}\text{ spread coefficient)} \quad (14.4.39)$$

$$C_{i,r,n}\left(\tau_i; s_{n,r}\right) = \frac{s_{n,r}}{\tau_i}\left(1 - e^{-\tau_i/s_{n,r}}\right) - e^{-\tau_i/s_{n,r}}; n > 1, \text{ (Base rate curvature coefficients)} \quad (14.4.40)$$

$$C_{i,sp_j,n}\left(\tau_i; s_{n,sp_j}\right) = \frac{s_{n,sp_j}}{\tau_i}\left(1 - e^{-\tau_i/s_{n,spj}}\right) - e^{-\tau_i/s_{n,spj}}; n > 1, \text{ and (}j^{\text{th}}\text{ spread curvatures coefficients)} \quad (14.4.41)$$

$b_{n,t}$ denotes the fitted parameters for level, slope, and curvatures.

The scalars, s_n, are deterministic and have several constraints. The scalar essentially applies various weights to different locations on the term structure. Note that s_0 is not used but allows for the simple summation expression of the general equation. Based on the work of Nelson and Siegel (1987) and Svensson (1995), we require $s_1 = s_2$. Thus,

$$V_{l,t} = \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-k_{l,i,t}\tau_{l,i}}$$

$$= \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left(r_{\tau_i,t}^{LSC} + sp_{l,\tau_i,t}^{LSC} + \varepsilon_{l,t}\right)\tau_{l,i}} > 0$$ (Value of generic instrument l at time t). (14.4.42)

and the residual error is defined as (hopefully near zero or assumed to be zero when instrument values are not available)

$$\varepsilon_{l,t} \ni V_{l,t} = \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left(r_{\tau_i,t}^{LSC} + sp_{l,\tau_i,t}^{LSC} + \varepsilon_{l,t}\right)\tau_{l,i}} > 0 : l = 1, \ldots, M.$$ (Residual error definition)

Enterprise value at time t as a function of the base rate curve and set of spread curves can be expressed as

$$S_t = \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-k_{l,i,t}\tau_{l,i}}$$

$$= \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left(r_{\tau_i,t}^{LSC} + sp_{l,\tau_i,t}^{LSC} + \varepsilon_{l,i,t}\right)\tau_{l,i}} \quad \text{or} \qquad (14.4.43)$$

$$= \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left(r_{\tau_i,t}^{LSC} + \sum_{j=1}^{N_{sp}} \alpha_{l,j} sp_{j,\tau_i,t}^{LSC} + \varepsilon_{l,i,t}\right)\tau_{l,i}}$$

$$S_t = \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left[\sum_{n=0}^{N_F^r} C_{i,r,n}\left(\tau_i; s_{r,n}\right) b_{n,r,t} + \sum_{j=1}^{N_{sp}} \alpha_{l,j} \sum_{n=0}^{N_F^{spj}} C_{i,spj,n}\left(\tau_i; s_{spj,n}\right) b_{n,spj,t} + \varepsilon_{l,i,t}\right]\tau_{l,i}}$$ (Enterprise value at time t) (14.4.44)

where the LSC factors are ordered with base rate factors first and then sets of spread factors and

$N_F = N_F^r + N_F^{sp}$, (Total number of factors) (14.4.45)

$x_i \in \left(x_i : i = 1, \ldots, N_F^r\right)$, and (Base rate factors) (14.4.46)

$x_i \in \left(x_i : i = N_F^r + 1, \ldots, N_F\right)$. (Spread factors across all spreads) (14.4.47)

The enterprise values can be expressed succinctly as

$$S_t = \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}\left(\tau_i; s_n\right) b_{n,t} + \varepsilon_{l,i,t}\right]\tau_{l,i}}$$ and (Enterprise value at time t) (14.4.48)

$$\tilde{S}_{t+\Delta} = \sum_{l=1}^{M} Q_{l,t} \sum_{i=0}^{N_l} E_{t+\Delta}\left(C\tilde{F}_{l,i} \middle| \Im_{t+\Delta}\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}\left(\tau_i - \Delta; s_n\right) \tilde{b}_{n,t+\Delta} + \tilde{\varepsilon}_{l,i,t+\Delta}\right]\left(\tau_{l,i} - \Delta\right)}$$, (Enterprise value at time $t + \Delta$)(14.4.49)

where

$\hat{C}_{i,n}\left(\tau_i; s_n\right) = \alpha_{l,n} C_{i,n}\left(\tau_i; s_n\right)$. (Alpha adjusted generic LSC coefficient) (14.4.50)

