Module 12.1: DRM GBM-Based Binomial Models

Learning objectives

- Apply Monte Carlo simulation to explore interactions between various inputs to the geometric Brownian motion binomial option valuation model
- Illustrate the insights gained from Monte Carlo simulation with a focus on correlation between the underlying stock price and volatility as well as interest rates and volatility

Executive summary

Based on the material presented in Module 5.2 and Module 8.1, we illustrate applying Monte Carlo simulation to analyzing the value-at-risk within the GBM binomial option valuation model (GBM BOVM) for both European-style and American-style options.

Central finance concepts

There are numerous ways to illustrate dynamic risk measures with options. We chose to focus on interactions among input parameters of a single instrument.

The main idea is once we have a robust valuation model (Module 5.2) as well as an understanding of static risk measures (Module 8.1), we are now able to explore various dynamic risk measures. For a review of the valuation models used here see Module 5.2. We focus here on dividend yields and plain vanilla options.

GBM-based European-style binomial option valuation models

Recall the GBM-based binomial option framework is designed to converge to a lognormal distribution in the limit to be consistent with the GBMOVM. This binomial framework has several objectives:

1. Multiplicative
2. Recombining
3. Incorporate dividends
4. Address early exercise with American-style options

Multiplicative and recombining are incorporated using *u* and *d* parameters at each node.

There are several GBM-based multiperiod valuation models including when there are no dividends, when a dividend yield is assumed, and when discrete dividends are assumed. Further, there are several alternative ways to frame these models such as based on digital valuation models.

GBM-based American-style binomial option valuation models

For American-style options, the early exercise potential must be incorporated. As discussed below, the approach typically taken is known as backward induction. At each node, we must compare the following values, the model option value, the early exercise value, and the lower boundary condition. The existence of various forms of dividends simply changes the required formulas.

With Monte Carlo simulations, the processing speed becomes a bit of a challenge. Fortunately, with modern computing power, it is easy to implement.

Binomial option valuation models and value-at-risk

In the quantitative materials below, we explore in detail VaR metrics related to the following 19 option-related strategies:

- Long stock (LS)
- Long call (LC, in-, at-, and out-of-the-money)
- Long put (LP, in-, at-, and out-of-the-money)
- Covered call writing (CCW, in-, at-, and out-of-the-money)
- Protective put buying (PPB, in-, at-, and out-of-the-money)
- Leveraged calls (LC, in-, at-, and out-of-the-money)
- Leveraged puts (LP, in-, at-, and out-of-the-money)

Covered call writing comprises long stock and short calls. Protective put buying comprises long stock and long put. Leveraged calls comprises long stock and long calls. Leveraged puts comprises long stock and short puts.

To illustrate this analysis, we assume the following inputs:

- Stock price = \$100
- Strike price = \$90, \$100, and \$110
- Interest rate = 5%
- Dividend yield = 0%
- Volatility = 30%
- Time to maturity = 1 year
- Number of steps = 250
- Style = European
- Payout type = Plain vanilla
- EMM probability = 50%

For illustration, we assume the stock price, interest rate, and volatility are subsequently random. Note that the option valuation framework assumes volatility and interest rates are constant. Dynamic risk management often requires a balance between theoretical models and practical implementation. Thus, we assume options are valued based on geometric Brownian motion and the binomial framework while simultaneously assuming the desired quantitative analysis is based on professional judgment within the firm.

We assume the following parameterizations:

- Horizon = 1 month
- Confidence level = 90%
- Number of simulations = 2,000
- Means (annualized, continuously compounded, percentage change)
  - Stock = 5%
  - Rate = 0%
  - Volatility = 0%
- Standard deviations
  - Stock = 30%
  - Rate = 10%
  - Volatility = 40%
- Correlations
  - Stock, Rate = –0.3
  - Rate, Volatility = 0.0
  - Stock, Volatility = –0.5

In the tables presented below, XL denotes the low strike price (\$90), X denotes the mid strike price (\$100), and XH denotes the high strike price (\$110). Thus, LCXH denotes the long call with a high strike price. Note that these various strategies require different levels of dollar investment; hence, for ease of analysis we report only return VaR (distance from \$0) as opposed to dollar VaR.

Table 12.1.1 presents the results of the simulation based on the initial parameterization given above and allowing the correlation between stock returns and stock volatility to range from –0.75 to +0.75 incrementing by 0.25. Panel A presents European-style (ES) and Panel B presents American-style (AS).

