Module 14.2: Return Attribution

Learning objectives

- Define and illustrate return attribution
- Analysis permits ex post return attribution and ex ante return expectation
- Return attribution decomposes the total excess return into a variety of statistics including excess return by stock, sector, sector allocation decision, security selection decision, and interaction
- Return attribution techniques are typically flexible enough to incorporate manager-specific decisions, including various screens and optimizers

Note that R code has not been developed for this module.

Executive summary

Return attribution is typically focused on decomposing the alpha (defined here as simply the deviation of the managed portfolios return from some prespecified benchmark portfolio return). In Module 14.3 and 14.4, we extend performance attribution to risk attribution (14.3) and risk-adjusted return attribution (14.4). We use the term performance attribution as inclusive of return attribution, risk attribution, risk-adjusted return attribution, and any other analysis within this analytical genre.

Return attribution seeks to allocation a portion of the measured alpha into three categories, sector allocation decision (SAD), security selection decision (SSD), and an interaction term (I). Return attribution can be extended to explore a variety of portfolio management decisions. For example, a manager may apply two separate screens, such as momentum and PE ratio, and desire a return attribution analysis of each of these screens as well as the interaction term.

Although not covered in this module, performance attribution can also be applied to enterprise value, such as a corporation. For example, a company where both assets and liabilities are sensitive to interest rates and credit risk, such as a bank, performance attribution can be applied to enterprise value changes based on say enterprise duration and credit exposures compared to some benchmark.

Central finance concepts

Return attribution is best explained with a simple example. Consider a managed fund that over-weights stocks and under-weights bonds as illustrated in Figure 14.2.1.

Figure 14.2.1. Inputs for return attribution

	A	B	C	D	E
1	**Performance Attribution**				
2	**Asset Allocation**				
3		**Benchmark**		**Managed**	
4		**Weights**	**Return**	**Weights**	**Return**
5	**Stocks**	60.00%	6.00%	70.00%	7.00%
6	**Bonds**	40.00%	3.00%	25.00%	2.50%
7	**Cash**	0.00%	1.00%	5.00%	1.20%
8	**Total Weights**	100.00%		100.00%	
9	**Return**		4.8000%		5.5850%
10	**Excess Return**				0.7850%

The portfolio manager also has a cash drag of 5%. Return attribution seeks to decompose the 78.5 basis points alpha into the managers sector allocation decision (SAD) and security selection decision (SSD). Although the technical details are provided in the quantitative finance materials section below, Figure 14.2.2 illustrates the results. From these results, we see that 20 basis points is attributable to the sector allocation decision, 40 basis points is attributable to the security selection decision, and 18.5 basis points is attributable to the interaction term (I).

Figure 14.2.2. Return attribution analysis

	A	B	C	D	E	F	G	H
1	Performance Attribution							
2	Asset Allocation							
3		Benchmark		Managed				
4		Weights	Return	Weights	Return	SAD	SSD	Interaction
5	Stocks	60.00%	6.00%	70.00%	7.00%	0.6000%	0.6000%	0.1000%
6	Bonds	40.00%	3.00%	25.00%	2.50%	-0.4500%	-0.2000%	0.0750%
7	Cash	0.00%	1.00%	5.00%	1.20%	0.0500%	0.0000%	0.0100%
8	Total Weights	100.00%		100.00%				
9	Return		4.8000%		5.5850%	0.2000%	0.4000%	0.1850%
10	Excess Return				0.7850%			0.7850%

With a single period model, the interaction term captures the remaining alpha. The interaction term accounts for the gain or loss from the combined effect. We explore the mathematical details below.

Note that multiperiod analysis is much more complicated and we assume no portfolio turnover during the single period. Alternatively, some approximations must be made, and these approximations inevitably result in an error term. We incorporate a time series error term to fully account for these measurement frictions.

Quantitative finance materials

We first identify the key results from the quantitative analysis. Next, we provide the mathematical justification for these key results. Finally, we rehash the various statistics that could be produced with R code.

Key results

We review the main results of return attribution in this section. First, we identify one raw attribution measure.