As previously noted, the expected cash flows could be modeled as $E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) = CF_{l,0} e^{g_{l,i,t}\tau_{l,i}}$. Thus, the expected cash flow growth rates could simply be incorporated as another spread component. Although easy to incorporate, it is beyond our purposes here.

Note alpha provides flexibility should one want to ascribe only a portion of one spread to a particular cash flow. The approach here clearly handles multiple spreads. For example, a particular instrument may be valued based on a spot rate, spread over spot rate to a BBB credit spread, and one half of spread between BB and BBB.

When assets and liabilities are combined,

$$S_t = \sum_{l=1}^{M} Q_{l,t} V_{l,t} \text{ and (Enterprise value at time t)} \tag{14.4.51}$$

$$\tilde{S}_{t+\Delta} = \sum_{l=1}^{M} Q_{l,t} \tilde{V}_{l,t+\Delta} = \sum_{l=1}^{M} Q_{l,t} V_{l,t} \left(1 + \tilde{R}_{l,\Delta}\right), \text{(Enterprise value at time t + } \Delta) \tag{14.4.52}$$

where the discretely compounded HPR for underlying instrument l over period Δ is

$$\tilde{R}_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta} - V_{l,t}}{V_{l,t}}. \tag{14.4.53}$$

Appendix B. Enterprise Change Performance Attribution

The goal here is to attribute changes in enterprise value to various fundamental risk factors related to the spot rate and spreads. We introduce simple notation for generic individual instrument values:

$$\tilde{V}_{l,t+\Delta}\left(\tilde{r}_\Delta, s\tilde{p}_\Delta\right) = V_{l,t}\left(r, sp\right)\left(1 + \tilde{R}_{l,\Delta}\right), \text{(Value at time t + } \Delta) \tag{14.4.54}$$

$$V_{l,t}\left(r, sp\right) = \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}(\tau_i; s_n) b_{n,t} + \varepsilon_{l,i,t}\right]\tau_{l,i}}, \text{(Value as function of factors, } t) \tag{14.4.55}$$

$$V_{l,t+\Delta}\left(r, sp\right) = \sum_{i=0}^{N_l} E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}(\tau_i - \Delta; s_n) b_{n,t} + \varepsilon_{l,i,t}\right](\tau_{l,i} - \Delta)}, \text{(Value as a function of factors, } t + \Delta) \tag{14.4.56}$$

$$\tilde{V}_{l,t+\Delta}\left(\tilde{r}_\Delta, sp\right) = \sum_{i=0}^{N_l} E_{t+\Delta}\left(C\tilde{F}_{l,i} \middle| \Im_{t+\Delta}\right) e^{-\left[\sum_{n=0}^{N_F^r} \hat{C}_{i,n}(\tau_i - \Delta; s_n) \tilde{b}_{n,t+\Delta} + \sum_{n=N_F^r+1}^{N_F} \hat{C}_{i,n}(\tau_i - \Delta; s_n) b_{n,t} + \varepsilon_{l,i,t}\right](\tau_{l,i} - \Delta)},$$

(At time t + Δ, value based on time t + Δ factors for the base rate and time t factors for spreads) (14.4.57)

$$\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_\Delta\right) = \sum_{i=0}^{N_l} E_{t+\Delta}\left(C\tilde{F}_{l,i} \middle| \Im_{t+\Delta}\right) e^{-\left[\sum_{n=0}^{N_F^r} \hat{C}_{i,n}(\tau_i - \Delta; s_n) b_{n,t} + \sum_{n=N_F^r+1}^{N_F} \hat{C}_{i,n}(\tau_i - \Delta; s_n) \tilde{b}_{n,t+\Delta} + \varepsilon_{l,i,t}\right](\tau_{l,i} - \Delta)}, \text{ and}$$