Table 12.1.1 Return VaR Based on GBM BOVM Stock Return and Volatility Correlation

Panel A: European-style

Strategy\Correlation	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75
LS	2.54	2.64	2.74	2.67	2.82	2.58	2.86
LCXL	11.49	11.99	13.19	13.55	14.31	13.71	15.53
LCX	13.70	14.54	16.15	16.66	17.56	17.32	19.60
LCXH	16.08	17.44	19.58	20.48	21.64	21.21	24.01
LPXL	26.78	25.66	23.72	22.73	21.26	19.29	16.86
LPX	20.86	20.14	18.45	17.92	16.79	15.58	13.73
LPXH	16.41	16.05	14.77	14.31	13.48	12.56	11.31
LCCWXL	0.48	0.44	0.34	0.19	0.19	0.03	-0.12
LCCWX	0.84	0.79	0.70	0.51	0.53	0.31	0.20
LCCWXH	1.20	1.12	1.05	0.83	0.90	0.63	0.59
LPPBXL	1.81	1.91	2.13	2.22	2.34	2.22	2.57
LPPBX	1.43	1.54	1.75	1.82	1.94	1.90	2.20
LPPBXH	1.05	1.16	1.35	1.41	1.52	1.49	1.73
LLCXL	4.00	4.16	4.46	4.43	4.65	4.40	4.91
LLCX	3.91	4.07	4.39	4.39	4.63	4.36	4.91
LLCXH	3.75	3.92	4.25	4.23	4.50	4.23	4.75
LLPXL	3.42	3.47	3.46	3.15	3.38	3.04	3.25
LLPX	3.99	4.04	4.02	3.65	3.90	3.51	3.74
LLPXH	4.68	4.75	4.71	4.33	4.62	4.17	4.45

Panel B American-style

Strategy\Correlation	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75
LS	2.83	2.84	2.79	2.69	2.68	2.88	2.71
LCXL	12.26	12.60	13.17	13.24	13.61	15.29	14.81
LCX	14.46	14.97	16.33	16.48	17.17	19.12	18.67
LCXH	16.83	17.70	19.58	20.04	20.90	23.46	23.15
LPXL	27.23	26.24	24.54	23.87	20.91	19.12	17.83
LPX	21.42	20.48	19.32	19.07	16.78	15.48	14.80
LPXH	16.90	16.20	15.37	15.21	13.71	12.90	12.56
LCCWXL	0.62	0.50	0.39	0.23	0.15	0.06	-0.16
LCCWX	1.01	0.89	0.77	0.55	0.49	0.40	0.13
LCCWXH	1.40	1.26	1.15	0.92	0.86	0.79	0.50
LPPBXL	1.94	2.00	2.12	2.13	2.23	2.51	2.44
LPPBX	1.50	1.56	1.73	1.76	1.85	2.07	2.04
LPPBXH	1.05	1.13	1.27	1.31	1.37	1.58	1.56
LLCXL	4.40	4.46	4.47	4.33	4.47	4.93	4.68
LLCX	4.26	4.37	4.39	4.29	4.41	4.88	4.67
LLCXH	4.09	4.19	4.24	4.16	4.27	4.74	4.55
LLPXL	3.90	3.81	3.66	3.40	3.24	3.41	3.07
LLPX	4.58	4.47	4.29	3.98	3.80	3.99	3.59
LLPXH	5.42	5.28	5.07	4.76	4.53	4.80	4.39

There are several insights that can be drawn from the table. First, the Long Stock (LS) row illustrates that Monte Carlo simulation with 2,000 simulation results in variation of return value-at-risk (RVaR) at the 90% confidence level. RVaR ranges from 2.54% (ES, $\rho = -0.75$) to 2.88% (AS, $\rho = 0.50$). As the number of simulations increase, distribution parameters tend to stabilize, but the tails of the distribution are much slower to converge. We selected 90% confidence level as it converges faster than 95% or 99%.

Second, focusing on the uncorrelated ES case ($\rho = 0.0$), RVaR increases with the strike price for Long Call (LC) ranging from 13.5% for the low strike price (XL) to 20.48% for the high strike price (XH). Recall the higher the strike price, the higher the implied leverage and hence, the higher the RVaR. We see the opposite pattern with puts. RVaR decreases with the strike price for Long Put (LP) ranging from 22.73% for the low strike price (XL) to 14.31% for the high strike price (XH). With puts, the higher the strike price, the lower the implied leverage (further in-the-money). Note that the patterns are similar for AS options but higher in magnitude for puts due to the additional early exercise premium.