Raw attribution

Key statistics: (M – Manager, B – Benchmark, k – sector, t – period, SAW – sector allocation weight)

Excess sector allocation weights (EW) by subperiod and individual sector

$$EW_{k,t} = \left(SAW_{M,k,t} - SAW_{B,k,t}\right); t, k. \qquad (14.2.1)$$

Key relationships (I – indicator function, N – number of shares, P – price per share)

$$\sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right) = 0 \text{ sum of weights equal to 1, total difference is 0,} \qquad (14.2.2)$$

$$SAW_{M,k,t} = \sum_{j=1}^{n_M} w_{M,j,t} I_{\text{Sector}_k}(j) \text{ weight of managed portfolio allocated to sector k,} \qquad (14.2.3)$$

$$SAW_{B,k,t} = \sum_{j=1}^{n_B} w_{B,j,t} I_{\text{Sector}_k}(j) \text{ weight of benchmark portfolio allocated to sector k,} \qquad (14.2.4)$$

$$w_{M,j,t} = \frac{N_{M,j,t} P_{M,j,t}}{\sum_{j=1}^{n_M} N_{M,j,t} P_{M,j,t}} \text{ weight of managed portfolio allocated to instrument j, and} \qquad (14.2.5)$$

$$w_{B,j,t} = \frac{N_{B,j,t} P_{B,j,t}}{\sum_{j=1}^{n_B} N_{B,j,t} P_{B,j,t}} \text{ weight of benchmark portfolio allocated to individual instrument j.} (14.2.6)$$

Return attribution
Key statistics: (R – rate of return)
Excess sector returns (ER) by subperiod and individual sector is

$$ER_{k,t} = \left(R_{M,k,t} - R_{B,k,t}\right); t, k. \tag{14.2.7}$$

Sector allocation decision (SAD) by subperiod and individual sector is

$$SAD_{k,t} = \left(SAW_{M,k,t} - SAW_{B,k,t}\right)\tilde{R}_{B,k,t}. \tag{14.2.8}$$

Security selection decision (SSD) by subperiod and individual sector is

$$SSD_{k,t} = SAW_{B,k,t}\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right). \tag{14.2.9}$$

Interaction (I) term by subperiod and individual sector is

$$I_{k,t} = \left(SAW_{M,k,t} - SAW_{B,k,t}\right)\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right). \tag{14.2.10}$$

Discrete return properties

$$\tilde{R}_{j,t} = \frac{\tilde{P}_{j,t+1}}{P_{j,t}} - 1 \text{ discretely compounded rate of return over time t for instrument j,} \tag{14.2.11}$$

$$\tilde{R}_{M,t} = \sum_{j=1}^{n_M} w_{M,j,t}\tilde{R}_{j,t} \text{ rate of return over time t for the managed portfolio, and} \tag{14.2.12}$$

$$\tilde{R}_{B,t} = \sum_{j=1}^{n_B} w_{B,j,t}\tilde{R}_{j,t} \text{ rate of return over time t for the benchmark portfolio.} \tag{14.2.13}$$

Excess return is only additive across time (t) when forced with a time series adjustment term. The decomposition terms (sector allocation decision SAD, security selection decision SSD, and interaction I) are additive across sectors (k) at a point in time:

$$ER = SAD + SSD + I + I_{TS} = \sum_{t=1}^{T} ER_t = \sum_{t=1}^{T} SAD_t + \sum_{t=1}^{T} SSD_t + \sum_{t=1}^{T} I_t + I_{TS}, \tag{14.2.14}$$

where

$$SAD_t = \sum_{k=1}^{K} SAD_{k,t}, \tag{14.2.15}$$

$$SSD_t = \sum_{k=1}^{K} SSD_{k,t}, \text{ and} \tag{14.2.16}$$

$$I_t = \sum_{k=1}^{K} I_{k,t}. \tag{14.2.17}$$

We define the time series adjustment term as simply a plug figure to account for the non-additivity of the decomposition process or

$$I_{TS} = ER - SAD - SSD - I = \sum_{t=1}^{T} ER_t - \sum_{t=1}^{T} SAD_t - \sum_{t=1}^{T} SSD_t - \sum_{t=1}^{T} I_t. \tag{14.2.18}$$