(At time t + Δ, value based on time t factors for the base rate and time t + Δ factors for spreads) (14.4.58)

$$\tilde{V}_{l,t+\Delta}\left(\tilde{r}_\Delta, s\tilde{p}_\Delta\right) = \sum_{i=0}^{N_l} E_{t+\Delta}\left(C\tilde{F}_{l,i} \middle| \Im_{t+\Delta}\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}(\tau_i - \Delta; s_n) \tilde{b}_{n,t+\Delta} + \tilde{\varepsilon}_{l,i,t+\Delta}\right](\tau_{l,i} - \Delta)}.$$

(At time t + Δ, value based on time t + Δ factors for the base rate and time t + Δ factors for spreads) (14.4.59)

Discretely compounded holding period return decomposition

Assuming initial values are positive, and quantity can be negative, the discretely compounded HPR decomposition can be expressed as

$$\tilde{R}_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - V_{l,t}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)}$$

$$= \frac{1}{V_{l,t}\left(r, sp\right)} \left\{ \begin{array}{l} \left[V_{l,t+\Delta}\left(r, sp\right) - V_{l,t}\left(r, sp\right)\right] + \left[\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - V_{l,t+\Delta}\left(r, sp\right)\right] \\ + \left[\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)\right] \\ + \left[\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - \tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) + V_{l,t+\Delta}\left(r, sp\right) - \tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right)\right] \end{array} \right\}. \quad (14.4.60)$$

Or alternatively the discretely compounded HPR decomposition can be expressed as

$$\begin{aligned} \tilde{R}_{l,\Delta} = & \frac{V_{l,t+\Delta}\left(r, sp\right) - V_{l,t}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)} + \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)} \\ & + \frac{\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)} \\ & + \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - \tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - \left[\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)\right]}{V_{l,t}\left(r, sp\right)} \end{aligned}. \quad (14.4.61)$$

Factoring the horizon component to focus on spot rate and spread factors, we have

$$\begin{aligned} \tilde{R}_{l,\Delta} = & \frac{V_{l,t+\Delta}\left(r, sp\right) - V_{l,t}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)} \\ & + \frac{V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)} \left\{ \begin{array}{l} \dfrac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)} + \dfrac{\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)} \\ + \dfrac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - \tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - \left[\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)\right]}{V_{l,t+\Delta}\left(r, sp\right)} \end{array} \right\} \end{aligned} \quad (14.4.62)$$

Thus, we can simplify in the following manner:

$$\tilde{R}_{l,\Delta} = R^h_{l,\Delta} + \left(1 + R^h_{l,\Delta}\right)\tilde{R}^F_{l,\Delta} \text{ or} \quad (14.4.63)$$

$$\tilde{R}_{l,\Delta} = R^h_{l,\Delta} + \left(1 + R^h_{l,\Delta}\right)\left(\tilde{R}^r_{l,\Delta} + \tilde{R}^{sp}_{l,\Delta} + \tilde{I}_{l,\Delta}\right). \quad (14.4.64)$$

The decomposition returns are defined as follows

$$R^h_{l,\Delta} = \frac{V_{l,t+\Delta}\left(r, sp\right) - V_{l,t}\left(r, sp\right)}{V_{l,t}\left(r, sp\right)}, \text{ (Horizon return attributable to the passage of time, } \Delta) \quad (14.4.65)$$

$$\tilde{R}^F_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)} = \tilde{R}^r_{l,\Delta} + \tilde{R}^{sp}_{l,\Delta} + \tilde{I}_{l,\Delta}, \text{ (Return attributable to risk factors)} \quad (14.4.66)$$

$$\tilde{R}^r_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)} \text{ (Return attributable to the rates)} \quad (14.4.67)$$

$$\tilde{R}^{sp}_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)}, \text{ and (Return attributable to the spread)} \quad (14.4.68)$$

$$\tilde{I}_{l,\Delta} = \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, s\tilde{p}_{\Delta}\right) - \tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - \left[\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)\right]}{V_{l,t+\Delta}\left(r, sp\right)}. \tag{14.4.69}$$

