# Module 14.3: Risk Attribution

**Learning objectives**
- Define and illustrate risk attribution
- Analysis permits ex post risk attribution and ex ante risk expectation
- Risk attribution decomposes the total variance of excess return into a variety of statistics including percentage marginal contribution to risk by stock, sector, sector allocation decision, security selection decision, and an interaction term

*Note that R code has not been developed for this module.*

## Executive summary
This module is like module 14.2 focused on return attribution. The material here is quite quantitative.

## Central finance concepts
Risk attribution decomposes the total variance of excess return into a variety of statistics including percentage marginal contribution to risk by stock, sector, sector allocation decision, security selection decision, and interaction.

The risk attribution statistics are not additive; hence, subperiod results do not aggregate to entire measurement period. Typically, risk is not additive across time, however percentage marginal contribution to risk is additive across sectors or securities as will be demonstrated in the next section.

## Quantitative finance materials
**Introduction to risk attribution**
Again, risk attribution decomposes the total variance of excess return into a variety of statistics. As before,

$$SAW_{M,k,t} = \sum_{j=1}^{n_M} w_{M,j,t} I_{\text{Sector}_k}(j) \text{ weight of managed portfolio allocated to sector k,} \quad (14.3.1)$$

$$SAW_{B,k,t} = \sum_{j=1}^{n_B} w_{B,j,t} I_{\text{Sector}_k}(j) \text{ weight of benchmark portfolio allocated to sector k,} \quad (14.3.2)$$

$$w_{M,j,t} = \frac{N_{M,j,t} P_{M,j,t}}{\sum_{j=1}^{n_M} N_{M,j,t} P_{M,j,t}} \text{ weight of managed portfolio allocated to individual instrument j,} \quad (14.3.3)$$

$$w_{B,j,t} = \frac{N_{B,j,t} P_{B,j,t}}{\sum_{j=1}^{n_B} N_{B,j,t} P_{B,j,t}} \text{ weight of benchmark portfolio allocated to individual instrument j,} \quad (14.3.4)$$

$$\tilde{R}_M = \sum_{j=1}^{n_M} w_{M,j} \tilde{R}_j = \sum_{k=1}^{K} SA\tilde{R}_{M,k} = \sum_{k=1}^{K} SAW_{M,k} \tilde{R}_{M,k} \text{ return on managed portfolio,} \quad (14.3.5)$$

$$SA\tilde{R}_{M,k} = SAW_{M,k} \tilde{R}_{M,k} \text{ managed, weight-adjusted, sector allocation return,} \quad (14.3.6)$$

$$\tilde{R}_B = \sum_{j=1}^{n_B} w_{B,j} \tilde{R}_j = \sum_{k=1}^{K} SA\tilde{R}_{B,k} = \sum_{k=1}^{K} SAW_{B,k} \tilde{R}_{B,k} \text{ return on benchmark portfolio, and} \quad (14.3.7)$$

$$SA\tilde{R}_{B,k} = SAW_{B,k} \tilde{R}_{B,k} \text{ benchmark, weight-adjusted, sector allocation return.} \quad (14.3.8)$$

Extensive use of uniquely constructed beta coefficients is applied here, including

$$\beta_{j,\Pi} = \frac{Cov\left(\tilde{R}_j, \tilde{R}_\Pi\right)}{\sigma_\Pi^2} \text{ beta of j}^{\text{th}} \text{ instrument with the portfolio,} \tag{14.3.9}$$

$$\beta_{M,ER,k} = \frac{\operatorname{cov}\left(\tilde{R}_{M,k}, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the k}^{\text{th}} \text{ sector of the managed portfolio with ER,} \tag{14.3.10}$$

$$\beta_{B,ER,k} = \frac{\operatorname{cov}\left(\tilde{R}_{B,k}, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the k}^{\text{th}} \text{ sector of the benchmark portfolio with ER,} \tag{14.3.11}$$

$$\beta_{SAD,ER} = \frac{Cov\left(S\tilde{A}D, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the SAD component with excess return,} \tag{14.3.12}$$

$$\beta_{SSD,ER} = \frac{Cov\left(S\tilde{S}D, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the SSD component with excess return, and} \tag{14.3.13}$$

$$\beta_{I,ER} = \frac{Cov\left(\tilde{I}, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the interaction component with excess return.} \tag{14.3.14}$$