Third, the remaining option blended strategies have dramatically lower RVaRs when compared to long calls and puts. The primary reason is the dramatically higher investment required for the underlying long stock position that is unleveraged.

Fourth, the same patterns noted above hold for covered call writing and protective put buying. In both cases, the further out-of-the-money, the less risk mitigation and hence the higher RVaR. As expected, the opposite pattern holds for leveraged calls and puts.

Fifth, the correlation between stock returns and volatility does influence RVaR although it has no direct theoretical impact on the underlying instrument's (stock, calls, and puts) value. For long calls, the RVaR increases with correlation and for long puts, the RVaR decreases with correlation. For covered call writing, the RVaR decreases with correlation and for protective put buying, the RVaR increases with correlation. For leveraged calls, the RVaR increases with correlation and for leveraged puts, the RVaR decreases with correlation.

In summary, although perhaps not a focus when valuing options, correlation between the underlying instrument returns and volatility is an important determinant of various dynamic risk measures, such as RVaR.

Table 12.1.2 presents the results of the simulation allowing the correlation between stock returns and interest rates to range from –0.75 to +0.75 incrementing by 0.25. Panel A presents European-style (ES) and Panel B presents American-style (AS). As expected, this correlation does not have a material impact on the RVaR estimates.

Table 12.1.2 Return VaR Based on GBM BOVM Stock Return and Interest Rate Correlation

Panel A: European-style

Strategy\Correlation	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75
LS	2.75	2.72	2.75	2.72	2.74	2.79	2.70
LCXL	12.28	12.32	12.37	12.35	12.44	12.81	12.40
LCX	14.64	14.67	14.96	14.96	14.92	15.31	14.90
LCXH	17.26	17.59	17.75	18.02	17.61	18.21	17.72
LPXL	25.31	25.53	26.16	26.23	25.87	25.63	25.20
LPX	19.61	20.01	20.40	20.44	20.25	20.15	19.80
LPXH	15.34	15.75	16.09	16.18	15.87	16.05	15.64
LCCWXL	0.48	0.43	0.44	0.39	0.49	0.45	0.42
LCCWX	0.87	0.81	0.82	0.77	0.86	0.83	0.81
LCCWXH	1.25	1.14	1.16	1.12	1.21	1.21	1.15
LPPBXL	1.99	1.99	1.99	1.99	1.98	2.03	1.96
LPPBX	1.58	1.57	1.60	1.58	1.56	1.61	1.54
LPPBXH	1.16	1.18	1.18	1.18	1.15	1.18	1.14
LLCXL	4.33	4.32	4.32	4.25	4.32	4.47	4.28
LLCX	4.24	4.24	4.23	4.13	4.22	4.36	4.15
LLCXH	4.05	4.06	4.07	3.96	4.06	4.18	4.02
LLPXL	3.63	3.59	3.56	3.50	3.58	3.67	3.62
LLPX	4.25	4.17	4.15	4.07	4.17	4.27	4.20
LLPXH	4.94	4.89	4.87	4.79	4.91	5.01	4.95

Panel B American-style

Strategy\Correlation	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75
LS	2.88	2.72	2.75	2.67	2.68	2.78	2.60
LCXL	12.77	12.34	12.52	12.19	12.32	12.57	12.29
LCX	15.52	14.84	14.91	14.54	14.87	14.96	14.57
LCXH	18.12	17.51	17.72	17.43	18.07	17.66	17.56
LPXL	25.43	26.26	25.44	25.67	26.69	26.08	26.32
LPX	20.14	20.70	20.05	20.24	21.14	20.46	20.87
LPXH	16.03	16.43	15.83	15.91	16.91	16.23	16.59
LCCWXL	0.51	0.46	0.50	0.39	0.43	0.45	0.37
LCCWX	0.92	0.83	0.88	0.74	0.80	0.84	0.75
LCCWXH	1.28	1.17	1.28	1.08	1.16	1.20	1.11
LPPBXL	2.07	1.99	1.99	1.93	1.94	1.97	1.91
LPPBX	1.65	1.56	1.56	1.51	1.54	1.53	1.46
LPPBXH	1.17	1.12	1.12	1.10	1.14	1.09	1.07
LLCXL	4.48	4.32	4.36	4.20	4.27	4.38	4.21
LLCX	4.37	4.20	4.28	4.13	4.18	4.28	4.13
LLCXH	4.19	4.03	4.13	3.97	4.02	4.11	3.99
LLPXL	3.84	3.59	3.79	3.62	3.56	3.66	3.48
LLPX	4.51	4.22	4.44	4.24	4.17	4.30	4.08
LLPXH	5.29	4.98	5.27	4.99	4.94	5.10	4.84

Table 12.1.3 presents the results of the simulation allowing the correlation between volatility and interest rates to range from –0.75 to +0.75 incrementing by 0.25. Panel A presents European-style (ES) and Panel B presents American-style (AS). As expected, this correlation does not have a material impact on the RVaR estimates.