(Return attributable to interaction term)

Let the present value expected cash flow weights for a given instrument cash flow discounted based on time $t + \Delta$ be defined as

$$w_{l,i,\tau_i-\Delta,t} = \frac{E_t\left(C\tilde{F}_{l,i} \middle| \Im_t\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}(\tau_i-\Delta; s_n) b_{n,t} + \varepsilon_{l,i,t}\right](\tau_{l,i}-\Delta)}}{V_{l,t+\Delta}\left(r, sp\right)}, \tag{14.4.70}$$

and define the change in LSC factors over the next time period (assumed to be Δ units of time) as

$$\Delta\tilde{b}_n = \tilde{b}_{n,t+\Delta} - b_{n,t}. \tag{14.4.71}$$

Therefore, the discretely compounded rate of return attributable to the fitted spot rate or spreads for instrument *l* can be approximated, based on Taylor series applied to the instrument's value, as (detailed proof available in Appendix C, note subscript "*r*" indicates the spot rate),

$$\begin{aligned} \tilde{R}_{1,\Delta}^{r} &= \frac{\tilde{V}_{l,t+\Delta}\left(\tilde{r}_{\Delta}, sp\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)} \\ &= -\sum_{n=0}^{N_F^r} FD_{n,r,l}\Delta\tilde{b}_{n,r} + \frac{1}{2}\sum_{n=0}^{N_F^r} FC_{n,r,l}\Delta\tilde{b}_{n,r,l}^2 + \sum_{n=0}^{N_F^r}\sum_{n'=n+1}^{N_F^r} FCC_{n,n',r,l}\Delta\tilde{b}_{n,r}\Delta\tilde{b}_{n',r} + \tilde{R}_{l,r}^{\eta} \end{aligned}, \tag{14.4.72}$$

where

$\tilde{R}_{l,r}^{\eta}$ – Estimation error from the Taylor series approximation procedure,

$FD_{n,r,l}$ – Factor duration, n[th] factor associated with interest rates (r), instrument l, horizon Δ,

$FC_{n,r,l}$ – Factor convexity, n[th] factor associated with interest rates (r), instrument l, horizon Δ, and

$FCC_{n,n',r,l}$ – Factor cross convexity, n[th] factor associated with interest rates (r), instrument l, horizon Δ.

The discretely compounded rate of return attributable to the fitted spread for instrument *l* can be approximated, based on Taylor series applied to the instrument's value, as (detailed proof available in Appendix C, note subscript "*sp*" indicates the spread rate)

$$\begin{aligned} \tilde{R}_{1,\Delta}^{sp} &= \frac{\tilde{V}_{l,t+\Delta}\left(r, s\tilde{p}_{\Delta}\right) - V_{l,t+\Delta}\left(r, sp\right)}{V_{l,t+\Delta}\left(r, sp\right)} \\ &= -\sum_{n=N_F^r+1}^{N_F} FD_{n,sp,l}\Delta\tilde{b}_{n,sp} + \frac{1}{2}\sum_{n=N_F^r+1}^{N_F} FC_{n,sp,l}\Delta\tilde{b}_{n,sp,l}^2 + \sum_{n=N_F^r+1}^{N_F}\sum_{n'=n+1}^{N_F} FCC_{n,n',sp,l}\Delta\tilde{b}_{n,sp}\Delta\tilde{b}_{n',sp} + \tilde{R}_{l,sp}^{\eta} \end{aligned}, \tag{14.4.73}$$

where

$\tilde{R}_{l,sp}^{\eta}$ – Estimation error from the Taylor series approximation procedure applied to spread,

$FD_{n,sp,l}$ – Factor duration, n^{th} factor associated with spread (*sp*), instrument *l*, horizon Δ,

$FC_{n,sp,l}$ – Factor convexity, n^{th} factor associated with spread (*sp*), instrument *l*, horizon Δ, and

$FCC_{n,n',sp,l}$ – Factor cross convexity, n^{th} factor associated with spread (*sp*), instrument *l*, horizon Δ.