**Key results**

*Key statistics*

$$\beta_{SAD,ER,k} = \left(SAW_{M,k} - SAW_{B,k}\right)\beta_{B,k,ER}, \tag{14.3.15}$$

$$\beta_{SSD,ER,k} = SAW_{B,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right), \text{ and} \tag{14.3.16}$$

$$\beta_{I,ER,k} = \left(SAW_{M,k} - SAW_{B,k}\right)\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right). \tag{14.3.17}$$

*Variance properties*

$$\sigma^2\left(\tilde{R}_M\right) = \sum_{j=1}^{n_M} w_{M,j} \operatorname{cov}\left(\tilde{R}_j, \tilde{R}_M\right) = \sum_{k=1}^{K} \operatorname{cov}\left(SA\tilde{R}_{M,k}, \tilde{R}_M\right) = \sum_{k=1}^{K} SAW_{M,k} \operatorname{cov}\left(\tilde{R}_{M,k}, \tilde{R}_M\right), \tag{14.3.18}$$

$$\sigma^2\left(\tilde{R}_B\right) = \sum_{j=1}^{n_B} w_{B,j} \operatorname{cov}\left(\tilde{R}_j, \tilde{R}_B\right) = \sum_{k=1}^{K} \operatorname{cov}\left(SA\tilde{R}_{B,k}, \tilde{R}_B\right) = \sum_{k=1}^{K} SAW_{B,k} \operatorname{cov}\left(\tilde{R}_{B,k}, \tilde{R}_B\right), \text{ and} \tag{14.3.19}$$

$$\begin{aligned} \sigma^2\left(ER\right) &= \operatorname{cov}\left(\tilde{R}_M - \tilde{R}_B, ER\right) = \sum_{j=1}^{n_M} w_{M,j} \operatorname{cov}\left(\tilde{R}_j, ER\right) - \sum_{j=1}^{n_B} w_{B,j} \operatorname{cov}\left(\tilde{R}_j, ER\right) \\ &= \operatorname{cov}\left(SAD + SSD + I, ER\right) = \operatorname{cov}\left(SAD, ER\right) + \operatorname{cov}\left(SSD, ER\right) + \operatorname{cov}\left(I, ER\right) \end{aligned}. \tag{14.3.20}$$

Betas are defined as

$$\beta_{j,\Pi} = \frac{Cov\left(\tilde{R}_j, \tilde{R}_\Pi\right)}{\sigma_\Pi^2} \text{ beta of j}^{\text{th}} \text{ instrument with the portfolio,} \tag{14.3.21}$$

$$\beta_{M,ER,k} = \frac{\operatorname{cov}\left(\tilde{R}_{M,k}, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the k}^{\text{th}} \text{ sector of the managed portfolio with ER, and} \tag{14.3.22}$$

$$\beta_{B,ER,k} = \frac{\operatorname{cov}\left(\tilde{R}_{B,k}, E\tilde{R}\right)}{\sigma_{ER}^2} \text{ beta of the k}^{\text{th}} \text{ sector of the benchmark portfolio with ER.} \quad (14.3.23)$$

Note

$$1 = \frac{\sum_{j=1}^{n} MCTR_j}{\sigma_\Pi^2} = \frac{\sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j, \tilde{R}_\Pi\right)}{\sigma_\Pi^2} = \sum_{j=1}^{n} w_j \beta_{j,\Pi} = \sum_{j=1}^{n} \% MCTR_j, \quad (14.3.24)$$

$$\% MCTR_j = w_j \beta_{j,\Pi} \text{ percentage marginal contribution to risk for instrument j,} \quad (14.3.25)$$

$$1 = \frac{Cov\left(\tilde{R}_M - \tilde{R}_B, E\tilde{R}\right)}{\sigma_{ER}^2} = \frac{Cov\left(\tilde{R}_M, E\tilde{R}\right)}{\sigma_{ER}^2} - \frac{Cov\left(\tilde{R}_B, E\tilde{R}\right)}{\sigma_{ER}^2} = \beta_{M,ER} - \beta_{B,ER}$$

$$1 = \frac{Cov\left(S\tilde{A}D + S\tilde{S}D + \tilde{I}, E\tilde{R}\right)}{\sigma_{ER}^2} = \frac{Cov\left(S\tilde{A}D, E\tilde{R}\right)}{\sigma_{ER}^2} + \frac{Cov\left(S\tilde{S}D, E\tilde{R}\right)}{\sigma_{ER}^2} + \frac{Cov\left(\tilde{I}, E\tilde{R}\right)}{\sigma_{ER}^2}, \text{ and} \quad (14.3.26)$$

$$= \beta_{SAD,ER} + \beta_{SSD,ER} + \beta_{I,ER}$$

$$\beta_{SAD,ER} + \beta_{SSD,ER} + \beta_{I,ER} = \beta_{M,ER} - \beta_{B,ER} = 1. \quad (14.3.27)$$

**Risk attribution analysis**

Again, the statistics are not additive; hence, subperiod results do not aggregate to entire measurement period. Typically, risk is not additive across time, however percentage marginal contribution to risk is additive across sectors or securities.