Table 12.1.3 Return VaR Based on GBM BOVM Volatility and Interest Rate Correlation

Panel A: European-style

Strategy\Correlation	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75
LS	2.75	2.77	2.73	2.82	2.76	2.77	2.59
LCXL	12.60	12.63	12.28	12.62	12.70	12.31	11.98
LCX	15.18	15.07	14.94	15.03	15.34	14.85	14.39
LCXH	18.20	17.90	17.91	17.76	18.20	17.91	17.26
LPXL	26.16	25.76	25.77	25.52	26.28	24.99	25.48
LPX	20.49	20.30	20.11	20.06	20.53	19.62	19.98
LPXH	16.15	16.13	16.06	15.85	16.21	15.36	15.63
LCCWXL	0.46	0.49	0.46	0.51	0.46	0.47	0.44
LCCWX	0.82	0.88	0.84	0.90	0.84	0.83	0.81
LCCWXH	1.16	1.23	1.21	1.27	1.19	1.16	1.13
LPPBXL	2.04	2.03	1.98	2.02	2.04	1.97	1.92
LPPBX	1.64	1.61	1.60	1.60	1.64	1.57	1.52
LPPBXH	1.24	1.21	1.19	1.19	1.23	1.18	1.13
LLCXL	4.41	4.42	4.30	4.45	4.30	4.30	4.13
LLCX	4.26	4.31	4.20	4.36	4.24	4.22	4.05
LLCXH	4.11	4.14	4.04	4.18	4.09	4.06	3.90
LLPXL	3.59	3.66	3.61	3.69	3.62	3.60	3.45
LLPX	4.18	4.24	4.22	4.28	4.23	4.19	4.01
LLPXH	4.90	4.98	4.94	5.05	4.91	4.93	4.70

Panel B American-style

Strategy\Correlation	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75
LS	2.72	2.80	2.82	2.88	2.77	2.79	2.76
LCXL	12.52	12.80	12.38	12.59	12.58	12.93	12.56
LCX	14.90	15.32	14.93	15.04	15.41	15.44	14.89
LCXH	17.66	18.31	17.78	17.99	18.52	18.11	18.08
LPXL	26.44	26.04	24.91	25.74	25.65	26.37	25.66
LPX	20.70	20.42	19.68	20.29	20.03	20.66	20.32
LPXH	16.40	16.19	15.57	16.04	15.92	16.53	16.14
LCCWXL	0.44	0.44	0.48	0.52	0.42	0.50	0.44
LCCWX	0.81	0.81	0.84	0.91	0.79	0.87	0.82
LCCWXH	1.20	1.14	1.21	1.29	1.11	1.23	1.19
LPPBXL	2.00	2.05	1.98	2.00	2.01	2.08	1.99
LPPBX	1.55	1.61	1.55	1.57	1.61	1.62	1.55
LPPBXH	1.13	1.17	1.13	1.15	1.18	1.15	1.15
LLCXL	4.30	4.47	4.40	4.43	4.36	4.47	4.35
LLCX	4.18	4.36	4.33	4.32	4.26	4.40	4.25
LLCXH	4.04	4.19	4.14	4.16	4.12	4.21	4.09
LLPXL	3.66	3.67	3.67	3.81	3.67	3.73	3.70
LLPX	4.29	4.30	4.31	4.47	4.29	4.38	4.33
LLPXH	5.08	5.07	5.12	5.29	5.09	5.17	5.12

In summary, the ability to conduct RVaR analyses under different sets of simulation assumptions dramatically increases the types of analysis possible for risk managers.

Quantitative finance materials

The quantitative analysis is based on prior materials covered in Modules 5.2 and 8.1. For convenience, we provide selected key formulas here.