Hence for each measurement period (e.g., a quarter), reported statistics include excess return (by subperiods and by sectors) and allocation decisions (also by subperiods and by sectors).

$$\begin{aligned}ER_t &= SAD_t + SSD_t + I_t = \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\tilde{R}_{B,k,t} \\ &+\sum_{k=1}^{K} SAW_{B,k,t}\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right) + \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right). \\ &= \sum_{k=1}^{K} SAW_{M,k,t}\tilde{R}_{M,k,t} - \sum_{k=1}^{K} SAW_{B,k,t}\tilde{R}_{B,k,t}\end{aligned} \tag{14.2.19}$$

Return attribution analysis

For the managed securities portfolio, the value of the managed portfolio at time t ($\Pi_{M,t}$) can be represented as

$$\Pi_{M,t} = \sum_{j=1}^{n_M} N_{M,j,t} P_{j,t}, \tag{14.2.20}$$

where $N_{M,j,t}$ is the number of shares of security j held in the managed portfolio at time t (for example, shares of stock) and $P_{j,t}$ is the value of one share of security j held in the managed portfolio observed at time t (for example, stock price per share). Let n_M denote the total number of securities owned in the managed portfolio at point in time t.

Managed security portfolios are evaluated against a benchmark portfolio that is also known as a bogey. Following the notation above with B denoting benchmark or bogey,

$$\Pi_{B,t} = \sum_{j=1}^{n_B} N_{B,j,t} P_{j,t}, \tag{14.2.21}$$

where $N_{B,j,t}$ is the number of shares of security j represented in the benchmark portfolio at time t and $P_{j,t}$ is the value of one share of security j held in the benchmark portfolio observed at time t. Let n_B denote the total number of securities owned in the benchmark portfolio at point in time t.

The percentage rate of return over period t (length of time unspecified, formally from time t to time $t+1$) of the managed portfolio ($\tilde{R}_{M,t}$) and the benchmark portfolio ($\tilde{R}_{B,t}$) can be expressed as:

$$\tilde{R}_{M,t} = \sum_{j=1}^{n_M} w_{M,j,t}\tilde{R}_{j,t} \text{ and} \tag{14.2.22}$$

$$\tilde{R}_{B,t} = \sum_{j=1}^{n_B} w_{B,j,t}\tilde{R}_{j,t}, \tag{14.2.23}$$

where $w_{M,j,t}$ ($w_{B,j,t}$) denotes the proportion of managed portfolio Π_M (Π_B) invested in security j at point in time t and $\tilde{R}_{j,t}$ denotes the rate of return on security j over period t. Formally,

$$w_{M,j,t} = \frac{N_{M,j,t}P_{M,j,t}}{\sum_{j=1}^{n_M} N_{M,j,t}P_{M,j,t}}, \tag{14.2.24}$$

$$w_{B,j,t} = \frac{N_{B,j,t}P_{B,j,t}}{\sum_{j=1}^{n_B} N_{B,j,t}P_{B,j,t}}, \text{ and} \tag{14.2.25}$$

$$\tilde{R}_{j,t} = \frac{\tilde{P}_{j,t+1}}{P_{j,t}} - 1. \tag{14.2.26}$$

Define the excess return as

$$ER_t = \tilde{R}_{M,t} - \tilde{R}_{B,t} = \sum_{j=1}^{n_M} w_{M,j,t}\tilde{R}_{j,t} - \sum_{j=1}^{n_B} w_{B,j,t}\tilde{R}_{j,t}. \tag{14.2.27}$$