*Basic risk setup*

The total risk of the portfolio can be measured as the variance of the rate of return on the portfolio or

$$\sigma_\Pi^2 \equiv E\left\{\left[\tilde{R}_\Pi - E\left(\tilde{R}_\Pi\right)\right]^2\right\} = Cov\left(\tilde{R}_\Pi, \tilde{R}_\Pi\right) = Cov\left(\sum_{j=1}^{n} w_j \tilde{R}_j, \tilde{R}_\Pi\right). \quad (14.3.28)$$

Variance is not the only risk measure. Other candidates include downside risk, value-at-risk, conditional value-at-risk, beta, and so forth.

Recall covariance [ $Cov(\;)$ ] is the expectation of the product of the deviations from the mean of which variance is a special case. The last equality is a direct substitution from the definition of return. From the properties of covariance, we have:

$$\sigma_\Pi^2 = Cov\left(\sum_{j=1}^{n} w_j \tilde{R}_j, \tilde{R}_\Pi\right) = \sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j, \tilde{R}_\Pi\right). \quad (14.3.29)$$

Thus the marginal contribution to risk of any given security $j$ (denote $MCTR_j$) within a portfolio is:

$$MCTR_j = w_j Cov\left(\tilde{R}_j, \tilde{R}_\Pi\right). \quad (14.3.30)$$

Clearly the sum of $MCTR_j$ is equal to the portfolio variance. Also note that if the covariance is negative, the contribution to risk is negative (assuming a long position or $w_j > 0$). Dividing both sides by the portfolio variance we can compute the percentage marginal contribution to risk as

$$1=\frac{\sum_{j=1}^{n} MCTR_j}{\sigma_{\Pi}^2}=\frac{\sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j,\tilde{R}_{\Pi}\right)}{\sigma_{\Pi}^2}=\sum_{j=1}^{n} w_j \beta_{j,\Pi}=\sum_{j=1}^{n} \% MCTR_j\,. \tag{14.3.31}$$

Therefore,

$$\% MCTR_j = w_j \beta_{j,\Pi}, \tag{14.3.32}$$

where

$$\beta_{j,\Pi}=\frac{Cov\left(\tilde{R}_j,\tilde{R}_{\Pi}\right)}{\sigma_{\Pi}^2}. \tag{14.3.33}$$

Beta could be estimated historically as the slope of an ordinary least squares regression. The total risk as measured by variance can be decomposed into the risk related to each security.

*Application to Brinson attribution analysis*

Decomposition of tracking error risk (excess return risk): ( $\sigma_{ER}^2 > 0$ )

$$1=\frac{Cov\left(\tilde{R}_M-\tilde{R}_B,E\tilde{R}\right)}{\sigma_{ER}^2}=\frac{Cov\left(\tilde{R}_M,E\tilde{R}\right)}{\sigma_{ER}^2}-\frac{Cov\left(\tilde{R}_B,E\tilde{R}\right)}{\sigma_{ER}^2}=\beta_{M,ER}-\beta_{B,ER}. \tag{14.3.34}$$

Decomposition of sector allocation decision, security selection decision and interaction:

$$\begin{aligned}1&=\frac{Cov\left(S\tilde{A}D+S\tilde{S}D+\tilde{I},E\tilde{R}\right)}{\sigma_{ER}^2}=\frac{Cov\left(S\tilde{A}D,E\tilde{R}\right)}{\sigma_{ER}^2}+\frac{Cov\left(S\tilde{S}D,E\tilde{R}\right)}{\sigma_{ER}^2}+\frac{Cov\left(\tilde{I},E\tilde{R}\right)}{\sigma_{ER}^2},\\&=\beta_{SAD,ER}+\beta_{SSD,ER}+\beta_{I,ER}\end{aligned} \tag{14.3.35}$$