Appendix C. N-Dimensional Taylor Series and Bond Risk Measures

The *N*-dimensional Taylor series is applied to identify the appropriate equivalent to modified duration, convexity, and cross convexity within the LSC model with discretely compounded HPRs. Taylor series is a well-known approximating technique and can be found in many mathematics books. See, for example, Aramanovich (1965).[6]

N-dimensional Taylor series

Assume a continuous function $f\left(\underline{x}\right)$, where $\underline{x}\left(x_1, x_2, \ldots, x_n\right)$ is a vector with n elements, and $-\infty < x_i < \infty, i = 1, \ldots, n$. Also assume at $f\left(\underline{x}^0\right)$ has derivatives of all orders. Let $D_i = \partial / \partial x_i$ be the operators of partial differentiation where $D_i f = \partial f / \partial x_i$, $D_i^m f = \partial^m f / \partial x_i^m$ and in the multidimensional case

$$D_{i_1} D_{i_2} \ldots D_{i_k} f = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_K}}, \tag{14.4.74}$$

where the required partial derivatives are assumed to exist. Then the Taylor series of f about the point $\underline{x}^0$ is

$$f\left(\underline{x}\right) = \sum_{i=0}^{\infty} \frac{\left\{ \sum_{j=1}^{n} \left(x_j - x_j^0\right) D_j \right\}^i}{i!} f\left(\underline{x}^0\right). \tag{14.4.75}$$

N-dimensional Taylor series applied to underlying values

The focus here is on estimating the portion of the HPRs attributable to moves in LSC factors related to the base rate and spreads. Thus, the HPR portion attributable to the holding period or horizon, Δ, is already addressed. Hence, we address the HPR based on the underlying instrument's value at time *t* + Δ but based on the LSC factor values at time *t* or $V_{l,t+\Delta}\left(\underline{b}_t\right)$.

The i' partial derivatives of the instrument *l* value with respect to LSC factor *n*, either related to rates or spreads, at time *t* (denoted generically as $b_{n,t}$) evaluated at time *t* +Δ, is

$$\frac{\partial^{i'} V_{l,t+\Delta}}{\partial b_{n,t}^{\;i'}} = \left(-1\right)^{i'} \sum_{i=1}^{N_l} \left(\tau_{l,i} - \Delta\right)^{i'} \hat{C}_{i,n}\left(\tau_{l,i} - \Delta; s_n\right)^{i'} E_t\left(C\tilde{F}_{l,i}\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}\left(\tau_{l,i} - \Delta; s_n\right) b_{n,t} + \varepsilon_{l,t}\right]\left(\tau_{l,i} - \Delta\right)}, \tag{14.4.76}$$

$$\frac{\partial^{i'} V_{l,t+\Delta}}{\partial b_{n,t}^{\;j'} \partial b_{n',t}^{\;i'-j'}} = \left(-1\right)^{i'} \sum_{i=1}^{N_l} \begin{matrix} \left(\tau_{l,i} - \Delta\right)^{i'} \hat{C}_{j,n}\left(\tau_{l,i} - \Delta; s_{r,n}\right)^{j'} \hat{C}_{i,n'}\left(\tau_{l,i} - \Delta; s_{r,n'}\right)^{i'-j'} \\ E_t\left(C\tilde{F}_{l,i}\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}\left(\tau_{l,i} - \Delta; s_n\right) b_{n,t} + \varepsilon_{l,t}\right]\left(\tau_{l,i} - \Delta\right)} \end{matrix}, \tag{14.4.77}$$

and so forth.