Note that the excess return can be decomposed into the sector allocation decision, the security selection decision, and an interaction term

$$\begin{aligned} ER_t = SAD_t + SSD_t + I_t &= \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\tilde{R}_{B,k,t} \\ &+ \sum_{k=1}^{K} SAW_{B,k,t}\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right) + \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right), \\ &= \sum_{k=1}^{K} SAW_{M,k,t}\tilde{R}_{M,k,t} - SAW_{B,k,t}\tilde{R}_{B,k,t} \end{aligned} \tag{14.2.28}$$

where

$$SAW_{M,k,t} = \sum_{j=1}^{n_M} w_{M,j,t} I_{\text{Sector}_k}\left(j\right) \text{ and} \tag{14.2.29}$$

$$SAW_{B,k,t} = \sum_{j=1}^{n_B} w_{B,j,t} I_{\text{Sector}_k}\left(j\right), \tag{14.2.30}$$

where $I_{\text{Sector}_k}\left(j\right)$ denotes an indicator function that equals 1.0 when stock j falls within sector k and zero otherwise. Note, by definition,

$$1.0 = \sum_{k=1}^{K} SAW_{M,k,t} \text{ and} \tag{14.2.31}$$

$$1.0 = \sum_{k=1}^{K} SAW_{B,k,t}, \tag{14.2.32}$$

where K denotes the number of sectors.

The weighted average return for each sector for the managed portfolio and the benchmark portfolio:

$$\tilde{R}_{M,k,t} = \sum_{j=1}^{n_M}\left(\frac{w_{M,j,t}}{SAW_{M,k,t}}\right)\tilde{R}_{j,t} I_{\text{Sector}_k}\left(j\right) \text{ and} \tag{14.2.33}$$

$$\tilde{R}_{B,k,t} = \sum_{j=1}^{n_B}\left(\frac{w_{B,j,t}}{SAW_{B,k,t}}\right)\tilde{R}_{j,t} I_{\text{Sector}_k}\left(j\right). \tag{14.2.34}$$

The excess return allocated to the sector allocation decision can be estimated as

$$SAD = \sum_{k=1}^{K} \left(SAW_{M,k,t} - SAW_{B,k,t} \right) \tilde{R}_{B,k,t} . \tag{14.2.35}$$

The excess return allocated to the security selection decision can be estimated as

$$SSD = \sum_{k=1}^{K} SAW_{B,k,t} \left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t} \right). \tag{14.2.36}$$

The excess return allocated to the interaction term can be estimated as

$$I = \sum_{k=1}^{K} \left(SAW_{M,k,t} - SAW_{B,k,t} \right) \left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t} \right). \tag{14.2.37}$$

Measures the portion of the excess return attributed to both the sector allocation decision and the security selection decision.

Potential reported return statistics (note time subscript suppressed for clarity)
All the statistics below can be reported by entire measurement period, subperiods, or ex ante based on given theoretical model.

Return on the *managed* portfolio in total and by sectors:

$$\tilde{R}_M = \sum_{j=1}^{n_M} w_{M,j} \tilde{R}_j = \sum_{k=1}^{K} SA\tilde{R}_{M,k} = \sum_{k=1}^{K} SAW_{M,k} \tilde{R}_{M,k} , \tag{14.2.38}$$

$$SA\tilde{R}_{M,k} = SAW_{M,k} \tilde{R}_{M,k} \text{ weight-adjusted, sector allocation return,} \tag{14.2.39}$$

$$SAW_{M,k} = \sum_{j=1}^{n_M} w_{M,j} I_{\mathrm{Sec}tor_k} \left(j \right) \text{ sector allocation weight, and} \tag{14.2.40}$$

$$\tilde{R}_{M,k} = \sum_{j=1}^{n_M} \left(\frac{w_{M,j}}{SAW_{M,k}} \right) \tilde{R}_j I_{\mathrm{Sec}tor_k} \left(j \right) \text{ return to sector k.} \tag{14.2.41}$$

Return on the *benchmark* portfolio in total and by sectors:

$$\tilde{R}_B = \sum_{j=1}^{n_B} w_{B,j} \tilde{R}_j = \sum_{k=1}^{K} SA\tilde{R}_{B,k} = \sum_{k=1}^{K} SAW_{B,k} \tilde{R}_{B,k} , \tag{14.2.42}$$

$$SA\tilde{R}_{B,k} = SAW_{B,k} \tilde{R}_{B,k} \text{ weight-adjusted, sector allocation return,} \tag{14.2.43}$$

$$SAW_{B,k} = \sum_{j=1}^{n_B} w_{B,j} I_{\mathrm{Sec}tor_k} \left(j \right) \text{ sector allocation weight, and} \tag{14.2.44}$$

$$\tilde{R}_{B,k} = \sum_{j=1}^{n_B} \left(\frac{w_{B,j}}{SAW_{B,k}} \right) \tilde{R}_j I_{\mathrm{Sec}tor_k} \left(j \right) \text{ return to sector k.} \tag{14.2.45}$$