where

$$\beta_{SAD,ER}=\frac{Cov\left(S\tilde{A}D,E\tilde{R}\right)}{\sigma_{ER}^2}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\beta_{B,k,ER}\,, \tag{14.3.36}$$

$$\beta_{SSD,ER}=\frac{Cov\left(S\tilde{S}D,E\tilde{R}\right)}{\sigma_{ER}^2}=\sum_{k=1}^{K}SAW_{B,k}\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right), \tag{14.3.37}$$

$$\beta_{I,ER}=\frac{Cov\left(\tilde{I},E\tilde{R}\right)}{\sigma_{ER}^2}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right), \tag{14.3.38}$$

$$\beta_{M,k,ER}=\frac{\operatorname{cov}\left(\tilde{R}_{M,k},E\tilde{R}\right)}{\sigma_{ER}^2},\text{ and} \tag{14.3.39}$$

$$\beta_{B,k,ER}=\frac{\operatorname{cov}\left(\tilde{R}_{B,k},E\tilde{R}\right)}{\sigma_{ER}^2}. \tag{14.3.40}$$

Note

$$\beta_{SAD,ER}+\beta_{SSD,ER}+\beta_{I,ER}=\beta_{M,ER}-\beta_{B,ER}=1. \tag{14.3.41}$$

*Potential reported risk statistics (note time subscript suppressed for clarity)*

Risk of the *managed* portfolio in total and by sectors:

$$\sigma^2\left(\tilde{R}_M\right)=\sum_{j=1}^{n_M} w_{M,j}\operatorname{cov}\left(\tilde{R}_j,\tilde{R}_M\right)=\sum_{k=1}^{K}\operatorname{cov}\left(SA\tilde{R}_{M,k},\tilde{R}_M\right)=\sum_{k=1}^{K} SAW_{M,k}\operatorname{cov}\left(\tilde{R}_{M,k},\tilde{R}_M\right). \quad (14.3.42)$$

Percentage risk of the *managed* portfolio in total and by sectors:

$$1=\sum_{j=1}^{n_M} w_{M,j}\beta_{j,M}=\sum_{k=1}^{K}\beta_{SAR_{M,k},M}=\sum_{k=1}^{K} SAW_{M,k}\beta_{R_{M,k},M}, \quad (14.3.43)$$

$$\beta_{SAR_{M,k},M}=\frac{\operatorname{cov}\left(SA\tilde{R}_{M,k},\tilde{R}_M\right)}{\sigma^2\left(\tilde{R}_M\right)} \text{ weight-adjusted, sector allocation percentage risk,} \quad (14.3.44)$$

$$SAW_{M,k}=\sum_{j=1}^{n_M} w_{M,j} I_{\text{Sec}tor_k}\left(j\right) \text{ sector allocation weight, and} \quad (14.3.45)$$

$$\beta_{R_{M,k},M}=\frac{\operatorname{cov}\left(\tilde{R}_{M,k},\tilde{R}_M\right)}{\sigma^2\left(\tilde{R}_M\right)} \text{ percentage risk for sector k.} \quad (14.3.46)$$

Risk of the *benchmark* portfolio in total and by sectors:

$$\sigma^2\left(\tilde{R}_B\right)=\sum_{j=1}^{n_B} w_{B,j}\operatorname{cov}\left(\tilde{R}_j,\tilde{R}_B\right)=\sum_{k=1}^{K}\operatorname{cov}\left(SA\tilde{R}_{B,k},\tilde{R}_B\right)=\sum_{k=1}^{K} SAW_{B,k}\operatorname{cov}\left(\tilde{R}_{B,k},\tilde{R}_B\right). \quad (14.3.47)$$

Percentage risk of the *benchmark* portfolio in total and by sectors:

$$1=\sum_{j=1}^{n_B} w_{B,j}\beta_{j,B}=\sum_{k=1}^{K}\beta_{SAR_{B,k},B}=\sum_{k=1}^{K} SAW_{B,k}\beta_{R_{B,k},B},$$

$$\beta_{SAR_{B,k},B}=\frac{\operatorname{cov}\left(SA\tilde{R}_{B,k},\tilde{R}_B\right)}{\sigma^2\left(\tilde{R}_B\right)} \text{ weight-adjusted, sector allocation percentage risk,} \quad (14.3.48)$$

$$SAW_{B,k}=\sum_{j=1}^{n_B} w_{B,j} I_{\text{Sec}tor_k}\left(j\right) \text{ sector allocation weight, and} \quad (14.3.49)$$

$$\beta_{SAR_{B,k},B}=\frac{\operatorname{cov}\left(\tilde{R}_{B,k},\tilde{R}_B\right)}{\sigma^2\left(\tilde{R}_B\right)} \text{ percentage risk for sector k.} \quad (14.3.50)$$