We apply Taylor series to the discretely compounded bond returns attributable to factors or

$$\tilde{R}_{l,\Delta}^F\left(\underline{\tilde{b}}_{t+\Delta}\right) = \frac{\tilde{V}_{l,t+\Delta}\left(\underline{\tilde{b}}_{t+\Delta}\right) - V_{l,t+\Delta}\left(\underline{b}_t\right)}{V_{l,t+\Delta}\left(\underline{b}_t\right)}. \tag{14.4.78}$$

We define the stochastic change in the generic factor as

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[6] Aramanovich, I. G., R. S. Guter, L. A. Lyusternik, I. L. RaukhVaRger, M. I. Skanavi, and Yanpol'skii, A. R. (1965). Mathematical Analysis Differentiation and Integration. New York: Pergamon Press.

$$\Delta \tilde{b}_n = \tilde{b}_{n,t+\Delta} - b_{n,t}. \tag{14.4.79}$$

Thus, the Taylor series is applied to the underlying value,

$$V_{l,t+\Delta}\left(\underline{\tilde{b}}_{t+\Delta}\right) \cong \sum_{i'=0}^{\infty} \frac{\left(\sum_{n=1}^{N_F} \Delta \tilde{b}_n D_n\right)^{i'}}{i'!} V_{l,t+\Delta}\left(\underline{b}_t\right) \text{ or (Sum begins at 0)} \tag{14.4.80}$$

$$V_{l,t+\Delta}\left(\underline{\tilde{b}}_{t+\Delta}\right) - V_{l,t+\Delta}\left(\underline{b}_t\right) \cong \sum_{i'=1}^{\infty} \frac{\left(\sum_{n=1}^{N_F} \Delta \tilde{b}_n D_n\right)^{i'}}{i'!} V_{l,t+\Delta}\left(\underline{b}_t\right). \text{ (Sum begins at 1)} \tag{14.4.81}$$

(C-7)

Approximating the discretely compounded bond returns with up to the first and second derivatives, we have

$$\tilde{R}_{l,\Delta}\left(\underline{\tilde{b}}_{t+\Delta}\right) = \frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)} \left[\sum_{n=0}^{N_F} \frac{\partial V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}} \Delta \tilde{b}_n + \tilde{\eta}_{l,t+\Delta}\right] \text{ and} \tag{14.4.82}$$

$$\tilde{R}_{l,\Delta}\left(\underline{\tilde{b}}_{t+\Delta}\right) = \frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)} \left\{\sum_{n=0}^{N_F} \frac{\partial V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}} \Delta \tilde{b}_n + \frac{1}{2}\left(\sum_{n=0}^{N_F} \Delta \tilde{b}_n D_n\right)^2 + \tilde{\eta}_{2,t+\Delta}\right\}. \tag{14.4.83}$$

Note that factor durations are defined as

$$FD_{n,l} \equiv -\frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)} \frac{\partial V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}}. \tag{14.4.84}$$

Further, factor durations based on the underlying instrument's value expressions are

$$\begin{aligned} FD_{n,l} &= -\frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)}(-1)\sum_{i=1}^{N_l} \frac{\left(\tau_{l,i} - \Delta\right)\hat{C}_{i,n}\left(\tau_{l,i} - \Delta; s_n\right)}{E_t\left(C\tilde{F}_{l,i}\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}\left(\tau_{l,i}-\Delta; s_n\right) b_{n,t} + \varepsilon_{l,t}\right]\left(\tau_{l,i}-\Delta\right)}}, \\ &= \sum_{i=1}^{N_l}\left(\tau_{l,i} - \Delta\right)\hat{C}_{i,n}\left(\tau_{l,i} - \Delta; s_n\right) w_{l,i,\tau_{l,i}-\Delta,t} \end{aligned} \tag{14.4.85}$$

where

$$w_{l,i,\tau_i-\Delta,t} \equiv \frac{E_t\left(C\tilde{F}_{l,i}\right) e^{-\left[\sum_{n=0}^{N_F} \hat{C}_{i,n}\left(\tau_{l,i}-\Delta; s_n\right) b_{n,t} + \varepsilon_{l,t}\right]\left(\tau_{l,i}-\Delta\right)}}{V_{l,t+\Delta}\left(\underline{b}_t\right)}. \tag{14.4.86}$$

Thus, focusing solely on factor durations, the discretely compounded rate of return can be approximated with factor durations and factor changes, we have

$$\tilde{R}_{l,\Delta}\left(\underline{\tilde{b}}_{t+\Delta}\right) \cong -\sum_{n=0}^{N_F} FD_{n,l} \Delta \tilde{b}_n. \tag{14.4.87}$$