Excess return in total and by decisions:

$$ER = \tilde{R}_M - \tilde{R}_B = \sum_{j=1}^{n_M} w_{M,j} \tilde{R}_j - \sum_{j=1}^{n_B} w_{B,j} \tilde{R}_j = SAD + SSD + I \text{ excess return.} \tag{14.2.46}$$

Sector allocation decision in total and by sector:

$$SAD = \sum_{k=1}^{K} SAD_k = \sum_{k=1}^{K} \left(SAW_{M,k} - SAW_{B,k} \right) \tilde{R}_{B,k} \text{ total sector allocation decision,} \quad (14.2.47)$$

$$SAD_k = \left(SAW_{M,k} - SAW_{B,k} \right) \tilde{R}_{B,k} \text{ sector allocation decision attributable to sector k, and} \quad (14.2.48)$$

$$SAW_{M,k} - SAW_{B,k} \text{ excess weight allocated to sector k.} \quad (14.2.49)$$

Security selection decision in total and by sector:

$$SSD = \sum_{k=1}^{K} SSD_k = \sum_{k=1}^{K} SAW_{B,k} \left(\tilde{R}_{M,k} - \tilde{R}_{B,k} \right) \text{ total security selection decision,} \quad (14.2.50)$$

$$SSD_k = SAW_{B,k} \left(\tilde{R}_{M,k} - \tilde{R}_{B,k} \right) \text{ security selection decision attributable to sector k, and} \quad (14.2.51)$$

$$\tilde{R}_{M,k} - \tilde{R}_{B,k} \text{ excess return allocated to sector k.} \quad (14.2.52)$$

Interaction in total and by sector:

$$I = \sum_{k=1}^{K} I_k = \sum_{k=1}^{K} \left(SAW_{M,k} - SAW_{B,k} \right) \left(\tilde{R}_{M,k} - \tilde{R}_{B,k} \right) \text{ total interaction,} \quad (14.2.53)$$

$$I_k = \left(SAW_{M,k} - SAW_{B,k} \right) \left(\tilde{R}_{M,k} - \tilde{R}_{B,k} \right) \text{ interaction attributable to sector k,} \quad (14.2.54)$$

$$SAD_k = \left(SAW_{M,k} - SAW_{B,k} \right) \tilde{R}_{B,k} \text{ sector allocation decision attributable to sector k,} \quad (14.2.55)$$

$$SSD_k = SAW_{B,k} \left(\tilde{R}_{M,k} - \tilde{R}_{B,k} \right) \text{ security selection decision attributable to sector k,} \quad (14.2.56)$$

$$SAW_{M,k} \left(\tilde{R}_{M,k} - \tilde{R}_{B,k} \right) \text{ managed portfolio weight-adjusted excess return attributable to sector k, and} \quad (14.2.57)$$

$$\left(SAW_{M,k} - SAW_{B,k} \right) \tilde{R}_{M,k} \text{ managed return adjusted by excess weight allocated to sector k.} \quad (14.2.58)$$

Subperiod aggregation:

$$ER_t = \tilde{R}_{M,t} - \tilde{R}_{B,t} = \sum_{j=1}^{n_M} w_{M,j,t} \tilde{R}_{j,t} - \sum_{j=1}^{n_B} w_{B,j,t} \tilde{R}_{j,t} \, . \quad (14.2.59)$$

Thus, based on return attribution analysis, there is a vast trove of results that may aid in improving portfolio management processes in a variety of ways.

Summary

Return attribution here is focused on decomposing the alpha (defined here as simply the deviation of the managed portfolios return from some prespecified benchmark portfolio return). Return attribution allocates a portion of the measured alpha into three categories, sector allocation decision (SAD), security selection decision (SSD), and an interaction term (I). We explored return attribution in both a single period context as well as a multiperiod context.

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