Risk of the excess return in total and by sectors:

$$\begin{aligned}\sigma^2\left(ER\right)&=\operatorname{cov}\left(\tilde{R}_M-\tilde{R}_B,ER\right)=\sum_{j=1}^{n_M} w_{M,j}\operatorname{cov}\left(\tilde{R}_j,ER\right)-\sum_{j=1}^{n_B} w_{B,j}\operatorname{cov}\left(\tilde{R}_j,ER\right).\\&=\operatorname{cov}\left(SAD+SSD+I,ER\right)=\operatorname{cov}\left(SAD,ER\right)+\operatorname{cov}\left(SSD,ER\right)+\operatorname{cov}\left(I,ER\right)\end{aligned} \quad (14.3.51)$$

Percentage risk of the excess return by portfolio and by decisions:

$$\begin{aligned}1&=\beta_{M,ER}-\beta_{B,ER}=\sum_{j=1}^{n_M} w_{M,j}\beta_{j,ER}-\sum_{j=1}^{n_B} w_{B,j}\beta_{j,ER}=\sum_{j=1}^{n_M\cup n_B}\left(w_{M,j}-w_{B,j}\right)\beta_{j,ER}.\\&=\beta_{SAD,ER}+\beta_{SSD,ER}+\beta_{I,ER}\end{aligned} \quad (14.3.52)$$

Sector allocation decision percentage risk in total and by sector:

$$\beta_{SAD,ER}=\sum_{k=1}^{K}\beta_{SAD,k,ER}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\beta_{B,k,ER} \text{ total SAD percentage risk,} \quad (14.3.53)$$

$$\beta_{SAD,k,ER} = \left(SAW_{M,k} - SAW_{B,k}\right)\beta_{B,k,ER} \text{ SAD attributable to sector k percentage risk,} \quad (14.3.54)$$

$$SAW_{M,k} - SAW_{B,k} \text{ excess weight allocated to sector k.} \quad (14.3.55)$$

Security selection decision percentage risk in total and by sector:

$$\beta_{SSD,ER} = \sum_{k=1}^{K}\beta_{SSD,k,ER} = \sum_{k=1}^{K} SAW_{B,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right) \text{ total SSD percentage risk,} \quad (14.3.56)$$

$$\beta_{SSD,k,ER} = SAW_{B,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right) \text{ SSD percentage risk attributable to sector k, and} \quad (14.3.57)$$

$$\beta_{M,k,ER} - \beta_{B,k,ER} \text{ excess percentage risk allocated to sector k.} \quad (14.3.58)$$

Interaction percentage risk (IPR) in total and by sector:

$$\beta_{I,ER} = \sum_{k=1}^{K}\beta_{I,k,ER} = \sum_{k=1}^{K}\left(SAW_{M,k} - SAW_{B,k}\right)\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right) \text{ total IPR,} \quad (14.3.59)$$

$$\beta_{I,k,ER} = \left(SAW_{M,k} - SAW_{B,k}\right)\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right) \text{ IPR attributable to sector k,} \quad (14.3.60)$$

$$\beta_{SAD,k,ER} = \left(SAW_{M,k} - SAW_{B,k}\right)\beta_{B,k,ER} \text{ SAD attributable to sector k percentage risk,} \quad (14.3.61)$$

$$\beta_{SSD,k,ER} = SAW_{B,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right) \text{ SSD percentage risk attributable to sector k,} \quad (14.3.62)$$

$$SAW_{M,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right) \text{ managed portfolio weight-adjusted excess \% risk attributable to sector k, and} \quad (14.3.63)$$

$$\left(SAW_{M,k} - SAW_{B,k}\right)\beta_{M,k,ER} \text{ managed percentage risk adjusted by excess weight allocated to sector k.} \quad (14.3.64)$$

# Summary

Risk attribution decomposed the total variance of excess return into a variety of statistics including percentage marginal contribution to risk by stock, sector, sector allocation decision, security selection decision, and interaction. It was demonstrated that the risk attribution statistics are not additive; hence, subperiod results do not aggregate to entire measurement period.

# References

See module 14.4.