Factor convexity is defined and expressed as

$$FC_{n,l} \equiv \frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)} \frac{\partial^2 V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}^2} = \sum_{i=1}^{N_l}\left(\tau_{l,i}-\Delta\right)^2 \hat{C}_{i,n}\left(\tau_{l,i}-\Delta; s_n\right)^2 w_{l,i,\tau_i-\Delta,t}, \quad (14.4.88)$$

and factor cross convexity is defined and expressed as

$$FCC_{n,n',l} \equiv \frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)} \frac{\partial^2 V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}\partial b_{n',t}} = \sum_{i=1}^{N_l}\left(\tau_{l,i}-\Delta\right)^2 \hat{C}_{i,n}\left(\tau_{l,i}-\Delta; s_n\right)\hat{C}_{i,n'}\left(\tau_{l,i}-\Delta; s_{n'}\right) w_{l,i,\tau_i-\Delta,t}. \quad (14.4.89)$$

Let $A_n \equiv \Delta\tilde{b}_n D_n$, then

$$\begin{aligned}
\left(\sum_{n=0}^{N_F}\Delta\tilde{b}_n D_n\right)^2 &= \left(\sum_{n=0}^{N_F} A_n\right)^2 = \sum_{n=0}^{N_F}\sum_{n'=0}^{N_F} A_n A_{n'} \\
&= \sum_{n=0}^{N_F} A_n^2 + 2\sum_{n=0}^{N_F}\sum_{n'=n+1}^{N_F} A_n A_{n'} \\
&= \sum_{n=0}^{N_F} \Delta\tilde{b}_n^2 D_n^2 + 2\sum_{n=0}^{N_F}\sum_{n'=n+1}^{N_F} \Delta\tilde{b}_n\Delta\tilde{b}_{n'} D_n D_{n'} \\
&= \sum_{n=0}^{N_F}\frac{\partial^2 V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}^2}\Delta\tilde{b}_n^2 + 2\sum_{n=0}^{N_F}\sum_{n'=n+1}^{N_F}\frac{\partial^2 V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}\partial b_{n',t}}\Delta\tilde{b}_n\Delta\tilde{b}_{n'}
\end{aligned} \quad . \quad (14.4.90)$$

Thus, the discretely compounded rate of return can be approximated by factor risk measures along with changes in the factors. Specifically,

$$\begin{aligned}
\tilde{R}_{l,\Delta}^{dc}\left(\tilde{\underline{b}}_{t+\Delta}\right) &\cong \frac{1}{V_{l,t+\Delta}\left(\underline{b}_t\right)}\left[\sum_{n=0}^{N_F}\frac{\partial V_{l,t+\Delta}\left(\underline{b}_t\right)}{\partial b_{n,t}}\Delta\tilde{b}_n + \frac{1}{2}\left(\sum_{n=0}^{N_F}\Delta\tilde{b}_n D_n\right)^2\right] \\
&= -\sum_{n=0}^{N_F} FD_{n,l}\Delta\tilde{b}_n + \frac{1}{2}\sum_{n=0}^{N_F} FC_{n,l}\Delta\tilde{b}_n^2 + \sum_{n=0}^{N_F}\sum_{n'=n+1}^{N_F} FCC_{n,n',l}\Delta\tilde{b}_n\Delta\tilde{b}_{n'}
\end{aligned} \quad .$$

Appendix D. Selected Calculations from Cases

Selected calculations are provided as an aid to understanding how to deploy the keel model in practice. The focus here is on the small bank case but the process is the same for the other two cases. Based on the LSC model and the data given in Tables 14.4.4, 14.4.11, and 14.4.22, the continuously compounded spot rate and spread are computed for every 0.25 years to maturity out to 30 years. Figures 14.4.2a-c illustrates the data given in Table 14.4.4. Based on the spot rate and the all-in rate (spot plus spread), the appropriate discount factors are computed. These rates and discount factors are used throughout.