The current value of an option is equal to the present value of the expected terminal payout as we assume European-style options where the underlying instrument is adjusted for a continuously compounded cash flow yield.

$$O_0 = PV\left[E_\pi\left(O_T\right)\right] = \iota_U S e^{-\delta T} Bin_{1,\iota_U} - \iota_U X e^{-rT} Bin_{2,\iota_U},$$

where the binomial summations are

$$Bin_{1,1} \equiv Bin_{1,j>a,n} = \sum_{j>a}^{n} \left(\frac{n!}{j!(n-j)!} \right) \pi_1^j \left(1-\pi_1\right)^{n-j},$$

$$Bin_{2,1} \equiv Bin_{2,j>a,n} = \sum_{j>a}^{n} \left(\frac{n!}{j!(n-j)!} \right) \pi_2^j \left(1-\pi_2\right)^{n-j},$$

$$Bin_{1,-1} \equiv Bin_{1,0,j<a} = \sum_{j=0}^{j<a} \left(\frac{n!}{j!(n-j)!} \right) \pi_1^j \left(1-\pi_1\right)^{n-j},$$

$$Bin_{2,-1} \equiv Bin_{2,0,j<a} = \sum_{j=0}^{j<a} \left(\frac{n!}{j!(n-j)!} \right) \pi_2^j \left(1-\pi_2\right)^{n-j},$$

where the terms are as defined before except

$$\pi = \frac{e^{(r-\delta)T} - d}{u-d}.$$

Generically, the binomial option valuation model can be expressed as

$$O_0 = PV_r \left[\sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!} \right) \pi^j \left(1-\pi\right)^{n-j} \max\left(0, \iota_U u^j d^{n-j} S_0 - \iota_U X\right) \right],$$

where u and d are defined as

$$u = \frac{e^{(r-\delta)\Delta t + A}}{Den} \text{ and}$$

$$d = \frac{e^{(r-\delta)\Delta t}}{Den}.$$

For American-style options, we must rely on backward recursion. Working backward through the lattice, the process to compute the option value based on the two subsequent nodes or at time *i* for *j* up moves, the binomial model value (denoted with B superscript) can be expressed as

$$O_{i,j}^B = PV_{r,i,\Delta t}\left[\pi O_{i+1,j+1} + \left(1-\pi\right) O_{i+1,j}\right],$$

The binomial model value, however, may be lower than the early exercise value (denoted with superscript X) that can be expressed as

$$O_{i,j}^X = \max\left[0, \iota_U \left({}_{w/o}S_{i,j} + PV_{r,i,n-i}\left(\underline{D}\right) - X\right)\right],$$

where $\underline{D}$ denotes the vector of future dividend payments and $PV_{r,i,n-i}\left(\underline{D}\right)$ denotes its present value *i* periods from time 0. Recall the lower boundary condition (denoted with superscript *L*) is

$$O_{i,j}^L = \max\left\{0, \iota_U \left[PV_{\delta,i,n-i}\left(S_{i,j}\right) + PV_{r,i,n-i}\left(\underline{D}\right) - PV_{r,i,n-i}\left(X\right)\right]\right\}.$$

Thus, the fair value of the option at time *i* with *j* up moves is

$$O_{i,j} = \max\left[O_{i,j}^B, O_{i,j}^X, O_{i,j}^L\right].$$

The initial option value is obtained through backward induction along the binomial lattice for the underlying instrument.

Thus, the initial value of the various options is determined based on the lattice given above. The simulation is run, and the options are subsequently revalued incorporating the new values for the stock, rate, and volatility as well as the passage of calendar time. Once all the simulations are run, then return VaR is estimated and reported in the tables above.

There are several alternative strategies that could also be pursued (but not here). Selected potential strategies to consider include:

- Short stock (designated cash margin percentage (e.g., 100%), m_S)
- Short call (designated cash margin percentage of underlying stock (e.g., 10%), m_c)
- Short put (designated cash margin percentage of underlying stock (e.g., 10%), m_p)

- Short CCW: Short stock, long call (synthetic leveraged long put: designated cash margin percentage of underlying stock (e.g., 10%), m_{sCCW})
- Short PPB: Short stock, short put (synthetic leveraged short put: designated cash margin percentage of underlying stock (e.g., 10%), m_{sPPB})
- LSC: Short stock, short call (leveraged short call) (synthetic leveraged short put: designated cash margin percentage of underlying stock (e.g., 10%), m_{sLSC})
- LLP: Short stock, long put (leveraged long put) (synthetic leveraged short put: designated cash margin percentage of underlying stock (e.g., 10%), m_{sLLP})

Further, one could add additional stocks to explore various cross-correlations.

Summary

As illustrated with these simple simulations, the ability to conduct RVaR analyses under different sets of simulation assumptions dramatically increases the types of analysis possible for risk managers. Risk managers should be constantly exploring various interactions as well as stress testing parameter assumptions.

References

See modules 5.2 and 8.1.