The initial design for this small bank is zero equity duration. For simplicity, we assume the categories Cash and Due, Premises, Noninterest Bearing Deposits, and Borrowings have zero duration. Borrowings are assumed to be floating rate just prior to a reset date. Securities are assumed to result in \$5 million cash flow each semi-annual period starting at year 2 and ending at year 5.5. An additional cash flow of \$1.38267 million is assumed at year 6. This dollar amount was chosen to result in \$40 million in value today.

The Securities factor sensitivities is computed as follows:

$$FD_{n,Securities} = \sum_{i=1}^{9}\left(\tau_i - 0.25\right) C_{i,n}\left(\tau_i - 0.25; 2.0\right) w_{Securities,i,\tau_9-0.25,0}, \text{ (Factor durations)}$$

$$FC_{n,Securities} = \sum_{i=1}^{9} \left(\tau_i - 0.25\right)^2 C_{i,n}\left(\tau_i - 0.25; 2.0\right)^2 w_{Securities,i,\tau_9 - 0.25,0}$$, and (Factor convexities)

$$FCC_{n,n',Securities} = \sum_{i=1}^{9} \left(\tau_i - 0.25\right)^2 C_{i,n}\left(\tau_i - 0.25; 2.0\right) C_{i,n'}\left(\tau_i - 0.25; 2.0\right) w_{Securities,i,\tau_9 - 0.25,0}.$$

(Factor cross convexities)

Table 14.4.25 provides selected values for the three factor LSC model.

Table 14.4.25. Intermediate Calculations for Small Bank Securities Factor Sensitivities

					Factor Durations			Factor Convexities			Factor Cross-Convexities		
i	**τ_i**	**CF_i**	**k_i**	**w_i**	**L**	**S**	**C**	**L**	**S**	**C**	**L**	**S**	**C**
1	2.0	5.0	0.40	12.37	0.22	0.14	0.05	0.38	0.17	0.02	0.25	0.09	0.06
2	2.5	5.0	0.51	12.31	0.28	0.17	0.08	0.62	0.22	0.05	0.37	0.17	0.10
3	3.0	5.0	0.61	12.25	0.34	0.18	0.10	0.93	0.27	0.08	0.50	0.27	0.15
4	3.5	5.0	0.72	12.17	0.40	0.20	0.12	1.29	0.31	0.11	0.63	0.38	0.19
5	4.0	5.0	0.82	12.08	0.45	0.20	0.14	1.70	0.35	0.15	0.77	0.51	0.23
6	4.5	5.0	0.92	11.98	0.50	0.21	0.15	2.16	0.37	0.19	0.90	0.64	0.26
7	5.0	5.0	1.01	11.87	0.56	0.22	0.16	2.68	0.39	0.22	1.02	0.77	0.30
8	5.5	5.0	1.09	11.76	0.62	0.22	0.17	3.24	0.40	0.26	1.15	0.91	0.32
9	6.0	1.383	1.17	3.22	0.18	0.06	0.05	1.06	0.11	0.08	0.35	0.29	0.10
Sum				100.0	3.55	1.60	1.02	14.1	2.66	1.16	5.95	4.04	1.71

The procedure for computing these parameters remains the same across all instruments within all three cases. We provide further comments on selected other balance sheet items.

For Loans and Leases, we assume about 50% are floating rate and 50% are equivalent to 3 to 5 years to maturity. Thus, we impose a barbell structure. The category Interest Bearing Deposits (IBD) has a long history of debate regarding the appropriate rate sensitivity. Legally, the IBD duration is one day. In practice, IBD duration is longer. We assume, based on a Small Bank study, roughly 30% of IBD are equivalent to 0.5 to 0.75 year cash flows and 70% are equivalent to 5 year cash flows. Clearly, in practice, the management team of each bank would craft a structure suitable for their perspective on IBD.

By design, the bank's equity duration is zero. The goal in the paper is to accentuate the consequences of popular trading strategies and how the keel model identifies flaws in them.

Strategy #1 converts $40 million of the Borrowings into a synthetic fixed rate of 1.04% based on the LSC curve. Strategy #2 converts the floating rate loans to synthetic fixed rate of 1.41% based on the LSC curve. With the initial design of zero equity duration, both strategies must result in non-zero equity durations.